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ABSOLUTE CONTINUITY OF MARKOV PROCESSES

by Hiroshi Kunita

Introduction and summary.

A great deal of attentions has been devoted to multiplicative functionals of Markov processes and its transformations, and the absolute continuity of Markov processes. The problem we are concerned in this paper is this : Given two Markov processes which are equivalent on the germ field, find a criterion that they are equivalent up to the lifetime. As an earlier result for this direction, we refer to Dawson [2].

After introducing Lévy systems of Hunt process in §1, we get the representation of a terminal time ; the terminal time consists of a hitting time and a first jumping time for a suitable set. The result is very close to Walsh-Weil [18].

§3 is devoted to the representation of a multiplicative functional (MF). Then we study the relation of Lévy systems between the given Markov process and the one transformed by MF. These are generalizations of the work by Kunita-Watanabe [19]. The Lebesgue decomposition of two Markov processes is discussed in §5. The Radon-Nikodym derivative is defined as a MF.

Our central problem is discussed in §6. Assuming that two Hunt processes satisfy Hunt's hypothesis (K), we show that the equivalence on the germ field implies the equivalence up to the lifetime if and only if corresponding Lévy measures are equivalent.

Three examples are discussed in §7.

§1. Notations and definitions. Lévy system.

In this section, we introduce the basic notation and terminology of Hunt process, and then introduce a Lévy system. For a more information of the standard material, refer to the book of Blumenthal-Getoor [1].

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P_x)$ be a Hunt process with state space E. Let $\mathcal{F}_t^0 = \sigma(X_s : s \le t)$. Recall that $\mathcal{F}_t$ is the "completed" σ-field of $\mathcal{F}_t^0$. We denote by ζ the lifetime of the process as usual. Throughout this paper, we assume Meyer's hypothesis (L) : i.e., there exists a measure γ on E such that any α-excessive function with $\int u(x)\gamma(dx) = 0$ is identically 0. Also, the following Hunt's hypothesis (K) is often our basic assumption (Hunt [5]).

Hypothesis (K). If u is an α-excessive function, $u(X_t)$ is continuous in $t \in (0, \zeta)$ where X_t is continuous.

The event which is valid with P_x-probability 1 for all $x \subset E_\Delta$ is denoted as "a.s." (almost surely).

A right continuous $\mathcal{F}_t$-adapted process A_t is called an additive functional (AF) if $A_0 = 0$, $A_t = A_\zeta$ for $t \ge \zeta$ a.s. and satisfies $A_{t+s} = A_t + A_s \circ \theta_t$ a.s. for each $t, s \ge 0$. If the exceptional set $N_{t,s}$ that the above is not valid satisfies $P_x(\bigcup_{t,s} N_{t,s}) = 0$ for all x, A_t is called a perfect AF. In the sequel, we consider perfect AF's only.

Let A_t be a continuous increasing AF. A measure μ on E is called a canonical measure of A_t of $\int_0^t 1_F(X_s)dA_s = 0$ a.s. is equivalent to $\mu(F) = 0$. The existence of the canonical measure is known. Let A_t^1 and A_t^2 be continuous increasing AF's with canonical measures μ^1 and μ^2. Then $\mu^1 << \mu^2$ (absolutely continuous) if and only if there exists a $\mathcal{E}$-measurable function f such that $A_t^1 = \int_0^t f(X_s)dA_t^2$. (We write $A^1 << A^2$ in such a case).

Following S.Watanabe [18], we shall introduce a Lévy system. Let $f(x, y) \geq 0$ be a bounded $E \times E_\Delta$-measurable function such that $f(x,x) = 0$ on E. Set

$$P_t(f) = \sum_{s \leq t} f(X_{s-}, X_s).$$

If $P_t(f)$ is integrable, there exists a continuous increasing AF $\tilde{P}_t(f)$ such that $P_t(f) - \tilde{P}_t(f)$ is a P_x-martingale. A pair of a kernel $n(x, dy)$ on $E \times E_\Delta$ and a continuous increasing AF φ_t is called a <u>Lévy system</u> if

$$\tilde{P}_t(f) = \int_0^t n \circ f(X_s) d\varphi_s \qquad \text{a.s.}$$

holds for all f such that $P_t(f)$ is integrable. Here

$$n \circ f(x) = \int_{E_\Delta} n(x, dy) f(x, y).$$

Define

$$Q_t(f) = P_t(f) - \int_0^t n \circ f(X_s) d\varphi_s. \tag{1.1}$$

It is a martingale AF.

The Lévy system of a given process is not unique. Let $(\tilde{n}, \tilde{\varphi})$ be an another Lévy system with the canonical measure $\tilde{\mu}$ of $\tilde{\varphi}$. Then the measures on $E \times E_\Delta$;

$$\mu(dx)n(x, dy) \quad \text{and} \quad \tilde{\mu}(dx)\tilde{n}(x, dy)$$

are equivalent (mutually absolutely continuous). In fact, if $P_t(f) = 0$ a.s., then $\tilde{P}_t(f) = 0$ a.s.. This implies

$$\int \mu(dx)n(x, dy)f(x, y) = 0.$$

The same is valid for $\tilde{\mu}(dx)\tilde{n}(x, dy)$. Thus the null sets of the measure $\mu(dx)n(x, dy)$ does not depend on the choice of Lévy system. We call it the canonical measure.

Now let A_t be a discontinuous increasing AF. It is called quasi-left continuous if for any increasing sequence of stopping times T_n with limit T, it holds $\lim_{n\to\infty} A_{T_n} = A_T$ a.s.. The following representation theorem is due to Motoo and S. Watanabe [18].

Representation theorem. Let A_t be a discontinuous increasing quasi-left continuous AF. Suppose that A_t is locally integrable. Then there exists a nonnegative $\mathcal{E}\times\mathcal{E}_\Delta$-measurable function f such that $A_t = P_t(f)$ a.s.. The function f is unique except for the null set of the measure $(\mu(dx)n(x, dy))$.

In several points of later discussions, the classification of stopping times due to Meyer [12] is often used. Here we review it. A stopping time T is called totally inaccessible if for any increasing sequence of stopping times S_n with limit S less than T, it holds

$$P_x(S_n < S = T < \infty \quad \text{for all} \quad n) = 0$$

for all x. A stopping time T is called accessible (or predictable) if for each x there exists an increasing sequence of stopping times T_n with limit T such that $T_n < T$ for all n a.s. P_x on the set $0 < T$. An important characterization is that T is totally inaccessible if and only if $X_T \neq X_{T-}$ on $0 < T < \infty$ a.s. and T is accessible if and only if $X_T = X_{T-}$ ($X_0 = X_{0-}$ by convention). Let now T be a stopping time.

Define T^a by T if $X_T = X_{T-}$ and by ∞ if $X_T \neq X_{T-}$, then T^a is accessible. On the other hand, T^i defined by T if $X_T \neq X_{T-}$ and ∞ if $X_T = X_{T-}$ is totally inaccessible. We have the decomposition $T = T^a \wedge T^i$.

§2. Representation of terminal times.

In this section, we review some properties of terminal time and then obtain a representation of exact terminal time, that will play a basic role in §6.

A stopping time T is called a terminal time if $T\circ\theta_t + t = T$ holds on the set $T > t$ a.s. for each t. The exceptional set may depend on t. If the exceptional set does not depend on t, it is called a perfect terminal time.

Let T be a terminal time. It holds $T\circ\theta_t + t \geq T$ a.s. for each t and $T\circ\theta_t + t \geq T\circ\theta_s + s$ a.s. for each $s \leq t$. Define $\hat{T} = \lim_{s\downarrow 0} (T\circ\theta_s + s)$ (s : rationals). Then $\hat{T} \geq T$ a.s. and $\hat{T} = T$ on $T > 0$. If $\hat{T} = T$ a.s., T is called exact. It is easy to see that $\hat{T}$ defined above is exact. Walsh [17] and Meyer [13] proved that the exact terminal time is perfect.

Suppose now T is an exact terminal time. Then the function $u(x) = E_x(e^{-T})$ is 1-excessive. In fact,

$$e^{-t}P_t u(x) = E_x(e^{-(t+T\circ\theta_t)} \; ; \; \zeta > t)$$

and the right hand side increases to u as t decreases to 0. Let F be the set of regular points of T ;

$$F = \{x \in E_\Delta \; ; \; P_x\,(T = 0) = 1\}.$$

Then F is finely closed nearly Borel set, since F coincides with the set of x such that $u(x) = 1$.

Let T_F be the hitting time for the set F. Then it holds $T \le T_F$ a.s.. In fact, note that $T\circ\theta_{T_F} > 0$ on $T > T_F$: Then

$$P_x(T > T_F) = P_x(T\circ\theta_{T_F} > 0,\ T > T_F)$$
$$= E_x(E_{X_{T_F}}(T > 0)\ ;\ T > T_F)$$
$$= P_x(X_{T_F} \notin F,\ \ T > T_F).$$

Here we used the 0-1 law : $P_x(T > 0) = 0$ or 1. Since F is finely closed, $X_{T_F} \in F$ a.s. on $T_F < \infty$. This proves $P_x(T > T_F) = 0$.

The following representation of the terminal time was obtained by Walsh-Weil [18], under a different assumption.

Theorem 2.1. Assume hypothesis (K). Let T be an exact terminal time and let F be the set of regular points of T. Then there exists a $\mathcal{E}\times\mathcal{E}$-measurable set A included in $(\overline{E-F})\times(E-F)-\{(x,x)\ ;\ x \in E\}$ such that

$$T = \inf\{t > 0 : (X_{t-}, X_t) \in A \text{ or } X_t \in F\} \quad \text{a.s.} \tag{2.1}$$

on $T < \zeta$.

Before going to the proof, we prepare two lemmas.

Lemma 2.1. Define iterates of T as $T_0 = 0$, $T_1 = T$, $T_n = T_{n-1} + T\circ\theta_{T_{n-1}}$ and $T_\infty = \lim_{n\to\infty} T_n$. Then $T_\infty = T_F$ a.s. on $T_\infty < \zeta$.

Proof. The stopping time T_∞ is a terminal time. In fact, it is easy to see that $T_n\circ\theta_t + t = T_{n+k}$ if $T_k \le t < T_{k+1}$. Letting n tend to infinity, we obtain $T_\infty\circ\theta_t + t = T_\infty$. Moreover, it holds $\{T_\infty = 0\} = \{T = 0\}$, which can be checked easily by induction. This implies that T_∞ is exact and the set of regular points of T_∞ coincides with F. Therefore it holds $T_\infty \le T_F$ a.s..

We shall prove $T_\infty \ge T_F$ a.s.. Recall $u(x) = E_x(e^{-T})$. By the strong Markov property,

$$E_x(e^{-T_n}) = E_x(e^{-T_{n-1}}u(X_{T_{n-1}})).$$

Making n tend to ∞, we see that $\lim_{n\to\infty} u(X_{T_n}) = 1$ on $T_\infty < \infty$. Hypothesis (K) implies $\lim_{n\to\infty} u(X_{T_n}) = u(X_{T_\infty})$ on $T_\infty < \zeta$. Therefore, $X_{T_\infty} \in F$ on $T_\infty < \zeta$ a.s., i.e. $T_\infty \ge T_F$ on $T_\infty < \zeta$.

Define now

$$S = \inf_{n\ge 1} \{T_n \ ; \ X_{T_n} = X_{T_n-}\},$$
$$= T_\infty \quad \text{if} \ \{ \quad \} = \phi.$$

(As a convention, we put $X_0 = X_{0-}$.) Then S is a terminal time. In fact, if $t < S$, there exists a nonnegative integer k such that $T_k \le t < T_{k+1}$. Note the relation $T_n \circ \theta_t + t = T_{n+k}$. Then

$$S\circ\theta_t + t = \inf_n\{T_n\circ\theta_t + t \ ; \ X_{T_n\circ\theta_t}\circ\theta_t = X_{T_n\circ\theta_t-}\circ\theta_t\}$$
$$= \inf_n\{T_{n+k} \ ; \ X_{T_{n+k}} = X_{T_{n+k}-}\}$$
$$= S.$$

The exactness of S will be obvious.

<u>Lemma</u> 2.2. It holds $S = T_F$ a.s. on $S < \zeta$.

<u>Proof</u>. Let us define the accessible part of S as

$$S^a = S \quad \text{if} \quad X_S = X_{S-} \quad \text{and} \quad S < \infty$$

$= \infty$ if $X_S \neq X_{S-}$.

If holds $S^a = S$ on $S < T_\infty = T_F$. Then for each P_x, there exists an increasing sequence of stopping times S_n^a with limit S^a such that $S_n^a < S^a$ for all n on the set $S^a > 0$ a.s. P_x. Set $S_n = S_n^a \wedge S$. Then $\bigcup_n (S_n = S) \subset (S = T_F)$ a.s.. We shall show that $S \circ \theta_{S_n} + S_n = S$ a.s. for each n. If $S_n < S$, the relation is clear since S is a perfect terminal time. If $S_n = S$, then $S \circ \theta_{S_n} = 0$. In fact

$$P_x(S \circ \theta_{S_n} > 0,\ S_n = S) = E_x(P_{X_{S_n}}(S > 0)\ ;\ S_n = S)$$

$$\leq E_x(P_{X_{T_F}}(S > 0)\ ;\ S = T_F).$$

Note that $T \leq S \leq T_F$, then the set of regular points of S coincides with F. Then it holds $P_{X_{T_F}}(S > 0) = 0$ a.s. P_x, proving that the last member of the above is 0.

Define now $v(x) = E_x(e^{-S})$. It is a 1-excessive function. It holds

$$E_x(e^{-S}) = E_x(e^{-(S \circ \theta_{S_n} + S_n)}) = E_x(e^{-S_n} v(X_{S_n}))$$

$$= E_x(e^{-S} \lim_{n \to \infty} v(X_{S_n})).$$

Therefore, $\lim_{n\to\infty} v(X_{S_n}) = 1$ a.s. on $S < \infty$. Then we have $X_S \in F$ a.s. on $S < \zeta$, since the regular points of S coincides with F. We thus have $S \geq T_F$ on $S < \zeta$.

Proof of Theorem. From the previous lemma, it holds $X_{T-} \neq X_T$ a.s. on $0 < T < T_F$. Set

$$A_t = \sum_{T_n \leq t < T_\infty} 1_{E-F}(X_{T_n}).$$

It is easy to see $A_t + A_s \circ \theta_t = A_{t+s}$ if $t+s < T_\infty$. It is quasi-left continuous since it jumps at the discontinuous time of X_t. Then by the representation theorem of Motoo and Watanabe [18], there exists a $E \times E_\Delta$ measurable function such that $A_t = P_t(f)$ on $t < T_F$ a.s.. Since $\Delta A_t =$ 0 or 1, f takes values 0 and 1 a.s. $\mu(dx)n(x, dy)$. Set $A = \{(x, y)$; $f(x, y) = 1\}$. Then $A_t = P_t(1_A)$. Let R be the first jump time of A_t. Then

$$R = \inf\{t > 0; \ (X_{t-}, X_t) \in A\}.$$

Since $R = T$ on $T < T_F$, we obtain the representation (2.1). The set A is included in $(\overline{E-F}) \times (E-F)$ because $X_t \in E-F$ for $t < T_F$. The proof is complete.

§3. Representation of regular MF.

A F_t-adapted process α_t is called a <u>multiplicative functional</u> (MF) if for each $t, s \geq 0$, it holds $\alpha_t \alpha_s \circ \theta_t = \alpha_{t+s}$ a.s.. In this paper, we assume that MF α_t is nonnegative, right continuous and $E_x(\alpha_t) \leq 1$ for each x and t. Thus α_t is a supermartingale.

A MF α_t is called <u>perfect</u> if exceptional sets $N_{t,s}$ that $\alpha_t \alpha_s \circ \theta_t = \alpha_{t+s}$ are not valid satisfy $P_x(\underset{t,s}{\cup} N_{t,s}) = 1$ for all x. MF is not perfect in general, but we may modify it to a perfect one by Walsh [16] and Meyer [13]. Given a MF α_t, we define a new functional $\hat{\alpha}_t$ by

$$\hat{\alpha}_t = \text{ess} \varlimsup_{s \downarrow 0} \alpha_{t-s} \circ \theta_s \quad \text{if } t > 0$$

$$\hat{\alpha}_0 = \text{ess} \varlimsup_{s \downarrow 0} \hat{\alpha}_s,$$

where ess $\overline{\lim}$ is taken with respect to the Lebesgue measure of the time interval. Then $\hat{\alpha}_t$ is perfect MF. It holds $\alpha_t = \hat{\alpha}_t$ if $\alpha_0 > 0$.

A MF α_t is called exact if for each $t > 0$, $x \in E$ and $\varepsilon_n \downarrow 0$, $\alpha_{t-\varepsilon_n} \circ \theta_{\varepsilon_n}$ converges to α_t a.s. P_x. If α_t is exact, $\alpha_t = \hat{\alpha}_t$ a.s. so that α_t is perfect.

Now, let α_t be a MF. Set

$$T_\alpha = \inf\{t > 0 \ ; \ \alpha_t = 0\}$$

$$= \infty \qquad \text{if } \{ \ \} = \phi.$$

It is not difficult to see that T_α is a terminal time. If α_t is perfect, T_α is perfect. If α_t is exact, T_α is exact.

A MF α_t is called regular if for each x there exists an increasing sequence of stopping times T_n such that $\lim T_n \geq T_\alpha$ and that $\alpha_{t \wedge T_n}$ is a martingale with respect to P_x. The following factorization of MF is due to Itô-Watanabe [7].

Factorization theorem of MF. Let α_t be a perfect MF. Then there exists a regular perfect MF $\alpha_t^{(0)}$ and a decreasing perfect MF $\alpha_t^{(1)}$ which has no common jumps with X_t on $[0, T_\alpha)$, such that $\alpha_t = \alpha_t^{(0)}\alpha_t^{(1)}$ and that $T_\alpha = T_{\alpha^{(0)}} = T_{\alpha^{(1)}}$. The factorization is unique.

In order to state our theorem, we need a slight modification of the definition of a local martingale. Let T be an accessible stopping time. A right continuous $\mathcal{F}_t$-adapted process M_t defined for $t \in [0, T)$ is called a local martingale, if for each x, there exists an increasing sequence of stopping times T_n with limit T such that $T_n < T$ on $T > 0$ a.s. for all n and that $M_{t\wedge T_n}$ is a P_x-martingale for each n. For notations and basic properties of stochastic integral by local martingales, refer to Kunita-Watanabe [10].

Theorem 3.1. Let α_t be a regular perfect MF. Let T_α^a be the accessible part of T_α ($T_\alpha^a = 0$ if $T_\alpha = 0$). Then there exists a unique continuous local martingale M_t^c, $t \in [0, T_\alpha^a)$ such that $M_t^c + M_s^c \circ \theta_t = M_{t+s}^c$ for $t+s < T_\alpha$ a.s., and a unique nonnegative $E \times E_\Delta$-measurable function $g(x, y)$ with $g(x, x) = 0$ on E such that

$$(3.1) \qquad \int_0^t n \circ \left(\frac{|g-1|^2}{1+|g-1|}\right)(X_s)\,d\varphi_s < \infty \quad \text{for } t < T_\alpha \text{ a.s.},$$

and α_t is represented as

$$(3.2) \quad \alpha_t = \exp[M_t^c - \tfrac{1}{2}\langle M^c\rangle_t + Q_t(g-1)] \cdot \prod_{\substack{s \le t \\ X_{s-} \ne X_s}} g(X_{s-}, X_s) e^{-(g(X_{s-}, X_s)-1)} \quad \text{if } t < T_\alpha^a$$

$$= 0 \quad \text{if } t \ge T_\alpha^a.$$

Proof. In case where α_t is strictly positive, the proof is found in Kunita-Watanabe [10] and Deléans [3]. Also, a similar representation of a nonnegative supermartingale is in Kunita [8]. We give here a quick proof, leaving some details to the above references.

Given a positive number ε, let us define

$$T^\varepsilon = \inf\{t < T_\alpha \; ; \; \left|\frac{\Delta\alpha_t}{\alpha_{t-}}\right| > \varepsilon\}$$

$$= T_\alpha \quad \text{if } \{ \quad \} = \phi,$$

where $\Delta\alpha_t = \alpha_t - \alpha_{t-}$. It is a perfect terminal time. Since α_t is regular, T^ε is totally inaccessible. Define iterates of T^ε by $T_1^\varepsilon = T^\varepsilon$, $T_n^\varepsilon = T_{n-1}^\varepsilon + T^\varepsilon \circ \theta_{T_{n-1}^\varepsilon}$. Then T_n^ε are totally inaccessible. Set

$$A_t^{\varepsilon} = \sum_{n=1}^{\infty} \frac{\Delta\alpha_{T_n^{\varepsilon}}}{\alpha_{T_n^{\varepsilon}-}} 1_{T_n^{\varepsilon} \le t} .$$

It is a quasi-left continuous AF, which is then written as $A_t^{\varepsilon} = P_t(f^{\varepsilon})$.

Now the local martingale AF $Q_t(f^{\varepsilon})$ (defined by (1.1)) is the compensated sum of $\Delta\alpha_s/\alpha_{s-}$ such that $|\Delta\alpha_s/\alpha_{s-}| > \varepsilon$ for $t < T_{\alpha}$. It turns out that

$$\left(\sum_{s\le t} \left|\frac{\Delta\alpha_s}{\alpha_{s-}}\right|^2\right)^{1/2}, \quad t \in [0, T_{\alpha}^{a})$$

is locally integrable. Then it implies that $Q_t(f^{\varepsilon})$ converges in P_x-probability to a local martingale as $\varepsilon \to 0$. (See [8]). Then we can find a $E\times E_{\Delta}$-measurable function $f(x, y)$ with $f(x, x) = 0$ on E such that $Q_t(f) = \lim_{\varepsilon\to 0} Q_t(f^{\varepsilon})$ in P_x-probability.

It holds $f-1 \ge 0$ a.e. $\mu(dx)n(x, dy)$, since

$$\Delta Q_s(f) = \frac{\Delta\alpha_s}{\alpha_{s-}} \ge -1.$$

Set now $f_1 = f1_{|f|\le 1}$ and $f_2 = f1_{|f|>1}$. Then

$$\int_0^t n\circ f_1^2(X_s)d\varphi_s < \infty, \qquad \int_0^t n\circ|f_2|(X_s)d\varphi_s < \infty \quad \text{a.s.}$$

(See [10]). The above is equivalent to

$$\int_0^t n\circ\left(\frac{|f|^2}{1+|f|}\right)(X_s)d\varphi_s < \infty \quad \text{a.s..}$$

Setting $g = f+1$, we get the condition (3.1)

We shall next define the continuous local martingale M_t^c. The stochastic integral

$$N_t = \int_0^t \alpha_{s-}dQ_s(f), \qquad t \in [0, T_{\alpha}^{a})$$

is well defined as a local martingale. It holds $\Delta N_s = \Delta\alpha_s$. Then $\alpha_t^c \equiv \alpha_t - N_t$, $t \in [0, T_\alpha^a)$ is a continuous local martingale. Define now

$$M_t^c = \int_0^t \alpha_s^{-1} d\alpha_s^c, \quad t \in [0, T_\alpha^a).$$

It is a continuous local martingale.

Set now $M_t = M_t^c + Q_t(f)$. Then it holds

$$\alpha_t - 1 = \int_0^t \alpha_{s-} dM_s, \quad t \in [0, T_\alpha^a).$$

This functional equation has a unique solution. The solution is in fact represented as (3.2) by Doléans [3].

It remains to prove the additivity of M_t. Let $s, t \geq 0$ and let

$$\delta = \{0 = s_0 < s_1 < \cdots < s_n = s < s_{n+1} < \cdots < s_{n+m} = s+t\}$$

be a partition of $[0, s+t]$ such that $s_k - s_{k-1} = \varepsilon$. Set

$$M_u^\varepsilon = \sum_{s_n \leq u} \alpha_{s_n}^{-1} 1_{T_\alpha > s_n} (\alpha_{s_n} - \alpha_{s_{n-1}}).$$

Then M_u^ε converges to M_t in probability as $\varepsilon \to 0$. On the other hand, since

$$M_u^\varepsilon = \sum_{s_n \leq u} \alpha_\varepsilon \circ \theta_{s_n} 1_{T_\alpha > s_n},$$

it holds $M_s^\varepsilon + M_t^\varepsilon \circ \theta_s = M_{s+t}^\varepsilon$ for $s, t \in \delta$ such that $s, t < T_\alpha$. Letting $\varepsilon \to 0$, one has the additivety of M_t. The proof is complete.

<u>Remark</u> 1. Assume hypothesis (K). If α_t is exact, then T_α has the representation of Theorem 2.1. The set A then coincides with $\{(x, y) ; g(x, y) = 0\}$. In fact, the totally inaccessible part T_α^i

coincides on the set $T^i_\alpha < T_\alpha$ with the first time that

$$\prod_{s\le t} g(X_{s-}, X_s)e^{-(g(X_{s-}, X_s)-1)}$$

becomes 0. The above infinite product is convergent, since $\sum_{s\le t} (1-g(X_{s-}, X_s))^2$ is finite. Hence T^i_α coincides with the first time that $g(X_{t-}, X_t) = 0$ on the set $T^i_\alpha < T_\alpha$.

Remark 2. Assume hypothesis (K). Then any potential is regular. Hence natural increasing AF's are continuous. (See Meyer [11]). Let $\alpha_t = \alpha_t^{(0)}\alpha_t^{(1)}$ be the factorization of Itô-Watanabe. Then $\log \alpha_t^{(1)}$, $t \in [0, T_\alpha)$ is a natural increasing AF (up to $t < T_\alpha$), so that it is continuous. We can assume $\log \alpha_t^{(1)} << \varphi_t$ for $t < T_\alpha$. Then one has the representation

$$\alpha_t^{(1)} = \exp -\int_0^t c(X_s)d\varphi_s, \tag{3.3}$$

where $c(x)$ is a nonnegative function, unique up to the null set of the canonical measure μ of φ_t.

The following example will be used in section 6.

Example. Assume hypothesis (K). Let T be an exact terminal time and let F and A be sets of Theorem 2.1. Let $\alpha_t = 1_{T>t}$. It is an exact MF. The factorization of Itô-Wanatabe is

$$\alpha_t^{(0)} = \exp\int_0^t n\circ 1_A(X_s)d\varphi_s \; 1_{T_\alpha>t},$$

$$\alpha_t^{(1)} = \exp-\int_0^t n\circ 1_A(X_s)d\varphi_s \; 1_{T_\alpha>t}.$$

The representation (3.2) corresponds to

$$\alpha_t^{(0)} = \exp Q_t(1_A)\cdot 1_{A^c}(X_{T-}, X_T)e^{-1_A(X_{T-}, X_T)}, \quad t < T_\alpha^a.$$

§4. Transformation by MF and the Lévy system.

Let α_t be a perfect MF such that $\alpha_t = 0$ for $t \geq \zeta$ a.s.. Set

$$Q_t(x, A) = \int_{X_t \in A} \alpha_t dP_x \qquad \text{if } A \subset E$$

$$Q_t(x, \Delta) = 1 - Q_t(x, E).$$

It is well known that Q_t is a transition probability. We shall define a family of probability measures Q_x, $x \in E_\Delta$ on F_t^0 as

$$Q_x(X_{t_1} \in A_t, \cdots, X_{t_n} \in A_n)$$
$$= \int \cdots \int_{A_1 \times \cdots \times A_n} Q_{t_1}(x, dx_1) Q_{t_2 - t_1}(x_1, dx_2) \cdots Q_{t_n - t_{n-1}}(x_{n-1}, dx_n).$$

Then Q_x is a standard Markov process. It satisfies

(4.1) $$Q_x(B \cap \zeta > T) = E_x(\alpha_T ; B), \quad B \in F_T$$

for any stopping time T (see [9]).

The purpose of this section is to prove

Theorem 4.1. Assume hypothesis (K). Let $\alpha_t = \alpha_t^{(0)} \alpha_t^{(1)}$ be the decomposition of Itô-Watanabe, and let (3.2) and (3.3) be the representations of $\alpha_t^{(0)}$ and $\alpha_t^{(1)}$. Set

(4.2) $$n^Q(x, dy) = n(x, dy)g(x, y) \quad \text{on } E \times E$$

$$n^Q(x, \Delta) = c(x).$$

Then (n^{Q}, φ_t) is a Lévy system of the process Q_x.

We shall consider the case where α_t is regular and is decreasing separately, and next combine two cases. We assume $T_\alpha > 0$ a.s.. Detailed discussion for the case $T_\alpha \geq 0$ is omitted.

Lemma 4.2. The assertion of the theorem is valid if α_t is regular.

Proof. We shall first observe that the lifetime of Q_x is accessible. Let T_n be an increasing sequence of stopping times with limit $\geq T_\alpha$ such that $\alpha_{t\wedge T_n}$ is a P_x-martingale. Then

$$Q_x(\zeta > T_n \wedge t) = E_x(\alpha_{T_n \wedge t}) = 1,$$

$$Q_x(\zeta > T_\alpha) = E_x(\alpha_{T_\alpha}) = 0,$$

proving that ζ is accessible.

Now, set $M_t = M_t^c + Q_t(g-1)$ as before. Then it holds $\alpha_t - 1 = \int_0^t \alpha_{s-} dM_s$. Let Z_t be a P_x-local martingale and let $Y_t = Z_t - \langle Z, M\rangle_t$. We shall prove that Y_t, $t \in [0, \zeta)$ is a Q_x-local martingale. By Itô's formula ([10]).

$$\begin{aligned}\alpha_t Y_t &= (1 + \int_0^t \alpha_{s-} dM_s)(Z_t - \langle Z, M\rangle_t) \\ &= Z_t - \langle Z, M\rangle_t + \int_0^t \alpha_{s-}(Z_{s-} - \langle Z, M\rangle_{s-}) dM_s \\ &\quad + \int_0^t (\int_0^{s-} \alpha_{u-} dM_u) dZ_s - \int_0^t (\int_0^{s-} \alpha_{u-} dM_u) d\langle Z, M\rangle_s + \int_0^t \alpha_{s-} d\langle Z, M\rangle_s.\end{aligned}$$

The 1st, 3rd and 4-th terms are local martingales. The sum of the remaining terms is 0. Hence $\alpha_t Y_t$ is a local martingale.

Now, let $\{T_n\}$ be an increasing sequence of stopping times such that $\lim_{n\to\infty} T_n \geq T_\alpha$ and that $\alpha_{t\wedge T_n}$ and $\alpha_{t\wedge T_n} Y_{t\wedge T_n}$ are P_x-martingale for each n.

Then $T_n < \zeta$ and $\lim_{n\to\infty} T_n \geq \zeta$ a.s. Q_x as before. It holds

$$E_x(\alpha_{t\wedge T_n} Y_{t\wedge T_n} \ ; B) = E_x(\alpha_{s\wedge T_n} Y_{s\wedge T_n} \ ; B)$$

for any $B \in F_s$. Denote the expectation by Q_x as E_x^Q. Then the above is equivalent to

$$E_x^Q(Y_{t\wedge T_n} \ ; B) = E_x^Q(Y_{s\wedge T_n} \ ; B)$$

by (4.1). Therefore Y_t, $t \in [0, \zeta)$ is a Q_x-local martingale.

Now, let f be a bounded $E\times E$-measurable function such that $f(x, x) = 0$ and $\int_0^t n\circ|f|(X_s)d\varphi_s$ is integrable. Set $Z_t = Q_t(f)$. Then

$$\langle Q(f), M\rangle_t = \langle Q(f), Q(g-1)\rangle_t = \int_0^t n\circ(f(g-1))(X_s)d\varphi_s.$$

Therefore,

$$Q_t(f) - \langle Q(f), M\rangle_t = P_t(f) - \int_0^t n\circ(fg)(X_s)d\varphi_s, \quad t \in [0, \zeta),$$

is a Q_x-local martingale. Recall the definition of the Lévy system ; we obtain the assertion.

<u>Lemma</u> 4.3. The assertion of the theorem is valid if ζ is accessible (P_x) and α_t is decreasing.

<u>Proof.</u> Suppose that (3.3) is the representation of $\alpha_t = \alpha_t^{(1)}$. Set

$$\beta_t = \exp\int_0^t c(X_s)d\varphi_s.$$

Then $\beta_t 1_{\zeta>t} dQ_x = 1_{\zeta>t} dP_x$ on F_t. Therefore, $\beta_t 1_{\zeta>t}$, $t \in [0, \zeta)$ is a Q_x-local martingale. We shall prove that

$$(4.3) \qquad P_t(1_\Delta) - \int_0^t c(X_s)d\varphi_s = 1_{\zeta\leq t} - \int_0^t c(X_s)d\varphi_s$$

is a Q_x-local martingale, where $1_\Delta(x, y)$ is 0 or 1 according as $y \neq \Delta$ or $y = \Delta$. It holds

$$\int_0^t \beta_{s-} d1_{\zeta>s} = 1_{\zeta>t}\beta_t - \int_0^t 1_{\zeta>s} d\beta_s = 1_{\zeta>t}\beta_t - \beta_t$$

and

$$-\int_0^t \beta_{s-} c(x_s) d\varphi_s = \beta_t - 1.$$

Therefore,

$$\int_0^t \beta_{s-} dP_s(1_\Delta) - \int_0^t \beta_{s-} c(X_s) d\varphi_s = 1_{\zeta>t}\beta_t - 1 = Q_x\text{-local martingale.}$$

This implies that (4.3) is a local martingale. We then see that $n^Q(x, \Delta) = c(x)$.

We shall next show that $Q_t(f)$, $t \in [0, \zeta)$ is a Q_x-local martingale. Set $Z_t = \beta_t 1_{\zeta>t}$. Then $Z_t Q_t(f)$ is a Q_x-local martingale, because

$$E_x^Q[Z_T Q_T(f) \; ; \; B] = E_x[Q_T(f) \; ; \; B \cap \zeta > t],$$

for any stopping time T such that both are well defined. Since Z_t and $Q_t(f)$ are processes with bounded variation, it holds

$$Z_t Q_t(f) = \int_0^t Z_s dQ_s(f) + \int_0^t Q_{s-}(f) dZ_s.$$

Denote the first term of the right hand as Y_t. Then Y_t is a Q_x-local martingale. Since Z_t and $Q_t(f)$ have no common jumps, we can write as $Y_t = \int_0^t Z_{s-} dQ_s(f)$. Now, consider the stochastic integral $\int_0^t Z_{s-}^{-1} dY_s$. Obviously, it equals $Q_t(f)$, proving that $Q_t(f)$ is a Q_x-local martingale. The proof is complete.

Proof of Theorem. Let $Q_x^{(0)}$ be the standard process such that

$1_{\zeta>t}dQ_x^{(0)} = \alpha_t^{(0)}dP_x$ on F_t. Then $(n(x, dy)g(x,y)1_{E\times E}(x,y), \mathcal{P}_t)$ is a Lévy system of $Q_x^{(0)}$ by Lemma 4.2. Note the relation $1_{\zeta>t}dQ_x = \alpha_t^{(1)}dQ_x^{(0)}$. Then Lemma 4.3 proves the theorem.

§5. Lebesgue decomposition and Radon-Nikodym density of Markov processes.

Let us consider two Hunt processes P_x, Q_x defined on the same state space E and the same sample space $(\Omega, F^0, F^0_{t+}, X_t, \theta_t)$. Recall that $F^0_t = \sigma(X_s, s \le t)$ and $F^0_{t+} = \bigcap_{\varepsilon>0} F^0_{t+\varepsilon}$. The purpose of this section is to prove

Theorem 5.1. There exists a terminal time T (unique up to the measures $P_x + Q_x$ for all x) and a right continuous MF α_t relative to P_x such that

(i) $P_x(T = \infty) = 1 \quad \forall x$

(ii) $Q_x(\cdot \cap T \le t)$ is singular to P_x on F^0_{t+}

(iii) $Q_x(\cdot \cap T > t)$ is absolutely continuous with respect to P_x and

(5.1) $$Q_x(B \cap T > t) = \int_B \alpha_t dP_x \qquad \forall B \in F^0_{t+}.$$

(iv) The Lebesgue decomposition is written as

(5.2) $$P_x(B) = \int_B \alpha_t dP_x + Q_x(B \cap T \le t).$$

The proof is divided to 4 lemmas.

Lemma 5.2. There exists a stopping time T and a right continuous F^0_{t+}-adapted process α_t which satisfy (i) ∿ (iv) of Theorem 5.1.

Proof. Since $Q_x << \frac{1}{2}(P_x+Q_x)$ on F^0_t for each x, there exists a density function $h_t^{(x)}(\omega)$ such that

$$Q_x(B) = \frac{1}{2}\int_B h_t^{(x)} d(P_x+Q_x).$$

For each $t \geq 0$, we can define $h_t^{(x)}(\omega)$ as a $\mathcal{E}_\Delta \times \mathcal{F}_t^0$-measurable function by a standard argument. Set $h_t(\omega) = h_t^{(X_0(\omega))}(\omega)$. It is a $\frac{1}{2}(P_x+Q_x)$-martingale such that $0 \leq h_t \leq 2$ a.s.. Denote the right continuous modification as $\overline{h}_t$. Then $(\overline{h}_t, \mathcal{F}_{t+}^0)$ is a $\frac{1}{2}(P_x+Q_x)$-martingale for each x. Let

$$T = \inf\{t > 0 \ ; \ 2-\overline{h}_t = 0\}.$$

Then T is a $\mathcal{F}_{t+}^0$-stopping time. It holds $2-\overline{h}_t = 0$ for $t \geq T$ a.s. $\frac{1}{2}(P_x+Q_x)$, since $2-\overline{h}_t$ is a nonnegative martingale.

Remark now the relation $\overline{h}_t dP_x = (2-\overline{h}_t)dQ_x$. Then

$$P_x(T \geq t) = \int_{T\geq t} \overline{h}_t dP_x = \int_{T\geq t} (2-\overline{h}_t)dQ_x = 0.$$

On the other hand, it holds $\frac{1}{2-\overline{h}_t}(2-\overline{h}_t) = 1$ on $T > t$. Hence for $B \in \mathcal{F}_{t+}^0$,

$$Q_x(B \cap T > t) = \int_{B\cap T>t} \frac{1}{2-\overline{h}_t}(2-\overline{h}_t)dQ_x$$

$$= \int_{B\cap T>t} \frac{1}{2-\overline{h}_t}\overline{h}_t dP_x.$$

Define

$$\alpha_t = \frac{\overline{h}_t}{2-\overline{h}_t} \quad \text{if} \quad t < T$$

$$= 0 \quad \text{if} \quad t \geq T.$$

Then

$$Q_x(B \cap T > t) = \int_B \alpha_t dP_x.$$

We have thus the Lebesgue decomposition (5.2). The proof is complete.

Two σ-fields F_t^0 and F_{t+}^0 are different in general. Hence it may occur that $h_t \neq \overline{h}_t$ with positive probability. We denote as $N_t = \{h_t = 2\}$ and $f_t = \frac{h_t}{2-h_t}$ on N_t^c. Then the Lebesgue decomposition on F_t^0 is written as

$$P_x(B) = \int_B f_t dP_x + Q_x(B \cap N_t) \qquad B \in F_t^0.$$

f_t is a (F_t^0, P_x)-supermartingale (See Neveu [14]). It holds $\lim_{\varepsilon \downarrow 0} f_{t+\varepsilon} = \alpha_t$ a.s. P_x.

<u>Lemma</u> 5.3. The process f_t is a MF.

<u>Proof</u>. We follow the discussion of Dynkin [4]. Let $Q_t(x, dy)$ and $P_t(x, dy)$ be transition probabilities of Q_x and P_x, respectively. Consider the Lebesgue decomposition

$$Q_t(x, dy) = q_t(x, y)P_t(x, dy) + Q_t(x, dy)1_{N_t^{(x)}}(y).$$

We can assume that $1_{N_t^{(x)}}(y)$ is $E_\Delta \times E_\Delta$-measurable.

Let δ ; $0 = t_0 < t_1 < \cdots < t_n = t$ be a partition. Let $B_t^\delta = \sigma(X_{t_0}, X_{t_1}, \cdots, X_{t_n})$. Define

$$N_t^\delta = \{\omega \ ; \ X_{t_i}(\omega) \in N_{t_i - t_{i-1}}^{(X_{t_{i-1}}(\omega))} \quad \text{for some} \quad 1 \leq i \leq n\}$$

and

$$f_t^\delta = \prod_{i=1}^n q_{t_i - t_{i-1}}(X_{t_{i-1}}, X_{t_i}).$$

Then the Lebesgue decomposition on the σ-field B_t^δ is written as

$$Q_x(B) = \int_B f_t^\delta dP_x + Q_x(B \cap N_t^\delta), \quad B \in B_t^\delta.$$

Consider now a sequence of partitions $\delta_1 \prec \delta_2 \prec \cdots \prec \delta_n \prec \cdots (\delta_{n-1}$ is a subpartion of δ_n), such that $|\delta_n|$ (= $\max_i |t_i^{(n)} - t_{i-1}^{(n)}|$) tends to 0. Then $(f_t^{\delta_n}, \mathcal{B}_t^{\delta_n}, P_x)_{n=1,\cdots,}$ is a supermatingale for each t. It converges to f_t a.s. P_x. (Neveu [14, Proposition III-2-7]).

Now let δ' ; $0 = s_0 < \cdots < s_m = s$ be a partion of $[0, s]$ and let $\delta \cup \delta'$; $0 = t_0 < \cdots < t_n = t < t_n+s_1 < \cdots < t_n+s_m$ be a partion of $[0, t+s]$. Then it holds

$$f_{t+s}^{\delta\cup\delta'} = f_t^{\delta} f_s^{\delta'} \circ \theta_t$$

Letting $|\delta| \to 0$ and $|\delta'| \to 0$, we have $f_{t+s} = f_t \cdot f_s \circ \theta_t$ a.s. P_x. The proof is complete.

The set $\{T > 0\}$ belongs to the germ field F_{0+}^0. It holds $Q_x(T > 0) = 1$ or 0 by the 0-1 law. Also, $\alpha_0 = f_{0+}$ is F_{0+}^0-measurable, so that α_0 = const a.s. P_x by the 0-1 law. Since $E_x(\alpha_0) = Q_x(T > 0)$, we see that

$$P_x(\alpha_0 = 1) = 1 \Leftrightarrow Q_x(T > 0) = 1$$

$$P_x(\alpha_0 = 0) = 1 \Leftrightarrow Q_x(T = 0) = 1.$$

Define

$$\hat{E} = \{x \ ; \ Q_x(T > 0) = 1\}.$$

Then $\hat{E}$ is $\mathcal{E}$-measurable. It holds $\alpha_0 = 1_{\hat{E}}(X_0)$ a.s..

Lemma 5.4. The process α_t is a MF. It holds

(5.3) $$\alpha_t = f_t 1_{\hat{E}}(X_t) = \alpha_t 1_{\hat{E}}(X_t) \quad \text{a.s..}$$

Proof. Letting $\varepsilon \to 0$ in the equality $f_{t+\varepsilon} = f_t f_\varepsilon \circ \theta_t$, we have

$\alpha_t = f_t\alpha_0\circ\theta_t$. Let $B \in F^0_{t+}$. Then

$$E_x(\alpha_0\circ\theta_t\ ;\ B) = E_x(E_{X_t}(\alpha_0)\ ;\ B) = E_x(1_{\hat{E}}(X_t)\ ;\ B).$$

This implies $\alpha_0\circ\theta_t = 1_{\hat{E}}(X_t)$ a.s. P_x, proving (5.3).

We shall next prove that α_t is a MF. Let $t, s \geq 0$. If $s = 0$, $\alpha_t\alpha_0 = \alpha_t$ is obvious since $\alpha_0 = 1_{\hat{E}}(X_0)$. If $s > 0$ and $\alpha_t = 0$, then $\alpha_{t+s} = \alpha_t\alpha_s\circ\theta_t$ holds because both are 0. If $s > 0$ and $\alpha_t > 0$, then $\alpha_t = f_t$. Therefore,

$$\alpha_{t+s} = f_t f_s\circ\theta_t 1_{\hat{E}}(X_{t+s}) = \alpha_t\alpha_s\circ\theta_t.$$

The proof is complete.

<u>Lemma</u> 5.5. T is a terminal time.

<u>Proof.</u> Let $\hat{Q}_x$ be the standard process defined by the relation $\hat{Q}_x(B \cap \zeta > t) = E_x(\alpha_t\ ;\ B)$, $B \in F^0_{t+}$, where α_t is the MF of the Radon-Nikodym density. Then $\hat{Q}_x \leq Q_x$ on $F^0_{t+} \cap \zeta > t$. The Radon-Nikodym density $\beta_t = d\hat{Q}_x/dQ_x$ equals $1_{T>t}$ obviously. The multiplicatively of β_t implies $1_{T>t+s} = 1_{T>t}1_{T\circ\theta_t>s}$. This proves $T\circ\theta_t + t = T$ if $t < T$, i.e., T is a terminal time.

Theorem 5.1 follows immediately from Lemmas 5.2 $\sim$ 5.5.

A simple consequence of theorem is

<u>Proposition</u> 5.6. It holds

$$x \in \hat{E} \Leftrightarrow P_x = Q_x \text{ on } F^0_{0+}$$

$$x \notin \hat{E} \Leftrightarrow P_x \perp Q_x \text{ (singular) on } F^0_{0+}.$$

Proof. If $x \in \hat{E}$ then $P_x >> Q_x$ on F^0_{0+} and $dQ_x = \alpha_0 dP_x$. Since $\alpha_0 = 1$ a.s. P_x, we have $Q_x = P_x$ on F^0_{0+}. The converse is obvious. If $x \notin \hat{E}$, then there is no absolute consinuous part. Hence $P_x \perp Q_x$ on F^0_{0+}. Then converse is also clear.

§6. Absolute continuity of Markov process.

The preceding theorem shows that $Q_x << P_x$ on $F^0_{t+} \cap \{T > t\}$. We shall study the terminal time T in details and find a criterion that $T \geq \zeta$ a.s. Q_x. We assume for simplicity that $T > 0$ a.s. Q_x for $x \in E$ except for the last remark in this section. Hence T is an exact terminal time and hence it has the representation.

(6.1) $$T = \inf\{t > 0 \; ; \; (X_{t-}, X_t) \in A\} \quad \text{a.s. } Q_x \text{ on } T < \zeta.$$

Observe that the Radon-Nikodym density α_t is an exact and is a perfect MF.

We shall denote by $\mu^P n^P (= \mu^P(dx) n^P(x,dy) 1_{E\times E}(x,y))$ and $\mu^Q n^Q$ $(= \mu^Q(dx) n^Q(x,dy) 1_{E\times E}(x,y))$ canonical measures of the processes P_x and Q_x, respectively. Note that $\mu^P n^P$ and $\mu^Q n^Q$ are measures restricted to $E\times E$.

Theorem 6.1. Assume hypothesis (K) for P_x and Q_x. Then $1_A \mu^Q n^Q$ and $\mu^P n^P$ are mutually singular and $1_{A^c} \mu^Q n^Q$ is absolutely continuous with respect to $\mu^P n^P$.

Remark. Let $\hat{Q}_x$ be the standard process defined by

$$\hat{Q}_x(B \cap \zeta > t) = \int_{B\cap\zeta>t} \alpha_t dP_x = Q_x(B \cap T \wedge \zeta > t).$$

Then (gn^P, φ_t^P) and $(1_{A^c}n^Q, \varphi_t^Q)$ are Lévy systems of $\hat{Q}_x$ (restricted to E) by Theorem 4.1 and Example in §4. We can choose canonical measures of P_x and Q_x such that $g\mu^P n^P = 1_{A^c}\mu^Q n^Q$. Then the assertion of the theorem is equivalent to that

$$\mu^Q n^Q = g\mu^P n^P + 1_A \mu^Q n^Q,$$

is the Lebesgue decomposition of canonical measures.

The theorem states that the terminal time T is just the hitting time for the support of the singular part of $\mu^Q n^Q$ with respect to $\mu^P n^P$. Hence $\mu^Q n^Q << \mu^P n^P$ if and only if T is greater than ζ a.s. Q_x. This implies our main result.

Theorem 6.2. Assume hypothesis (K) for P_x and Q_x. Then $P_x = Q_x$ on F_{t+}^0 for each x implies $Q_x << P_x$ on $F_{0+}^0 \cap \{\zeta > t\}$ for each t and x if and only if $\mu^Q n^Q << \mu^P n^P$ on $E \times E$.

Remark 1. An immediate consequence of the theorem is $P_x = Q_x$ on F_{0+}^0 for each x implies $Q_x \cong P_x$ (equivalent) on $F_{t+}^0 \cap \{\zeta > t\}$ if and only if $\mu^Q n^Q \cong \mu^P n^P$ on $E \times E$. In particular, if P_x and Q_x are diffusions (up to ζ), then $P_x = Q_x$ on F_{t+}^0 implies $P_x \cong Q_x$ on $F_{t+}^0 \cap \{\zeta > t\}$, since $\mu^Q n^Q = \mu^P n^P = 0$. This fact was proved by Dawson [2] under a stronger condition.

Remark 2. Of course, it can occur that $Q_x << P_x$ on $F_{t+}^0 \cap \{\zeta > t\}$ but $P_x << Q_x$ on $F_{t+}^0 \cap \{\zeta > t\}$ fails. Such examples are given in §7. What we can show is that $P_x << Q_x$ on $F_{t+}^0 \cap \{S > t\}$, where

$$S = \inf\{t > 0 \ ; \ (X_{t-}, X_t) \in K\}, \quad K = \{(x, y) \ ; \ g(x, y) = 0\}.$$

In fact, $1_K\mu^P n^P$ is the singular part of $\mu^P n^P$ with respect to $\mu^Q n^Q$.

The idea of the proof is as follows. Let $\overline{g}(x, y)$ be the Radon-Nikodym density of the absolute continuous part of $\mu^P n^P$ with respact to $\mu^Q n^Q$: here $\mu^P n^P$ and $\mu^Q n^Q$ are chosen so that $g\mu^P n^P = 1_{A^c}\mu^Q n^Q$. Then it holds $\overline{g} \geq g$ a.s. $\mu^P n^P$ and $\overline{g} = g$ on A^c. Theorem 6.1 is equivalent to that $\overline{g} = g$ a.s. $\mu^P n^P$.

For the proof of this fact, let us define a prefact MF

$$\overline{\alpha}_t = \exp[M_t^c - \tfrac{1}{2}\langle M^c\rangle_t + Q_t(g-1)] \prod_{\substack{s\leq t \\ X_s\neq X_{s-}}} \overline{g}(X_{s-}, X_s)e^{-(\overline{g}(X_{s-}, X_s)-1)}$$

$$\cdot\exp - \int_0^t c(X_s)d\varphi_s^P,$$

where M_t^c is the continuous local martingale defined in the representation of $\alpha_t^{(0)}$. We can extend M_t^c so that it has the additive property for $t < \zeta$. $c(x)$ is the function defined in the representation of $\alpha_t^{(1)}$. Then it holds $\overline{\alpha}_t = \alpha_t$ for $t < T_\alpha$ and $\overline{\alpha}_t \geq \alpha_t$ a.s.. We have further.

<u>Lemma</u> 6.2. $E_x(\overline{\alpha}_t) \leq 1$ for each t and x.

<u>Proof</u>. $\overline{\alpha}_t$ has the factorization $\overline{\alpha}_t = \overline{\alpha}_t^{(0)}\overline{\alpha}_t^{(1)}$, where

$$\overline{\alpha}_t^{(0)} = \exp[M_t^c - \tfrac{1}{2}\langle M^c\rangle_t + Q_t(\overline{g}-1)] \prod_{\substack{s\leq t \\ X_s\neq X_{s-}}} \cdot\ \overline{g}(X_{s-}, X_s)e^{-(\overline{g}(X_{s-}, X_s)-1)}$$

$$\overline{\alpha}_t^{(1)} = \exp\int_0^t \{n^P\circ(\overline{g}-g)(X_s) - c(X_s)\}d\varphi_t^P.$$

The first one is a regular MF. The killing rate $n^{\hat{Q}}(x, \Delta)$ of the process $\hat{Q}_x$ is $c(x) = \int_E n^Q(x, dy)1_A(x, y)$ by Theorem 4.1. Therefore $n^P\circ(\overline{g}-g)(x) \leq c(x)$, proving that $\bar{\alpha}_t^{(1)}$ is decreasing. Then $\bar{\alpha}_t^{(0)}\bar{\alpha}_t^{(1)}$ is a supermartingale for each P_x. The proof is complete.

Let us now define an another standard process $\bar{Q}_x$ as

$$\bar{Q}_x(B \cap \zeta > t) = \int_{B\cap\{\zeta>t\}} \bar{\alpha}_t dP_x \qquad B \in F^0_{t+}.$$

Then $\bar{Q}_x \geq \hat{Q}_x$ on $F^0_{t+} \cap \{\zeta > t\}$ and $\bar{Q}_x = \hat{Q}_x$ on $F^0_{t+} \cap \{T \wedge \zeta > t\}$. The next lemma shows that $\bar{Q}_x$ is again a subprocess of Q_x. But since $\hat{Q}_x$ is the maximal subprocess among absolutely continuous ones, we get $\hat{Q}_x = \bar{Q}_x$. This proves $g = \bar{g}$ a.s. $\mu^P n^P$ and then the theorem is established.

<u>Lemma</u> 6.3. It holds $\bar{Q}_x \leq Q_x$ on $F^0_{t+} \cap \{\zeta > t\}$.

<u>Proof</u>. We may assume $T \leq \zeta$ without loss of generality. (Consider $T \wedge \zeta$ instead of T, if necessary). Define iterates of T as $T_1 = T$, $T_n = T_{n-1} + T\circ\theta_{T_{n-1}}$ and $T_\infty = \lim_{n\to\infty} T_n$. It holds $T_1 < T_2 < \cdots$ a.s., so that T_∞ is an accessible terminal time. The regular points of T_∞ is empty. Therefore $T_\infty \geq \zeta$ a.s. Q_x by Theorem 2.1. Hence it suffices to prove $\bar{Q}_x \leq Q_x$ on $F^0_{t+} \cap \{T_n > t\}$ for each n. We shall prove the case T_2 only since the discussion for the general T_n is similar.

Let F^0_{T-} be the σ-field generated by the sets $B \cap \{T > t\}$, where $B \in F^0_{t+}$. Let $\Phi(t, \omega)$ be a bounded positive left continuous F^0_{t+}-adapted process. Then $\Phi(T)$ is F^0_{T-}-measurable. Let Ψ be a bounded positive F^0_{T-}-measurable function. We shall prove

$$E_x^{\bar{Q}}(\Phi(T)\theta_T\circ\Psi \ ; \ T_2 < \zeta) \leq E_x^{Q}(\Phi(T)\theta_T\circ\Psi \ ; \ T_2 < \zeta). \tag{6.2}$$

Set $f(x) = E_x(\Psi \ ; \ T < \zeta)$. Then the right hand of the above equals

$E^Q_x(\Phi(T)f(X_T))$. Note that the Q_x-expectation of

$$\int_0^T \Phi(s)dQ_s(1_A f) = \Phi(T)f(X_T) - \int_0^T \Phi(s)n^Q\circ(1_A f)(X_s)d\varphi^Q_s$$

is 0. Then the right hand of (6.2) equals

$$E^Q_x[\int_0^T \Phi(s)n^Q\circ(1_A f)(X_s)d\varphi^Q_s]. \tag{6.3}$$

Similarly, the left hand of (6.2) equals

$$E^{\overline{Q}}_x[\int_0^T \Phi(s)n^{\overline{Q}}\circ(1_A f)(X_s)d\varphi^{\overline{Q}}_s], \tag{6.4}$$

since $f(x) = E^{\overline{Q}}_x(\Psi \; ; \; T < \zeta)$. Furthermore, we can choose a Lévy system $(n^{\overline{Q}}, \varphi^{\overline{Q}})$ such that $n^{\overline{Q}}(x, dy) = n^P(x, dy)\overline{g}(x, y)$ and $\varphi^{\overline{Q}}_t = \varphi^Q_t$ for $t < T$. Then $n^{\overline{Q}} \leq n^Q$. Therefore (6.3) is larger than (6.4), proving (6.2).

Now, the σ-field $F^0_{T_2-}$ is generated by elements of F^0_{T-} and $\theta_T F^0_{T-}$. The inequality (6.2) shows that $\overline{Q}_x \leq Q_x$ on $F^0_{T_2-} \cap \{T_2 < \zeta\}$ i.e., $\overline{Q}_x \leq Q_x$ on $F^0_{t+} \cap \{T_2 > t, \zeta > T\}$ for each x. While on the set $\zeta = T$, it holds $\zeta = T_2 = T$, so that $\{T_2 > t, \zeta = T\} = \{\zeta = T > t\}$. Hence $\overline{Q}_x = Q_x$ on $F_{t+} \cap \{T_2 > t, \zeta = T\}$. This proves $\overline{Q}_x \leq Q_x$ on $F^0_{t+} \cap \{T_2 > t\}$. The proof is complete.

Remark. In case where $\hat{E} \neq E$, the terminal time T may not be exact. Let $\hat{T}$ be the exact modification of T defined at the beginning of §2. $\hat{T}$ is represented as (2.1) with sets F and A. Since $\hat{T} \geq T$, the sets of regular points of $\hat{T}$ is included in the sets of regular points of T, i.e., one has $\hat{E} \subset E-F$. The set $(E-F) - \hat{E}$ is of Q_x-potential 0. One can show similarly that if $\mu^P n^P \ll \mu^Q n^Q$ on $(\overline{E-F})\times(E-F)$, then $P_x \ll Q_x$ on $F^0_{t+} \cap \{T_{\hat{E}^c} > t\}$ for $x \in \hat{E}$.

§7. Examples.

7.1. Additive processes.

Let P_x and Q_x be temporollary homogeneous additive processes. Let $X_t = X_t^c + X_t^d$ be the decomposition of continuous additive process and the discontinuous one. We assume for simplicity that the laws of X_t^c relative to P_x and Q_x are the same. We denote Lévy measures of P_x and Q_x by σ^P and σ^Q respectively. Then

Proposition 7.1. $P_x >> Q_x$ on F_{t+}^0 for all $x \in R^1$ if and only if $\sigma^P >> \sigma^Q$ and the density function $g(x)\sigma^P(dx) = \sigma^Q(dx)$ satisfies

$$\int \frac{|g(x)-1|^2}{1+|g(x)-1|} \sigma^P(dx) < \infty. \tag{7.1}$$

Further, $P_x \cong Q_x$ on F_{t+}^0 if and only if $g(x) > 0$ a.s. σ^P.

Proof. A Lévy system of P_x is defined as

$$n^P(x, dy) = \sigma^P(dy-x), \quad \varphi_t^P = t.$$

A Lévy system of Q_x is defined similarly. If $P_x >> Q_x$ on F_{t+}^0 for all x, then there exists a $R\times R$-measurable function $g(x, y)$ such that $n^Q(x, dy) = g(x, y)n^P(x, dy)$. Then $g(x, y)$ must be a function of $x-y$ a.s. n^Q. We shall write $g(x-y) = g(x, y)$. This g satisfies (7.1) in view of (3.1).

Conversely, let g be a function satisfying (7.1). We define a MF α_t by the right hand of (3.2) setting $M_t^c = 0$. Define the process $\hat{Q}_x$ by $\hat{Q}_x = \alpha_t P_x$. Then $\hat{Q}_x$ is an additive process with the Lévy measure $\sigma^P(dx)g(x) = \sigma^Q$ (See Skorohod [16]). Hence $\hat{Q}_x = Q_x$ holds, i.e. $P_x >> Q_x$. The last assertion will be obvious.

It should be noted that $\sigma^P >> \sigma^Q$ does not imply $P_x >> Q_x$ on F^0_{t+}. Actually the mass $\sigma^P(K)$ of the set $K = \{x ; g(x) = 0\}$ has to be finite if $P_x >> Q_x$. In fact, since

$$\frac{|g(x)-1|^2}{1+|g(x)-1|} \geq \frac{1}{2} 1_K(x).$$

The relation (7.1) implies $\sigma^P(K) < \infty$.

Conversely, if K is a set such that $\sigma^P(K) < \infty$ and $\sigma^Q(dx) = 1_{K^c}(x)\sigma^P(dx)$, then $P_x >> Q_x$ on F^0_{t+}, because

$$\frac{|1_{K^c}-1|^2}{1+|1_{K^c}-1|} = \frac{1}{2} 1_K$$

and the left hand side is integrable relative to σ^P.

<u>Corollary.</u> Let P^W_x be a Wiener process and let P_x be an additive process of the form "Wiener process + discontinuous additive process". Let σ be its Lévy measure. Then $P_x >> P^W_x$ on F^0_{t+}, $\forall t$ if and only if $\sigma(R^1) < \infty$. The Radon-Nikodym density is $\alpha_t = 1_{T>t}$, where T is the first jumping time of the process Q_x ; $T = \inf\{t > 0 : |X_t - X_{t-}| > 0\}$.

7.2. One dimensional diffusion.

Let P_x, $x \in R^1$ be a one dimensional regular diffusion and let

$$\mathcal{G}u(\xi)m(d\xi) = dD_s u(\xi) - u(\xi)k(d\xi), \tag{7.2}$$

be its generator, where $s(x)$ is the canonical scale and $D_s u(\xi)$ is the derivative relative to $s(x)$. m is the speed measure and k is the killing measure. (Itô-McKean [6]).

<u>Proposition</u> 7.2. $P_x \underset{=}{\sim} P^W_x$ on $F^0_{t+} \cap \{\zeta > t\}$ for all x if and only

if there exists a function $f \in L^2_{loc}(R^1)$ such that $dm = 2e^B d\xi$, $ds = e^{-B} d\xi$, where $B(\xi) = \int^{\xi} f(y)dy$. The Radon-Nikodym density $\alpha_t P^W_x = P_x$ is represented as

$$(7.3) \qquad \alpha_t = \exp\left(\int_0^t f(X_s)dX_s - \frac{1}{2}\int_0^t |f(X_s)|^2 ds\right)\exp - \int_0^t \ell(t, x)k(dx),$$

where $\ell(t, x)$ is the local time at the point x.

Proof. Consider the case where P_x has no killing, i.e. $k \equiv 0$. If $P_x \underset{=}{\sim} P^W_x$ on $F^0_{t+} \cap \{\zeta > t\}$ for all x, α_t is a regular MF. Hence there exists a continuous local martingale AF M^c_t such that $\alpha_t = \exp(M^c_t - \frac{1}{2}\langle M^c\rangle_t)$. It is well known that M^c_t is represented as $M^c_t = \int_0^t f(X_s)dX_s$ with $\int_0^t |f(X_s)|^2 ds < \infty$ a.s. P^W_x. Also, it holds $\int_0^t |f(X_s)|^2 ds < \infty$ a.s. (P^W_x), if and only if $f \in L^2_{loc}(R^1)$ (Orey [15]). Then the generator of P_x is

$$\mathcal{J}u(\xi) = \frac{1}{2}\frac{d^2}{d\xi^2}u(\xi) + f(\xi)\frac{d}{d\xi}.$$

This proves the "only if" part. "If" part is well known.

It is easy to prove the case with killing. We omit the detail.

The process P^W_x is conservative. Hence $P_x >> P^W_x$ on F^0_{t+} ($^\forall x$) under the condition of the proposition. But $P_x << P^W_x$ on F_{t+} is not true if P_x is not conservative. It holds $P_x \underset{=}{\sim} P^W_x$ on F^0_{t+} if and only if $k \equiv 0$, $f \in L^2_{loc}(R^1)$ and that P_x is conservative. (Orey [15]).

7.3. Markov chain.

Let P_x and Q_x be Markov chains on a countable set E and let $T_1 = \inf\{t > 0 ; X_t \neq X_0\}$. Set $(q^P_x)^{-1} = E_x[T_1]$ and $\pi^P(x, y) = P_x(x_{T_1} = y)$.

Then

$$n^P(x, dy) = q_x^P \pi^P(x, y)$$

and $\varphi_t^P = t$ is a Lévy system of P_x. Lévy system of Q_x is defined similarly. Then $P_x >> Q_x$ on $\mathcal{F}_{t+}^0 \cap \{\zeta > t\}$ if and only if $n^P(x, dy) >> n^Q(x, dy)$. In fact, since n^P and n^Q are finite measures, condition (3.1) is always satisfied.

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