MICHAEL J. SHARPE

Homogeneous extensions of random measures

Séminaire de probabilités (Strasbourg), tome 9 (1975), p. 496-514
<http://www.numdam.org/item?id=SPS_1975__9__496_0>

HOMOGENEOUS EXTENSIONS OF RANDOM MEASURES

by M. J. Sharpe

1. Introduction. Let S denote a terminal time for a Markov process X. In questions concerning the decomposition of the process in terms of the subprocess (X, S), one encounters problems of the following type: being given some sort of functional of (X, S) which possesses some homogeneity relative to the shift operator, find a means of extending that functional to one which is homogeneous for the entire process. For examples, one may consult [4] and [7]. The same sort of problem arises in a different framework in [1].

One of the main results of this paper, Theorem 4, was proved in [4]. A complete discussion of Theorems 1 and 3 may be found in [8].

We suppose that the Markov process $X = (\Omega, \mathfrak{F}, \mathfrak{F}_t, X_t, \theta_t, P^x)$ takes values in a separable metric space E. The family $(\mathfrak{F}_t)$ is defined by the usual completion procedure. It is assumed that X is right continuous and Markov relative to $(\mathfrak{F}_t)$, and that the family $(\mathfrak{F}_t)$ is right continuous. These conditions are certainly satisfied under the "hypothèses droites" of Meyer, as extended by Getoor [5].

2. Projections and Shifts. We denote by $\mathfrak{h}$ the family of evanescent processes: $Z \in \mathfrak{h}$ if and only if $E^{\mu}[\sup_t |Z_t|] = 0$ for all probabilities μ on $(E, \mathcal{E})$. We write $Z = W$ in case $\{Z \neq W\} \in \mathfrak{h}$. Let $\mathfrak{m}$ denote the measurable processes: $\mathfrak{m} = (\mathcal{B}^+ \times \mathfrak{F}) \vee \mathfrak{h}$, where $\mathcal{B}^+$ is the Borel σ- field on $\mathbb{R}^+ = [0, \infty)$.

We define a semigroup $(\Theta_t)_{t \geq 0}$ of operators on $\mathfrak{m}$ by setting $(\Theta_t Z)(s,\omega) = Z(s-t, \theta_t \omega) 1_{[t,\infty)}(s)$. If Z is adapted, so is $\Theta_t Z$. If $Z \in \mathfrak{h}$, $\Theta_t Z \in \mathfrak{h}$ also.

Let $\mathfrak{w}$(resp. $\mathcal{P}$) denote the σ-field of processes generated by $\mathfrak{h}$ and the family of adapted processes whose trajectories are a.s. right-continuous with left limits (resp. left continuous). These will be the appropriate classes of well measurable (resp. previsible) processes for the special theory of projections which is developed here. It is clear that if $Z \in \mathfrak{w}$ (resp. $\mathcal{P}$) then $\Theta_t Z \in \mathfrak{w}$ (resp. $\mathcal{P}$). It is not a difficult matter to prove, following ideas from [2], that special versions of the well measurable and previsible projections can be defined in such a way as to commute with the operators Θ_t. More precisely,

Theorem 1. There exist mappings $Z \to {}^1Z$ and $Z \to {}^3Z$ of $b\mathfrak{m}$ onto $b\mathfrak{w}$ and $b\mathcal{P}$ respectively such that

(i) For each initial law μ, ${}^1Z$(resp. ${}^3Z$) is the well measurable (resp. previsible) projection of Z relative to $(\Omega, \mathfrak{F}_t^{\mu}, P^{\mu})$: that is

$$E^{\mu}(Z_T;\ T<\infty) = E^{\mu}({}^1Z_T;\ T<\infty)\ (\text{resp. } E^{\mu}({}^3Z_T;\ T<\infty))$$

for all $(\mathcal{F}_t^{\mu})$ stopping times T (resp. previsible $(\mathcal{F}_t^{\mu})$ stopping times T).

$$\text{(ii)}\quad \Theta_t({}^1Z) = {}^1(\Theta_t Z)\ (\text{resp. } \Theta_t({}^3Z) = {}^3(\Theta_t Z)),\quad \text{for all } t\geq 0.$$

A measurable process Z is called homogeneous on $[0, \infty)$ (resp. $(0, \infty)$) in case for all $t\geq 0$, $Z_u(\theta_t\omega) = Z_{u+t}(\omega)$ for all $u\geq 0$ (resp. $u>0$) a.s. . This definition is equivalent to requiring that for all $t\geq 0$,

$$1_{[t,\infty)}\cdot \Theta_t Z = 1_{[t,\infty)}\cdot Z\ (\text{resp. } 1_{(t,\infty)}\cdot \Theta_t Z = 1_{(t,\infty)}\cdot Z)\ .$$

By Theorem 1, if Z is homogeneous and bounded (or positive) then ${}^1Z$ and ${}^3Z$ are also homogeneous.

The most elementary problem of homogeneous extension can now be treated. Let S denote a perfect terminal time for X. We call a measurable process Z, which vanishes on $[S, \infty)$, homogeneous on $[0, S)$ (resp. $(0, S)$) in case $1_{[t,S)}\cdot \Theta_t Z = 1_{[t,S)}\cdot Z$ (resp. $1_{(t,S)}\cdot \Theta_t Z = 1_{(t,S)}\cdot Z$) for all $t\geq 0$. That is to say that for all $t\geq 0$, $Z_u(\theta_t\omega) = Z_{u+t}(\omega)$ for all $u\in[0, S(\omega)-t)$ (resp. $u\in(0, S(\omega)-t)$) a.s.. An obvious example is $Z_t = f\circ X_t 1_{[0,S)}(t)$ $(f\in\mathcal{E})$, which is homogeneous on $[0, S)$.

Theorem 1 shows that if Z is homogeneous on $[0, S)$ (resp. $(0, S)$) then so is ${}^1Z$.

We let $S_t = t + S\circ\theta_t$, noting that $1_{[S_t]} = \Theta_t 1_{[S]}$. Taking account of perfection of S we see that for all $u \geq 0$, $t \geq 0$ and $\omega \in \Omega$, $S_{t+u}(\omega) = S_u(\omega)$ if $S_u(\omega) > u + t$. This means that for all ω, if $0 \leq t \leq u$ then either $(t, S_t(\omega)) \cap (u, S_u(\omega))$ is empty, or $S_t(\omega) = S_u(\omega)$. We define M to be the random set whose ω-section is the closure in $(0, \infty)$ of $\{S_t(\omega): t > 0\} \cap (0, \infty)$. The complement in $(0, \infty)$ of this ω-section is $\bigcup(t, S_t(\omega))$, where the union may be taken over $\mathbb{Q}$, the positive rationals. It is easy to see that the indicator of M is homogeneous on $(0, \infty)$, and it belongs to $\mathfrak{W}$. We let M^c denote the complement of M in $(0, \infty)$.

We shall say that a measurable process Z is perfectly homogeneous on $(0, S)$ in case $Z_t(\omega) = 0$ whenever $t \geq S(\omega)$ and for all ω, for all $t \geq 0$ and $u > 0$,

$$Z_{t+u}(\omega) = Z_u(\theta_t\omega) \quad \text{whenever} \quad t + u < S(\omega) \ .$$

One knows [6] that if Z is homogeneous on $(0, S)$ and Z is a.s. right continuous, there exists a perfect Z' which is indistinguishable from Z.

Theorem 2. Let Z be perfectly homogeneous on $(0, S)$. There exists a unique process $\bar{Z}$ which is perfectly homogeneous on $(0, \infty)$, vanishes on $M \cup \{0\}$, and is such that $1_{(0,S)} \cdot \bar{Z} = 1_{(0,S)} \cdot Z$. Moreover, if Z is bounded or positive and ${}^1Z$ can be chosen to be perfectly homogeneous on $(0, S)$ then ${}^1(\bar{Z}) = ({}^1Z)^-$.

Proof. For uniqueness, it will suffice to show that if W is homogeneous on $(0, \infty)$ and W vanishes on both $(0, S)$ and M then $W = 0$. However, we know that $M^c = \bigcup_{t \in \Theta} (t, S_t)$, so $1_{(0,S)} \cdot W = 0$ implies $0 = \Theta_t(1_{(0,S)} \cdot W) = 1_{(t,S_t)} \cdot \Theta_t W = 1_{(t,S_t)} \cdot W$ and consequently $W = 0$. To show existence, we may suppose $Z \geq 0$ and that Z vanishes on $(0, S)^c$. We define $\bar{Z}_s(\omega) = \sup\{(\Theta_t Z)(s, \omega) : t \in \Theta\}$. Obviously $\bar{Z} \in \mathfrak{m}$ and $\bar{Z}$ vanishes on $M \cup \{0\}$. From the definition of perfect homogeneity on $(0, S)$, we have, for all $t, u \geq 0$,

$$\Theta_u(1_{(t,S)} \cdot \Theta_t Z) = \Theta_u(1_{(t,S)} \cdot Z) ,$$

from which we obtain

$$1_{(t+u,S_u)} \cdot \Theta_{t+u} Z = 1_{(t+u,S_u)} \cdot \Theta_u Z .$$

From the remarks preceding the definition of M we may conclude that

$\Theta_{t+u}Z$ and $\Theta_u Z$ are identical on $(t+u, S_u) = (u, S_u) \cap (t+u, S_{t+u})$. Since $\Theta_u Z$ vanishes off (u, S_u) it follows that $\overline{Z} = \Theta_u Z$ on (u, S_u) for all real $u \geq 0$. The homogeneity of M shows that $\Theta_t \overline{Z}$ vanishes off $(t, \infty) \cap M^c = \bigcup \{(u, S_u) : u \geq t\}$ and so $\Theta_t \overline{Z} = \Theta_{t+u} Z$ on $\Theta_t(u, S_u) = (u+t, S_{u+t})$. Hence $\Theta_t \overline{Z} = 1_{(t,\infty)} \cdot \overline{Z}$.

Since M is well measurable, ${}^1(\overline{Z})$ may be assumed to vanish identically on M. In order to show that ${}^1(\overline{Z}) = ({}^1Z)^-$ it is sufficient, because of the uniqueness result, to show them equal on $(0, S)$. Since $(0, S)$ is well measurable, we have

$$1_{(0,S)}{}^1(\overline{Z}) = {}^1(1_{(0,S)} \cdot \overline{Z}) = {}^1Z = 1_{(0,S)} \cdot ({}^1Z)^- ,$$

completing the proof.

3. Random Measures.

A random measure $\varkappa$ is defined to be a positive kernel from $(\mathbb{R}^+, \mathcal{B}^+)$ to $(\Omega, \mathcal{F})$ satisfying

(i) $\omega \to \varkappa(\omega, B) \in \mathcal{F}$ for all $B \in \mathcal{B}^+$

(3.1) (ii) $B \to \varkappa(\omega, B)$ is a positive measure on $(\mathbb{R}^+, \mathcal{B}^+)$

(iii) There exists a strictly positive previsible process Y such that

$$E^x \int_0^\infty Y_t(\cdot)\varkappa(\cdot, dt) < \infty \text{ for all } x \in E .$$

We denote by $\mathfrak{m}^*$ the class of all random measures, and by $\mathfrak{m}_0^*$ the subclass of those which do not charge $\{0\}$. For any $Z \in \mathfrak{m}_+$, $Z * \varkappa$ is defined by

$$(Z * \varkappa)(\omega, B) = \int_0^\infty 1_B(t) Z_t(\omega)\varkappa(\omega, dt) .$$

If Z is bounded or previsible then $Z * \varkappa \in \mathfrak{m}^*$ if $\varkappa \in \mathfrak{m}^*$. We denote by $\mathfrak{w}^*$ (resp. $\mathfrak{P}^*$) the class of random measures $\varkappa$ such that for some strictly positive previsible Y, $(t, \omega) \to Y * \varkappa(\omega, [0, t])$ belongs to $\mathfrak{w}$ (resp. $\mathfrak{P}$) and $E^x(Y * \varkappa(\cdot, [0, \infty))) < \infty$ for all $x \in E$. We set $\mathfrak{w}_0^* = \mathfrak{w}^* \cap \mathfrak{m}_0^*$ and $\mathfrak{P}_0^* = \mathfrak{P}^* \cap \mathfrak{m}_0^*$.

One defines a semigroup $\hat{\Theta}_t$ of operators on $\mathfrak{m}^*$ by $(\hat{\Theta}_t\varkappa)(\omega, B) = \varkappa(\theta_t\omega, B - t)$, where it is supposed that $\varkappa(\omega, \cdot)$ is extended to a measure on $\mathbb{R}$ which doesn't charge $(-\infty, 0)$. It is easy to see that $\hat{\Theta}_t$ preserves $\mathfrak{m}_0^*$, $\mathfrak{w}^*$, $\mathfrak{P}^*$, $\mathfrak{w}_0^*$ and $\mathfrak{P}_0^*$. One has the identity

$$(3.2) \qquad \hat{\Theta}_t(Z * \varkappa) = \Theta_t Z * \hat{\Theta}_t \varkappa \quad .$$

(3.3) <u>Definition</u>. Let S be a perfect terminal time and let $\varkappa \in \mathfrak{m}^*$. We say that $\varkappa$ is homogeneous on $[0, S)$ (resp. $(0, S)$) if a.s.

(i) $\varkappa(\omega, \cdot)$ doesn't charge $[0, S(\omega))^c$ (resp. $(0, S(\omega))^c$)

(ii) for all $t \geq 0$

$$1_{[t,S)} * \hat{\Theta}_t \varkappa = 1_{[t,S)} * \varkappa$$

$$(\text{resp. } 1_{(t,s)} * \hat{\Theta}_t \varkappa = 1_{(t,S)} * \varkappa)$$

We say that $\varkappa$ is perfectly homogeneous on $[0, S)$ or $(0, S)$ if (i) and (ii) are identities in ω and t.

The simplest case arises when $\varkappa(\omega, dt) = dA_t(\omega)$ where A is a finite increasing process with $A_0 = 0$. One may check then that $\varkappa$ is homogeneous on $(0, S)$ if and only if A is a not necessarily adapted additive functional of (X, S). In the case where A_0 is not necessarily zero, one obtains that $\varkappa$ is homogeneous on $[0, S)$ if and only if A is a left additive functional of (X, S) in the sense of Azéma.

The same sort of method used in proving Theorem 1 can be applied to give the following dual result.

Theorem 3. There exist mappings $\varkappa \to \varkappa^1$ and $\varkappa \to \varkappa^3$ of $\mathfrak{M}^*$ onto $\mathfrak{W}^*$ and $\mathcal{P}^*$ respectively such that

(i) For each initial measure μ, $\varkappa^1$ (resp. $\varkappa^3$) is the dual well measurable (resp. previsible) projection of $\varkappa$ relative to $(\Omega, \mathcal{F}_t^{\mu}, P^{\mu})$: that is,

$$E^{\mu}\int_0^{\infty} Z_t(\cdot)\varkappa(\cdot, dt) = E^{\mu}\int_0^{\infty} Z_t(\cdot)\varkappa^1(\cdot, dt)$$

for all $Z \in \mathfrak{W}_+$ (resp. $\mathcal{P}_+$).

(ii) $\hat{\Theta}_t(\varkappa^i) = (\hat{\Theta}_t\varkappa)^i$ if $t \geq 0$, $(i = 1, 3)$.

One obtains, in particular, that if $\varkappa$ is homogeneous on $[0, S)$ (resp. $(0, S)$) then so is $\varkappa^1$. If S is previsible, the same is true for $\varkappa^3$.

The result which is dual to Theorem 2, and is perhaps more interesting is

Theorem 4. Let $\varkappa \in \mathfrak{M}_0^*$ be perfectly homogeneous on $(0, S)$. There exists a unique $\bar{\varkappa} \in \mathfrak{M}_0^*$ which is homogeneous on $(0, \infty)$ and carried by M^c such that $\varkappa = 1_{(0,S)} * \bar{\varkappa}$. If $\varkappa^1$ may be chosen to be perfectly homogeneous on $(0, S)$, then $(\varkappa^1)^- = (\bar{\varkappa})^1$. In particular, if $\varkappa \in \mathfrak{W}_0^*$ then $\bar{\varkappa} \in \mathfrak{W}_0^*$.

Proof. Suppose γ and $\nu \in \mathfrak{m}_0^*$ are homogeneous on $(0, \infty)$ and are both carried by M^c. If $1_{(0,S)} * \gamma = 1_{(0,S)} * \nu$ then for all $t \geq 0$, $\hat{\Theta}_t\left(1_{(0,S)} * \gamma\right) = \hat{\Theta}_t\left(1_{(0,S)} * \nu\right)$ so $1_{(t,S_t)} * \gamma = 1_{(t,S_t)} * \nu$. Since $M^c = \bigcup_{t \in \mathbb{Q}} (t, S_t)$ it follows that $\gamma = \nu$. This proves uniqueness of $\bar{\varkappa}$.

We define $\bar{\varkappa}$ by

$$\bar{\varkappa}(\omega, \cdot) = \bigvee_{t \in \mathbb{Q}} \hat{\Theta}_t \varkappa(\omega, \cdot)$$

where $\vee$ means supremum in the sense of measures. Properties (i) and (ii) of (3.1) are then satisfied by $\bar{\varkappa}$. Leaving aside (iii) for the moment, we observe, as in the proof of Theorem 2, that we have (identically)

$$\hat{\Theta}_t\left(1_{(u,S)} * \varkappa\right) = \hat{\Theta}_t(1_{(u,S)} * \hat{\Theta}_u \varkappa)$$

hence, using (3.2)

$$1_{(u+t,S_t)} * \hat{\Theta}_t \varkappa = 1_{(t+u,S_t)} * \hat{\Theta}_{t+u} \varkappa .$$

From this fact we obtain that $\hat{\Theta}_{t+u}\varkappa(\omega, \cdot)$ and $\hat{\Theta}_t \varkappa(\omega, \cdot)$ are identical on $(t+u, S_t(\omega)) = (t, S_t(\omega)) \cap (t+u, S_{t+u}(\omega))$. Consequently we have $\bar{\varkappa}(\omega, \cdot) = \hat{\Theta}_t \varkappa(\omega, \cdot)$ on $(t, S_t(\omega))$ for all $t \geq 0$. Since $\bar{\varkappa}$ is carried by M^c, the homogeneity of $\bar{\varkappa}$ is evident. We turn now to proving that condition (iii) of (3.1) is satisfied by $\bar{\varkappa}$. Let $Y \in \mathcal{P}_+$ be strictly positive and satisfy $E^x \int_0^\infty Y_t(\cdot)\varkappa(\cdot, dt) < \infty$ for all $x \in E$. We set $A_t(\omega) = \int_0^t Y_s(\omega)\varkappa^3(\omega, ds)$. We have $E^x A_\infty < \infty$ for all $x \in E$, and A is previsible. But

$$E^x \int_0^\infty Y_t e^{-A_t} \varkappa(dt) = E^x \int_0^\infty e^{-A_t} dA_t \leq 1 ,$$

showing that we may assume that $E^x \int_0^\infty Y_t(\omega)\varkappa(\cdot, dt) \leq 1$ for all $x \in E$.

Let $\{r_n\}$ be an enumeration of $\mathbb{Q}$, and define $Z_t = 1_M + \sum_{n=1}^\infty 2^{-n} 1_{\left(r_n, S_{r_n}\right)} \cdot \Theta_{r_n} Y$.

Obviously, $Z \in \mathcal{P}_+$ is strictly positive, and

$$\begin{aligned}
E^x \int_0^\infty Z_t(\cdot)\bar{\varkappa}(\cdot, dt) &\leq \sum 2^{-n} E^x \int_0^\infty \left(\Theta_{r_n} Y\right)_t \hat{\Theta}_{r_n} \varkappa(\cdot, dt) \\
&\leq \sum 2^{-n} E^x \left[\hat{\Theta}_{r_n} (Y * \varkappa)(\cdot, \mathbb{R}^+)\right] \\
&\leq \sum 2^{-n} E^x E^{X(r_n)} [(Y * \varkappa)(\cdot, \mathbb{R}^+)] \\
&\leq 1 .
\end{aligned}$$

To complete the proof we must show that $(\bar{\varkappa})^1 = (\varkappa^1)^-$. In view of uniqueness, it is sufficient to show that their restriction to $(0, S)$

are equal. This holds since $(0, S) \in \mathfrak{W}$ implies that

$$1_{(0,S)} * (\bar{\varkappa})^1 = (1_{(0,S)} * \bar{\varkappa})^1 = \varkappa^1 = 1_{(0,S)} * (\varkappa^1)^- .$$

Remark. The requirement in (3.1)(iii) that Y be previsible may be weakened to the condition that $Y \in \mathfrak{W}$ if one is interested only in well measurable projections. For sufficient conditions under which perfect versions of a homogeneous random measure exist, the reader should consult [9].

4. The Previsible Case.

If S is not previsible, the results of Theorems 2 and 4 are not valid for previsible projections. After a slight modification of the notion of projection, though, we can obtain essentially the same results.

Suppose Λ is a measurable set in $\mathbb{R}^+ \times \Omega$ which satisfies the condition

$$\{{}^3 1_\Lambda > 0\} \supset \Lambda \text{ up to evanescence.} \tag{4.1}$$

Let $\mathfrak{M}_\Lambda$, $\mathcal{P}_\Lambda$ and $\mathfrak{W}_\Lambda$ denote the traces of $\mathfrak{M}$, $\mathcal{P}$ and $\mathfrak{W}$ on Λ. We interpret $Y \in \mathcal{P}_\Lambda$, for example, to mean that $Y = Z \cdot 1_\Lambda$ where $Z \in \mathcal{P}$.

Similarly, $\mathcal{P}^*_\Lambda$ denotes the class of random measures of the form $1_\Lambda * \varkappa$, $\varkappa \in \mathcal{P}^*$, etc. We define, for $Z \in b\mathfrak{m}_\Lambda$ the previsible projection of Z on Λ by

$$ {}^3_\Lambda Z = (1_\Lambda / {}^3 1_\Lambda) \cdot {}^3 Z $$

and the dual previsible projection of $\varkappa \in \mathfrak{m}^*_\Lambda$ by

$$ \varkappa^3_\Lambda = (1_\Lambda / {}^3 1_\Lambda) * \varkappa^3 $$

In each case we set $0/0 = 0$. The following assertions are then routine, making use of (4.1):

(4.2) <u>Proposition</u>.

(a) For $Z \in b\mathfrak{m}_\Lambda$, ${}^3_\Lambda Z$ is the unique member of $\mathcal{P}_\Lambda$ with the property that for every initial measure μ, $E^\mu\{{}^3_\Lambda Z(T);\ T < \infty\} = E^\mu\{Z(T);\ T < \infty\}$ for every previsible stopping time T.

(b) For $\varkappa \in \mathfrak{m}^*_\Lambda$, $\varkappa^3_\Lambda$ is the unique member of $\mathcal{P}^*_\Lambda$ satisfying

$$ E^\mu \int_0^\infty W_t \, \varkappa^3_\Lambda(dt) = E^\mu \int_0^\infty W_t \varkappa(dt) $$

for all $W \in (\mathcal{P}_\Lambda)_+$ and every initial measure μ.

The cases of interest here are those in which $\Lambda = (0, S)$ and $\Lambda = M^c$.

(4.3) Lemma. The sets $(0, S)$ and M^c both satisfy the condition (4.1).

Proof. We have ${}^3 1_{(0,S)} = 1_{(0,S]} - {}^3 1_{[S]}$, so $\{{}^3 1_{(0,S)} = 0\} = [0] \cup (S, \infty) \cup \{1_{(0,S]} = {}^3 1_{[S]}\}$. Since $\{{}^3 1_{[S]} > 0\} \subset \cup [T_n]$ where $\{T_n\}$ is a sequence of previsible stopping times which englobe the accessible part of S, the set $\{1_{(0,S]} = {}^3 1_{[S]}\} = \cup [R_n]$, where $\{R_n\}$ is a sequence of previsible stopping times. For all n,
$P^\mu\{R_n < \infty\} = P^\mu\{R_n \le S,\ {}^3 1_{[S]}(R_n) = 1\} = P^\mu\{R_n \le S,\ R_n = S < \infty\} =$
$= P^\mu\{R_n = S < \infty\}$, hence $P^\mu\{R_n < S\} = 0$. It follows that $\{1_{(0,S]} = {}^3 1_{[S]}\}$ $\subset [S, \infty)$, and this shows that $(0, S)$ satisfies (4.1). We know, on the other hand, that $M^c = \cup \{(t, S_t) : t \in \mathbb{Q}\}$ and each (t, S_t) satisfies (4.1) by the above argument. Then $\{{}^3 1_{M^c} > 0\} \supset \{{}^3 1_{(t,S_t)} > 0\} \supset (t, S_t)$ for all $t \in \mathbb{Q}$, and so M^c satisfies (4.1).

Remark. The ideas above permit one to explain the difference between the previsible AF, A, generated by a natural potential u of (X, S) and the "natural" AF, B, generated by that potential. The Meyer decomposition gives the existence of the previsible AF A, which may charge S. The procedure explained in [3] determines a natural AF, B, of (X, S) which does not change S. It is not hard to check that A must be carried by $\{{}^3 1_{(0,S)} > 0\}$ and that $B = (1_{(0,S)} / {}^3 1_{(0,S)}) * A$.

We shall give only the dual projection version of the extension theorem, the projection version being entirely analogous.

Theorem 5. Let $\varkappa$ be a random measure which is homogeneous on $(0, S)$. Then $\gamma = \varkappa^3_{(0,S)}$ is homogeneous on $(0, S)$. If perfect versions of $\varkappa$ and γ can be found, then $\bar{\gamma} = (\bar{\varkappa})^3_{M^c}$. In particular, if $\varkappa \in \mathcal{P}^*_{(0,S)}$ then $\bar{\varkappa} \in \mathcal{P}^*_{M^c}$.

Proof. We start by showing that for all $t \geq 0$,

$$ {}^3 1_{(t,S_t)} \cdot 1_{(0,S)} = {}^3 1_{(t,S)} \cdot 1_{(0,S)} \quad . $$

For each μ and for each previsible stopping time T,

$$ \begin{aligned} E^{\mu}[{}^3 1_{(t,S_t)}(T) ; {}^3 1_{(0,S)}(T) > 0] &= E^{\mu}[1_{(t,S_t)}(T) ; {}^3 1_{(0,S)}(T) > 0] \\ &= E^{\mu}[1_{(t,S)}(T) ; {}^3 1_{(0,S)}(T) > 0] \end{aligned} $$

since $S_t = S$ on $\{T > t\}$. This last expression is equal to

$$E^{\mu}\left[{}^{3}1_{(t,S)}(T);\ {}^{3}1_{(0,S)}(T) > 0\right]$$

and so ${}^{3}1_{(t,S_t)} = {}^{3}1_{(t,S)}$ on $\{{}^{3}1_{(0,S)} > 0\}$, hence on (0,S), by (4.3).
We then have

$$1_{(t,S)} \cdot \Theta_t\left[1_{(0,S)}/{}^{3}1_{(0,S)}\right] = 1_{(t,S)} \cdot 1_{(t,S_t)}/\Theta_t\left({}^{3}1_{(0,S)}\right)$$

$$= 1_{(t,S)} \cdot 1_{(t,S_t)}/{}^{3}1_{(t,S_t)}$$

$$= 1_{(t,S)} \cdot 1_{(0,S)}/{}^{3}1_{(0,S)} \ .$$

That is, $1_{(0,S)}/{}^{3}1_{(0,S)}$ is homogeneous on (0, S). Observe now that if $\varkappa$ is any random measure carried by (0, S) then $\hat{\Theta}_t\varkappa(\omega, \{S(\omega)\}) = \varkappa(\theta_t\omega, \{S(\omega) - t\})$. If $S(\omega) > t$, this last term equals $\varkappa(\theta_t\omega, \{S(\theta_t\omega)\}) = 0$. Thus, if $\varkappa$ is homogeneous on (0, S), $1_{(t,S]} * \hat{\Theta}_t\varkappa = 1_{(t,S]} * \varkappa$ for all $t \geq 0$. Since $(t, S] \in \mathcal{P}$, we have $1_{(t,S]} * (\hat{\Theta}_t\varkappa)^3 = 1_{(t,S]} * \varkappa^3$, and from Theorem 3, one concludes that

$$1_{(t,S]} * \hat{\Theta}_t(\varkappa^3) = 1_{(t,S]} * \varkappa^3 \ .$$

Since $\{{}^{3}1_{(t,S)} > 0\} \subset (t, S]$, we have then

$$1_{(t,S)} \cdot \left[1_{(0,S)}/{}^{3}1_{(0,S)}\right] * \hat{\Theta}_t(\varkappa^3) = 1_{(t,S)} \cdot \left[1_{(0,S)}/{}^{3}1_{(0,S)}\right] * \varkappa^3$$

and hence, in view of the homogeneity of $1_{(0,S)}/{}^{3}1_{(0,S)}$ on $(0, S)$, and formula (3.2), we obtain

$$1_{(t,S)} * \hat{\Theta}_t\gamma = 1_{(t,S)} * \gamma .$$

We show next that ${}^{3}1_{M^c} = {}^{3}1_{(0,S)}$ on $(0, S)$. We have $1_{(0,S)} \cdot 1_{M^c} = 1_{(0,S)}$, so ${}^{3}\left(1_{(0,S]}\, 1_{M^c}\right) = {}^{3}1_{(0,S)}$. But $(0, S] \in \mathcal{P}$, and we obtain therefore $1_{(0,S]}\, {}^{3}1_{M^c} = {}^{3}1_{(0,S)}$, from which the above assertion is obvious. Since M^c is homogeneous on $(0, \infty)$, $1_{M^c}/{}^{3}1_{M^c}$ is homogeneous on $(0, \infty)$, using Theorem 1. By construction, $\bar{\varkappa}$ is homogeneous on $(0, \infty)$ and it is carried by M^c. Thus $(\bar{\varkappa})^3$ is homogeneous on $(0, \infty)$, thanks to Theorem 3. Since $\bar{\varkappa}$ is carried by the previsible set $\{{}^{3}1_{M^c} > 0\}$, the same is true of $(\bar{\varkappa})^3$. We now see that $(\bar{\varkappa})^3_{M^c} = \left(1_{M^c}/{}^{3}1_{M^c}\right) * (\bar{\varkappa})^3$ is homogeneous on $(0, \infty)$, and it is carried by M^c. In order to show that $\bar{\gamma} = (\bar{\varkappa})^3_{M^c}$, it will suffice to show that their restrictions to $(0, S)$ are equal. See the proof of Theorem 4.

We have

$$1_{(0,S]} * (\bar{\varkappa})^3_{M^c} = \left[1_{(0,S]} \cdot 1_{M^c} / {}^3 1_{M^c}\right] * \bar{\varkappa}^3$$

$$= \left(1_{M^c} / {}^3 1_{M^c}\right) * \left(1_{(0,S]} * \varkappa^-\right)^3$$

$$= \left(1_{M^c} / {}^3 1_{M^c}\right) * \varkappa^3 \ .$$

Since $\varkappa^3$ is carried by $\{{}^3 1_{(0,S)} > 0\} \subset (0, S]$ we have

$$1_{(0,S)} * (\bar{\varkappa})^3_{M^c} = \left(1_{(0,S)} \cdot 1_{M^c} / {}^3 1_{M^c}\right) * \varkappa^3$$

$$= \left(1_{(0,S)} / {}^3 1_{(0,S)}\right) * \varkappa^3$$

$$= \gamma = 1_{(0,S)} * \bar{\gamma} \ ,$$

completing the proof.

REFERENCES

1. J. Azéma and P.A. Meyer Une nouvelle représentation du type de Skorohod. Sem. de Probabilités de Strasbourg VIII, Springer Lecture Notes 381, 1974.

2. A. Benveniste and J. Jacod . Projection des fonctionelles additives et représentation des potentiels d'un processus de Markov. C.R. Acad. Sc. Paris, t. 276, 1365-1368 (1973).

3. R. Blumenthal and R.K. Getoor. Markov Processes and Potential Theory. Academic Press, 1968.

4. R.K. Getoor and M.J. Sharpe. Balayage and multiplicative functionals. Z. Wahrs. ver. Geb. 28, 139-164, 1974.

5. R.K. Getoor. Lecture notes on Markov processes: Ray processes and right processes. To appear.

6. P.A. Meyer. Quelques autres applications de la méthode de Walsh. Sém. de Probab. de Strasbourg VI. Springer Lecture Notes 258, 1972.

7. P.A. Meyer. Ensembles aléatoires Markoviens homogènes, I et II. Sém. de Probab. de Strasbourg VIII. Springer Lecture Notes 381, 1974.

8. M.J. Sharpe. Lecture Notes on Additive Functionals. To appear.

9. J.B. Walsh. The perfection of multiplicative functionals. Sém. de Probab. de Strasbourg VI. Springer Lecture Notes 258, 1972.