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KAI LAI CHUNG

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PEDAGOGIC NOTES ON THE BARRIER THEOREM

by Kai Lai Chung*

Let D be an open bounded set in R^d, $d \geq 1$; ∂D its boundary. Given $z \in \partial D$, a function f defined in D is called a barrier at z iff

(i) f is superharmonic and > 0 in D;

(ii) $\lim\limits_{D \ni x \to z} f(x) = 0.$

Let $\{X_t, t \geq 0\}$ be the standard Brownian motion in R^d. For any Borel subset B of R^d, let S_B denote the first exit time from B:

$$S_B = \inf\{t > 0: X_t \notin B\}.$$

D being fixed, we write S for S_D below. A point x is regular iff $P^x\{S = 0\} = 1$; otherwise $P^x\{S > 0\} = 1$ by the zero-one law.

Proposition 1. Let f be superharmonic in D and ≥ 0 in D. Extend f to $\overline{D}$ (= closure of D) as follows: for each $z \in \partial D$,

$$f(z) = \varliminf_{D \ni x \to z} f(x). \tag{1}$$

Then for each $x \in D$ we have

$$f(x) \geq E^x\{f(X(S))\}. \tag{2}$$

Proof. Let K_n be compact, $K_n \subset K^o_{n+1}$ (= interior of K_{n+1}) $\subset D$ such that $\bigcup_n K_n = D$. Then

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(3) $$S_{K_n} < S, \qquad S_{K_n} \uparrow S.$$

For each n, the process

(4) $$\{f(X_{t \wedge S_{K_n}});\ 0 \le t < \infty\}$$

is a supermartingale for each P^x, $x \in K_n^o$ (see Doob's lecture notes[1] for the latest proof of this result) . Letting $t \to \infty$ and using Fatou's lemma, we deduce that

(5) $$f(x) \ge E^x\{f(X(S_{K_n}))\}, \qquad x \in K_n^o.$$

Letting $n \to \infty$, $X(S_{K_n}) \to X(S) \in \partial D$, hence by the extended definition of f we have

$$\underline{\lim}_{n \to \infty} f(X(S_{K_n})) \ge f(X(S)).$$

Since $f \ge 0$ in $\overline{D}$, it follows from Fatou's lemma that

$$f(x) \ge E^x\{\underline{\lim}_{n \to \infty} f(X(S_{K_n}))\} \ge E^x\{f(X(S))\}.$$

This is true if $x \in K_n^o$, for every n; hence it is also true if $x \in D$.

Proposition 2. Let B_1 and B_2 be two open subsets of R^d, $B_1 \subset B_2$. Then for every $z \in \overline{D}$,

(6) $$E^z\{S_{B_2} < S;\ f(X(S_{B_2}))\} \le E^z\{S_{B_1} < S;\ f(X(S_{B_1}))\}.$$

1. See p.7 below for an alternative proof that doesn't use this result.

Proof. Writing S_1 for S_{B_1}, S_2 for S_{B_2}, we have

$$E^z\{S_1 < S;\ E^{X(S_1)}[S_2 < S;\ f(X(S_2))]\}$$

$$(7) \qquad = E^z\{S_1 < S;\ E^{X(S_1)}[S_2 < S;\ f(X(S_2 \wedge S))]\}$$

$$\leq E^z\{S_1 < S;\ E^{X(S_1)}[f(X(S_2 \wedge S))]\}$$

because $X(S_2 \wedge S) \in \overline{D}$ and $f \geq 0$ in $\overline{D}$. Now $X(S_1) \in B_2 \cap D$ on $\{S_1 < S\}$, hence we may apply Prop. 1 with D replaced by $B_2 \cap D$ to obtain

$$f(X(S_1)) \geq E^{X(S_1)}[f(X(S_{B_2 \cap D}))]$$

$$= E^{X(S_1)}[f(X(S_2 \wedge S))].$$

Substituting this into the last term of (7), we obtain (6).

<u>Theorem 1</u>. If there exists a barrier at $z \in \partial D$, then z is regular.

Proof. Let f be the barrier, extend it to $\overline{D}$ as in (1). Apply Prop. 2 with B_1 and B_2 two balls centered at z. Suppose z is not regular, so that $P^z\{S > 0\} = 1$. Since $S_{B_2} \downarrow 0$ as B_2 shrinks to z, we may choose B_2 so that

$$P^z\{S_{B_2} < S\} > 0.$$

Since $X(S_{B_2}) \in D$ on $\{S_{B_2} < S\}$, and $f > 0$ in D, we have

$$E^z\{S_{B_2} < S;\ f(X(S_{B_2}))\} > 0\,. \tag{8}$$

Now fix B_2 and let B_1 shrink to z. Then $X(S_{B_1}) \to z$, and on $\{S_{B_1} < S\}$, $X(S_{B_1}) \in D$; hence $f(X(S_{B_1})) \to 0$ by property (ii) of a barrier. Replacing f by $f \wedge 1$, which preserves (i) and (ii), we may assume that f is bounded. Hence by bounded convergence,

$$E^z\{S_{B_2} < S;\ f(X(S_{B_2}))\} \to 0\,. \tag{9}$$

The relations (6), (8) and (9) are incompatible. Hence z must be regular.

Remark. Theorem 1 is true for any continuous, strongly Markovian process in a nice topological space, provided that the definition of a "superharmonic function" will imply (5) above. This is essentially Dynkin's generalization (see [1], p. 35 ff.). The observation that Prop. 2 follows from Prop. 1 is due to R. Durrett.

Next, we define f in R^d as follows:

$$f(x) = E^x\{S\}\,. \tag{10}$$

<u>Proposition 3</u>. f is bounded in R^d and continuous in D.

Proof. $\{\|X_t\|^2 - dt,\ t \geq 0\}$ is a martingale, where $\|x\|^2 = \sum_{j=1}^{d} x_j^2$. Hence for any $x \in R^d$ and $n \geq 1$,

$$E^x\{\|X_{S\wedge n}\|^2 - d(S\wedge n)\} = \|x\|^2\,.$$

Letting $n \to \infty$, since $\|X_{S \wedge n}\|^2$ is bounded we obtain

(11) $$E^x\{\|X_S\|^2\} - dE^x\{S\} = \|x\|^2.$$

The first term in (11) is the stochastic solution to the Dirichlet problem for the domain D and the boundary function $x \to \|x\|^2$. Hence it is harmonic in D and therefore is in $C^\infty(D)$; hence so is f.

Let B be an open ball with center 0 and radius r. Apply (11) to S_B we obtain

$$E^x\{S_B\} = \frac{r^2 - \|x\|^2}{d}, \qquad x \in D.$$

Choose r so large that $\overline{D} \subset B$. It follows that $f \leq r^2/d$ in $\overline{D}$, hence in R^d because $f = 0$ in $R^d - \overline{D}$.

<u>Proposition 4</u>. The f in (10) is upper semi-continuous in R^d.

Proof. Let D_n be open bounded such that $D_n \supset \overline{D}_{n+1} \supset D$ and $\bigcap_n \overline{D}_n = \overline{D}$. Then for each $x \in R^d$, we have

(12) $$S_{D_n} \downarrow S \qquad P^x\text{-a.s.}$$

For each n, define f_n in R^d as follows:

$$f_n(x) = E^x\{S_{D_n}\}.$$

By Prop. 3, f_n is continuous in D_n. It follows from (12) and the boundedness of f_1 (by Prop. 3) that

(13) $$f_n(x) \downarrow f(x), \qquad x \in R^d.$$

The continuity of f_n in D_n, the fact that D_n is an open neighborhood of $\overline{D}$, and the relation (13) together imply that

(14) $$f(x) \geq \overline{\lim_{y \to x}}\ f(y), \qquad x \in R^d.$$

<u>Theorem 2</u>. Let $z \in \partial D$ and z be regular. Then the function f in (10), restricted to D, is a bounded continuous barrier at z.

Proof. This function is superaveraging over surfaces of closed balls in D, by a standard argument. It is bounded and continuous in D by Prop. 3. Hence it is superharmonic in D by the usual definition. It is clearly > 0 in D. Since z is regular, $f(z) = 0$. By Prop. 4, we have

$$\overline{\lim_{x \to z}}\ f(x) \leq f(z) = 0$$

even if x is not restricted to D. Hence f is a barrier at z.

Remark. To generalize Theorem 2 to a continuous, strongly Markovian process we need only to have Prop. 4. As its proof shows, it is sufficient to have the function f in (10) upper semi-continuous in D. (This will force f to be continuous in D if by "superharmonic" we include "lower semi-continuous" as habitually done.) If X has the strong Feller property, then $E^x\{S \circ \theta_t\}$ is continuous in D. Since

$$E^x\{S\} = \lim_{t \downarrow 0} \downarrow E^x\{t + S \circ \theta_t\}, \qquad x \in D,$$

the left member is upper semi-continuous. This is Dynkin's generalization.

Here is the alternative proof mentioned on p.2 (communicated by J.L.Doob).

Let $B(x)$ be the open ball with center x and radius half the distance from x to ∂D . Define $T_0 = 0$ and let T_{n+1} be the hitting time after T_n of $\partial B(X(T_n))$. Then T_n is optional and $\{X(T_n), \mathfrak{F}(T_n), n \geq 0\}$ is a Markov process with stationary transition probabilities. The transition distribution from x is the uniform distribution on $\partial B(x)$. It follows trivially that if f is positive and superharmonic the process $\{f(X(T_n)), \mathfrak{F}(T_n)\}$ is a positive supermartingale and that $T_n \longrightarrow S$ a.s.. Hence $f(x) \geq E^x[f(X(T_n))]$. By (1) this f is lower semicontinuous on $\overline{D}$ and so Fatou's lemma gives (2).