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On the uniqueness of solutions of stochastic differential equations with reflecting barrier conditions.

By Toshio Yamada.

Let $\sigma(t,x)$ and $b(t.x)$ be defined on $[0,\infty] \times R^1$, bounded continuous in (t,x).

We consider the following stochastic differential equation with reflecting barrier condition. (Skorohod equation) .

$$(1) \qquad \begin{cases} dx_t = \sigma(t,x_t)\, dB_t + b(t,x_t)dt + d\varphi_t \\ x_t \geq 0 \end{cases}$$

A precise formulation is as follows; by a probability space $(\Omega,\mathfrak{F},P)$ with an increasing family $\{\mathfrak{F}_t\}_{t\in[0,\infty)}$ which is denoted as $(\Omega,\mathfrak{F},P : \mathfrak{F}_t)$ we mean a probability space $(\Omega,\mathfrak{F},P)$ with a system $\{\mathfrak{F}_t\}_{t\in[0,\infty)}$ of sub-Borel fields of $\mathfrak{F}$ such that $\mathfrak{F}_t \subset \mathfrak{F}_s$ if $t < S$.

DEFINITION 1. - By a solution of the equation (1) , we mean a probability space with an increasing family of Borel fields $(\Omega, \mathfrak{F}, P : \mathfrak{F}_t)$ and a family of stochastic processes $X = \{x_t, B_t, \varphi_t\}$ defined on it such that

(i) with probability one, x_t, B_t and φ_t are continuous in t ,

(ii) they are adapted to $\mathfrak{F}_t$ i.e. ; for each t , x_t, B_t and φ_t are $\mathfrak{F}_t$- measurable,

(iii) B_t is a continuous $\mathfrak{F}_t$- martingale such that $E((B_t - B_s)^2/\mathfrak{F}_s) = t-s$, $t \geq s \geq 0$. $B_0 = 0$.

(iv) with probability one, φ_t is non-decreasing function and does not increase at any t where $x_t > 0$.

(v) $x = \{x_t, B_t, \varphi_t\}$ satisfies

$$x_t = x_o + \int_o^t \sigma(s.x_s)\ dB_s + \int_o^t b(s.x_s)ds + \varphi_t :\ x_t \geq 0\ .$$

where the integral by dB_s is understood in the sense of stochastic integral.

DEFINITION 2. - (pathwise uniqueness)

We shall say that pathwise uniqueness holds for (1) if, for any two solutions $x = (x_t, B_t, \varphi_t)$ and $\tilde{x} = (\tilde{x}_t, \tilde{B}_t, \tilde{\varphi}_t)$ defined on a same space $(\Omega, \mathfrak{F}, P : \mathfrak{F}_t)$ $x_o = \tilde{x}_o$ and $B_t \equiv \tilde{B}_t$ implt $x_t = \tilde{x}_t$ and $\varphi_t = \tilde{\varphi}_t$.

When σ and b are Lipschitz continuous, then, an is well known, by Skorohod theory [1] the pathwise uniqueness holds.

This can be strengthened and the uniqueness holds in certain non-Lipschitzian case.

In fact, we can prove the following. (cf. S. Nakao [2] , S. Manabe - T. Shiga [3]).

THEOREM. -

$$\text{Let (1)}\quad \begin{aligned} & dx_t = \sigma(t,x_t)\ dB_t + b(t,x_t)dt + d\varphi_t \\ & x_t \geq 0 \end{aligned}$$

be the Skorohod equation such that

(i) there exists a positive increasing function $\rho\ (u)$, $u \in [0,\infty)$ such that

$$|\sigma(t,x) - \sigma(t,y)| \leq \rho(|x-y|) \quad \forall x,y \in R^1$$

and

$$(2) \qquad \int_{0+} \rho^{-2}(u)\ du = +\infty$$

(ii) there exists a positive increasing concave function $K(u)$, $u \in [0,\infty)$ such that

$$|b(t,x) - b(t,y)| \leq K(|x-y|) \quad \forall x,y \in R^1$$

and

$$\int_{0+} K^{-1}(u)du = +\infty$$

Then, the pathwise uniqueness holds for (1)

(Proof.)

Let $1 = a_0 > a_1 > \dots > a_m > \dots \downarrow 0$ be defined by

$$\int_{a_1}^{a_0} \rho^{-2}(u)du = 1 \;, \;\dots\; \int_{a_m}^{a_{m-1}} \rho^{-2}(u)du = m, \dots$$

Then, thre exists a twice continuonsly differentiable function $\varphi_m(u)$ on $[0,\infty)$ such that $\varphi_m(0) = 0$

$$\varphi_m'(u) = \begin{cases} 0 & \text{if } 0 \le u \le a_m \\ \text{between } 0 \text{ and } 1 & a_m < u < a_{m-1} \\ 1 & u \ge a_{m-1} \end{cases}$$

and

$$\varphi_m''(u) = \begin{cases} 0 & 0 \le u \le a_m \\ \text{between } 0 \text{ and } \frac{2}{m}\rho^{-2}(u) & a_m < u < a_{m-1} \\ 0 & u \ge a_{m-1} \end{cases}$$

We extend $\varphi_m(u)$ on $(-\infty,\infty)$ symmetrically, i.e.; $\varphi_m(u) = \varphi_m(|u|)$. Clearly $\varphi_m(u)$ is a twice continously differentiable function on $(-\infty,\infty)$ such that $\varphi_m(u)\uparrow |u|$, $m\to\infty$.

Now, let $x^{(1)} = (x_t^{(1)}, B_t^{(1)}, \varphi_t^{(1)})$ and $x^{(2)} = (x_t^{(2)}, B_t^{(2)}, \varphi_t^{(2)})$ be two solutions on the same probability space with an incrasing family of Borel fields, such that $x_0^{(1)} = x_0^{(2)}$, $B_t^{(1)} = B_t^{(2)} = B_t$

Then

$$x_t^{(1)} - x_t^{(2)} = \int_0^t \{\sigma(s,x_s^{(1)}) - \sigma(s,x_s^{(2)})\}\, dB_s + \int_0^t \{b(s,x_s^{(1)}) - b(s,x_s^{(2)})\}ds + \varphi_t^{(1)} - \varphi_t^{(2)}$$

and by Ito's formula

$$\varphi_m(x_t^{(1)}-x_t^{(2)}) = \int_0^t \varphi_m'(x_s^{(1)}-x_s^{(2)})\{\sigma(s,x_s^{(1)})-\sigma(s,x_s^{(2)})\}dB_s$$

$$+\int_0^t \varphi_m'(x_s^{(1)}-x_s^{(2)})\{b(s,x_s^{(1)})-b(s,x_s^{(2)})\}ds$$

$$+\tfrac{1}{2}\int_0^t \varphi_m''(x_s^{(1)}-x_s^{(1)})\{\sigma(s,x_s^{(1)})-\sigma(s,x_s^{(2)})\}ds$$

$$+\int_0^t \varphi_m'(x_s^{(1)}-x_s^{(2)})d\varphi_s^{(1)}-\int_0^t \varphi_m'(x_s^{(1)}-x_s^{(2)})\,d\varphi_s^{(2)} = I_1+I_2+I_3+I_4-I_5\text{: say}$$

Then, $E[I_1] = 0$

and since φ_m' is uniformly bounded, $(|\varphi_m'(u)| \le 1)$ we get ,

$$|E[I_2]| \le \int_0^t E[K(|x_s^{(1)}-x_s^{(2)}|]ds \le \int_0^t K(E|x_s^{(1)}-x_s^{(2)}|)ds$$

by Jensen's inequality .

We have for I_3

$$|I_3| \le \tfrac{1}{2}\int_0^t \varphi_m''(x_s^{(1}-x_s^{(2)})\ \rho^2(|x_s^{(1)}-x_s^{(2)}|)ds$$

$$\le \tfrac{1}{2}\ t.\ \sup_{a_m \le |u| \le a_{m-1}} (\varphi_m''(u).\rho^2(u)) \le \tfrac{1}{2}.t.\tfrac{2}{m} \to 0 \ \text{as}\ m\to\infty .$$

For I_4 since $x_s^{(1)}$ and $x_s^{(2)}$ are non-negative functions and since $\varphi_m'(0)$ and $\varphi_m'(u) \le 0$ $(u \le 0)$ we can see the follings,

(i) when it occurs $x_s^{(1)} > x_s^{(2)} > 0$ then it follows $x_s^{(1)} > 0$ and $d\varphi_s^{(1)} = 0$

(ii) when it occurs $x_s^{(1)} = x_s^{(2)}$ then it follows $\varphi_m'(x_s^{(1)}-x_s^{(2)}) = 0$

(iii) when it occurs $x_s^{(1)}-x_s^{(2)} < 0$ then it follows $\varphi_m'(x_s^{(1)}-x_s^{(2)}) < 0$

Then we get $E[I_4] \le 0$

By the similar treatment we have $E[I_5] \ge 0$.

Also, $\varphi_m(x_t^{(1)}-x_t^{(2)}) \uparrow |x_t^{(1)}-x_t^{(2)}|$ as $m \to \infty$.

Then we have

$$E|x_t^{(1)}-x_t^{(2)}| \le \int_0^t K(E|x_s^{(1)}-x_s^{(2)}|)ds$$

As is well known, by the condition (ii) $\int_{0+} \frac{du}{K(u)} = +\infty$, this implies $E|x_t^{(1)} - x_t^{(2)}| = 0$ and therefore $x_t^{(1)} = x_t^{(2)}$, and hence we have $\varphi_t^{(1)} = \varphi_t^{(2)}$.

C.Q.F.D.

Remark. - For example, $\rho(u) = u^{\alpha} : \alpha \geq \frac{1}{2}$ satisfies the condition (i) .

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