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A PROPERTY OF CONFORMAL MARTINGALES

by

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Let Z be a complex-valued process and let X and Y be its real and imaginary parts respectively, so that $Z = X + iY$. Let $\{\mathcal{F}_t, t \geq o\}$ be an increasing family of complete σ-fields. We say that $\{Z_t, \mathcal{F}_t, t \geq o\}$ is a conformal martingale if both X and Y are continuous local martingales relative to $\{\mathcal{F}_t\}$, such that $\langle X,Y\rangle_t \equiv o$ and $\langle X,X\rangle_t \equiv \langle Y,Y\rangle_t$. If the point $t=o$ is not included in the parameter set, we will say that $\{Z_t, \mathcal{F}_t, t>o\}$ is a conformal martingale if for all $\delta>o$, $\{Z_t, \mathcal{F}_t, t \geq \delta\}$ is a conformal martingale. We refer the reader to (1) for the properties of conformal martingales. We want to call attention to the following property which, though elementary, is still curious.

Proposition : Let $\{Z_t, \mathcal{F}_t, t>o\}$ be a conformal martingale. Then, for a.e. ω, one of the following happens.

Either (i) $\lim_{t \to o} X_t(\omega)$ exists in the Riemann sphere,

or (ii) for each $\delta>o$, $\{X_t(\omega), o<t<\delta\}$ is dense in $\mathbb{C}$.

Remark : Both possibilities can occur. Indeed, if B_t is a complex Brownian motion from o and if f is holomorphic in $\mathbb{C} - \{o\}$, then $\{f(B_t) , t > o\}$ is a conformal martingale. If o is a removable singularity, $\lim_{t\to o} f(B_t)$ exists. If it is a pole, $\lim_{t\to o} f(B_t) = \infty$, and if it is an essential singularity, $\{f(B_t) , o<t<\delta\}$ is dense in $\mathbb{C}$ for each $\delta>o$. Thus the above proposition is the analogue for conformal martingales of Weierstrass' theorem.

Proof : All we must show is that if (i) doesn't happen , (ii) does. The only fact about conformal martingales we will need is that if $\{Z_t , t \geq t_o\}$ is a conformal martingale, it can be time-changed into a complex Brownian motion with a possibly finite lifetime([1] or [2] , p. 384). Thus all hitting probabilities for Z are dominated by those of Brownian motion.

Suppose (i) doesn't happen. Then there exist concentric circles C_1 , C_2 with a rational center z_o and rational radii $r_1 < r_2$ respectively, such that the number of incrossings of (C_1 , C_2) by $Z_t(\omega)$ is infinite. Here, the number of incrossings $\nu_{a,b}(\omega)$ of (C_1 , C_2) in (a,b) is defined to be the number of downcrossings (in the usual sense) of the interval $[r_1 , r_2]$ by the process $\{|Z_t(\omega) - z_o| , a<t<b\}$. Let D be a disc.

Suppose that D is not entirely contained in the interior of C_1 . (If it is, we merely talk about outcrossings rather that incrossings in what follows.) The proposition will be proved if we can show that for any $\delta>o$, $T_D < \delta$ a.s. on the set $\{\nu_{o\delta} = \infty\}$, where $T_D = \inf\{t > o : Z_t \in D\}$.

Let N be an integer.

$$(1)\qquad P\{\nu_{o\delta} = \infty \ , \ T_D > \delta\} \le \lim_{n\to\infty} \{\nu_{\frac{1}{n}\delta} > N \ , \ T_D > \delta\}$$
$$= \lim_{n\to\infty} P\{T_D > \delta | \nu_{\frac{1}{n}\delta} > N\} \ P\{\nu_{\frac{1}{n}\delta} > N\}$$
$$\le \lim_{n\to\infty} P\{T_D > \delta | \nu_{\frac{1}{n}\delta} > N\} \ .$$

But this last probability involves only hitting probabilities, and hence can be dominated by the corresponding probability for Brownian motion. If P_B^z is the probability measure of Brownian motion starting from z , let :

$$\rho = \sup_{z \in C_2} P_B^z \{T_D < T_{C_1}\} < 1 \ .$$

It is easy to see, using the strong Markov property, that

$$P_B^z\{B_t \notin D \ , \ \forall t \in (\tfrac{1}{n} \ , \ \delta) | \nu_{\frac{1}{n}\delta} \ge N\} \le \rho^{N-1}$$

Thus, from (1) we have :

$$(2)\qquad P\{\nu_{o\delta} = \infty \ , \ T_D < \delta\} \le \rho^{N-1} \to 0 \quad \text{as} \quad N \to \infty \ ,$$

and we are done.

References :

(1) R.K. GETOOR and M.J. SHARPE : Conformal martingales , Invent. Math. 16 , pp. 271-308 (1972).

(2) J.L. DOOB : Stochastic Processes , John Wiley and Sons , New York , 1953.