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On changing time

ON CHANGING TIME

by R. Cairoli and J.B. Walsh

Meyer's section theorem, Skorohod's embedding theorem,
and a number of time-change theorems are all aspects of a funda-
mental principle underlying general theory of processes, to wit :
there is a stopping time which will do almost anything one wants
it to do*).

The corresponding principle for multiparameter processes
operates only at a much-reduced level. It is not that there is
any lack of stopping times. To the contrary, there is a great,
even confusing, number of analogous objects. It is just that, by
and large, they are of limited usefulness. We propose to illus-
trate one of these limits in this note.

In two-dimensional time, one analogue (there are others)
of Brownian motion is the Brownian sheet $\{W_{s,t}, (s,t) \in \mathbb{R}_+^2\}$, which
is characterized by the fact that it is a zero-mean Gaussian pro-
cess with covariance function $\gamma(s,t;u,v) = (s \wedge u)(t \wedge v)$.

Question : can a given two-parameter martingale be time-
changed into a Brownian sheet ?

The answer to this in the one-parameter case, given
by the Dubins-Schwarz theorem, is "yes", and the time-change can

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*) Thus, while one can't find a stopping time which will boil an
egg, he can find one which will keep the egg from being hard boiled.

be constructed as follows. Let $\{M_t, t \in \mathbb{R}_+\}$ be a continuous martingale with unbounded paths and let $<M>_t$ be the continuous increasing process, with $<M>_o = o$, for which $M_t^2 - <M>_t$ is a martingale. If $T_t = \inf\{s : <M>_s > t\}$ is the inverse of $<M>_t$, then $\{M_{T_t}, t \in \mathbb{R}_+\}$ is a Brownian motion.

Notice that the time-change depends only on the increasing process. Thus, to make our question more specific, we ask if a given two-parameter martingale can be transformed into a Brownian sheet via a time-change which depends only on the increasing process $<M>_{s,t}$. (See [1] for a discussion of the increasing process associated with a two-parameter martingale.)

We will see that the answer to this question is "no", even if we restrict ourselves to strong martingales [1].

Let $\{W_{s,t}, (s,t) \in \mathbb{R}_+^2\}$ be a Brownian sheet, let

$$\phi(s,t) = \begin{cases} 1 & \text{if } st \leq 1, \\ 2 & \text{if } st > 1, \end{cases}$$

and define

(1)
$$M_{s,t} = \int\limits_o^s \int\limits_o^t \phi(u,v) dW_{u,v}.$$

M is a strong martingale with increasing process

(2)
$$<M>_{s,t} = \int\limits_o^s \int\limits_o^t \phi^2(u,v) du dv.$$

This process is deterministic, so that any time-change depending

only on it must be deterministic, i.e. of the form $(s,t) \to \Gamma(s,t)$, where Γ is a fixed mapping of $\mathbb{R}^2_+$ onto itself. Thus, the problem reduces to the simpler, but still not quite trivial, one of finding a mapping Γ of the positive quadrant onto itself such that $\{M_{\Gamma(s,t)}, (s,t) \in \mathbb{R}^2_+\}$ is a Brownian sheet.

Some notation : S will denote the open quadrant $\{(s,t): s > o, t > o\}$, H_c and S_c the sets $\{(s,t) \in S: st = c\}$ and $\{(s,t) \in S: st \leq c\}$ respectively $(c > o)$. We say $(s,t) \prec (u,v)$ if $s \leq u$ and $t \leq v$. When we write $z \wedge z'$ and $z \vee z'$ for elements of $\mathbb{R}^2_+$, we mean the inf and sup respectively relative to the partial order "$\prec$". Since all the processes we consider vanish on the axes, we need only consider mappings on the open set S. A mapping Γ of S onto itself is order-preserving if, for $z, z' \in S$, $z \prec z'$ if and only if $\Gamma(z) \prec \Gamma(z')$. An order-preserving map is necessarily one-to-one.

If we speak about a martingale without indicating the σ-fields, it is understood that the natural σ-fields are intended. $\mathcal{F}_z$ will always refer to the σ-fields generated by W, suitably completed. These fields satisfy the conditional independence hypothesis (F4) of [1]:

(F4) For each $(s,t) \in \mathbb{R}^2_+$, the fields $\mathcal{F}_{s,\infty}$ and $\mathcal{F}_{\infty,t}$ are conditionally independent given $\mathcal{F}_{s,t}$.

Lemma 1. Let Γ and Γ' be order-preserving mappings of S onto itself. If $\Gamma(z) = \Gamma'(z)$ for each z in some H_c, then $\Gamma \equiv \Gamma'$.

Proof. If $z \in S$, there are unique $z_1, z_2 \in H_c$ such that either $z = z_1 \vee z_2$ or $z = z_1 \wedge z_2$. If, for instance, $z = z_1 \vee z_2$, then $\Gamma(z) = \Gamma(z_1 \vee z_2) = \Gamma(z_1) \vee \Gamma(z_2)$, since Γ preserves order. But this equals $\Gamma'(z_1) \vee \Gamma'(z_2) = \Gamma'(z_1 \vee z_2) = \Gamma'(z)$.

Lemma 2. Let $\{X_z, \mathcal{G}_z, z \in S\}$ be a martingale whose σ-fields $\mathcal{G}_z$ satisfy (F4), and such that $P\{X_z = X_{z'}\} < 1$ if $z \neq z'$. Then

$$(3) \qquad E\{X_{z'} | X_z\} = X_z \text{ if and only if } z \prec z'.$$

Proof. Note that

$$(4) \qquad E\{X_{z'} | X_z\} = E\{E\{X_{z'} | \mathcal{G}_z\} | X_z\} = E\{X_{z \wedge z'} | X_z\},$$

where we have used (F4) to get the second equality. Suppose $E\{X_{z'} | X_z\} = X_z$. Then

$$(5) \qquad X_z = E\{X_{z \wedge z'} | X_z\}.$$

On the other hand, $z \wedge z' \prec z$ and X is a martingale, so

$$(6) \qquad X_{z \wedge z'} = E\{X_z | X_{z \wedge z'}\}.$$

By p.314 of [2], (5) and (6) together imply that $X_z = X_{z \wedge z'}$. If follows that $z = z \wedge z'$, so that $z \prec z'$. This establishes (3) in one direction. The other direction is clear, so we are done.

Lemma 3. Let $\{X_z, \mathcal{F}_z, z \in S\}$ be a martingale with the property that $P\{X_z = X_{z'}\} < 1$ if $z \neq z'$. Let Γ be a mapping of S onto itself and set $Y_z = X_{\Gamma(z)}$. If $\{Y_z, z \in S\}$ is a martingale with the same property and whose natural σ-fields satisfy (F4),

then Γ is order-preserving.

Proof. We apply Lemma 2 to both X and Y :

$$E\{Y_{z'}|Y_z\} = Y_z \text{ if and only if } z \prec z',$$

and

$$E\{X_{\Gamma(z')}|X_{\Gamma(z)}\} = X_{\Gamma(z)} \text{ if and only if } \Gamma(z) \prec \Gamma(z').$$

It follows that

$$z \prec z' \text{ if and only if } \Gamma(z) \prec \Gamma(z').$$

Remarks. 1) The mapping $\Gamma(s,t) = (st,t)$ is not order-preserving, even though $\{W_{\Gamma(z)}, z \in S\}$ is a martingale. Its natural σ-fields do not satisfy (F4), however.

2) Lemmas 1-3 have been stated for the parameter set S for simplicity. They hold, with no change in proof, if the parameter set is some S_c.

Lemma 4. Let $S_\infty = S$ and let $\mathbb{G}_c$ (o < c $\leq$ ∞) be the group of all mappings Γ of S_c onto itself which have the property that $\{W_{\Gamma(z)}, z \in S_c\}$ is a Brownian sheet. Then $\mathbb{G}_c$ is generated by the mappings

$$\Gamma_\lambda : \Gamma_\lambda(s,t) = (\lambda s, \frac{t}{\lambda}) \ (\lambda > o) \text{ and } \Gamma_+ : \Gamma_+(s,t) = (t,s)$$

on S_c, and, consequently, each of its elements can be uniquely extended to an element of $\mathbb{G}_\infty$.

Proof. Let $\mathbb{G}$ be the group generated by the mappings

Γ_λ and Γ_+ on S_c. W is a Gaussian process with covariance function $\gamma(s,t;u,v) = (s \wedge u)(t \wedge v)$. A mapping Γ is in $\mathbb{G}_c$ if and only if it leaves γ invariant on S_c. Γ_λ and Γ_+ do this, so $\mathbb{G} \subset \mathbb{G}_c$. We must show that $\mathbb{G}_c \subset \mathbb{G}$. If $\Gamma \in \mathbb{G}_c$, Γ preserves order (Lemma 3, remark 2) and is determined by its action on any one of the $H_{c'}(c' \leq c, c' < \infty)$ (Lemma 1). Furthermore, $\Gamma(H_{c'}) = H_{c'}$, since, if $\Gamma(s,t) = (s',t')$,

$$st = \gamma(s,t;s,t) = \gamma(s',t';s',t') = s't'.$$

Suppose for simplicity that $H_1 \subset S_c$. Let $o < a < b$, so that $(a,\frac{1}{a})$ and $(b,\frac{1}{b})$ are distinct points of H_1. Let their images be $(a',\frac{1}{a'})$ and $(b',\frac{1}{b'})$ respectively. Since γ is invariant under Γ,

$$(7) \qquad (a \wedge b)(\frac{1}{a} \wedge \frac{1}{b}) = (a' \wedge b')(\frac{1}{a'} \wedge \frac{1}{b'}).$$

There are two cases, according to whether $a' < b'$ or $b' < a'$.

Case 1 : $a' < b'$. In this case, (7) says that $\frac{a}{b} = \frac{a'}{b'}$, so that, if $\lambda = \frac{a'}{a}$, $\Gamma(a,\frac{1}{a}) = \Gamma_\lambda(a,\frac{1}{a})$ and $\Gamma(b,\frac{1}{b}) = \Gamma_\lambda(b,\frac{1}{b})$. It is not hard to verify that if z is a third point of H_1, $\Gamma(z) = \Gamma_\lambda(z)$, so that $\Gamma = \Gamma_\lambda$ on H_1, and hence, since Γ is determined by its action on H_1, on all of S_c.

Case 2 : $b' < a'$. Then (7) implies that $\frac{a}{b} = \frac{b'}{a'}$, so that, if $\lambda = aa'$, $b' = \lambda\frac{1}{b}$. Thus $\Gamma(a,\frac{1}{a}) = \Gamma_\lambda\Gamma_+(a,\frac{1}{a})$ and $\Gamma(b,\frac{1}{b}) = \Gamma_\lambda\Gamma_+(b,\frac{1}{b})$. It then follows as in case 1 that $\Gamma = \Gamma_\lambda\Gamma_+$ on S_c, and hence that $\Gamma \in \mathbb{G}$.

We can now come to the point. Lemmas 1, 3 and 4 show us that we have very few deterministic time-changes at our disposal,

so the following proposition comes as no surprise.

Proposition. The martingale M defined in (1) can not be transformed into a Brownian sheet by any time-change depending only on <M>.

Proof. As remarked before, <M> is deterministic, so that we need only consider deterministic time-changes. Thus, suppose there exists a mapping Γ of S onto itself which transforms M into a Brownian sheet. Now M is already a Brownian sheet on S_1, for $\phi \equiv 1$ there. Thus, by Lemma 4, there is a $\Lambda \in \mathbb{G}_\infty$ for which $\Lambda = \Gamma$ on S_1. Notice that $\Lambda^{-1}\Gamma$ must also transform M into a Brownian sheet. Clearly $P\{M_z = M_{z'}\} < 1$ if $z \neq z'$, so that, by Lemma 3, $\Lambda^{-1}\Gamma$ is order-preserving. But $\Lambda^{-1}\Gamma \equiv I$, the identity, on S_1. By Lemma 1, $\Lambda^{-1}\Gamma \equiv I$, and we are forced to conclude that M itself is already a Brownian sheet. This is a contradiction, and we are done.

References

[1] R. Cairoli and J.B. Walsh. Stochastic integrals in the plane. Acta mathematica, Vol. 134, 1975, p. 111-183.

[2] J.L. Doob. Stochastic processes, John Wiley & Sons, New York, 1953.