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MULTIPLICATIVE EXCESSIVE MEASURES AND DUALITY BETWEEN EQUATIONS OF BOLTZMANN AND OF BRANCHING PROCESSES

by Masao NAGASAWA [1]

Duality between Boltzmann's equations (corresponding Markov processes) and so-called S-equations of branching Markov processes (branching Markov processes) was discovered by H.Tanaka[11,12], Y.Takahashi[10] and the author(unpublished). This will be explained in the following sections. To make the duality to be realistic, existence of a special class of excessive measures of branching Markov processes will be discussed in the later sections.

I. Boltzmann's equation

Suppose we are observing the distribution of speed of a gas molecule. The speed is assumed to stay constant until collision occurs. Suppose after some time interval of the exponential distribution, two particles with speeds a_1 and a_2 collide and the speed of the first particle will be distributed in db with a probability distribution $\bar{\pi}_2(a_1,a_2;db)$ depending on a_1 and a_2. Then, if the initial distribution of speed of each particle is f(da), the speed distribution $u_t(da)$ at t satisfies

$$u_t = e^{-t}f + \int_0^t ds\, e^{-(t-s)} \iint u_s(da_1)u_s(da_2)\bar{\pi}_2(a_1,a_2;\cdot).$$

This is the so-called Boltzmann's equation of gas in a simple form. McKean[3,4,5] discussed probabilistic aspects of the equation and gave a model as a temporally inhomogeneous Markov processes(with non-constant transition mechanism in his terminology).

Let's generalize the equation allowing;(a) the speed before collision not to be constant but varrying as a right continuous

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Markov process $\bar{x}_t$ on a state space S with a collision time of the distribution $\exp(-c_t)$ where c_t is the Kac's additive functional of a non-negative function c, and (b) n-particle collision ($n = 2,3,\cdots$) with a distribution $\bar{\pi}_n(a_1,a_2,\cdots,a_n;db)$. Let $\bar{q}_n(a_1,\cdots,a_n)$ be a weight of n-collision and $\bar{P}^0_t$ be the transition probability of the Markov process, then a generalized Boltzmann's equation is

$$(1)\quad u_t=f\bar{P}^0_t+\int_0^t ds \sum_{n=2}^{\infty} \int_{S^n} \prod_{j=1}^{n} u_s(da_j)\bar{q}_n(\underline{\underline{a}})\int_S \bar{\pi}_n(\underline{\underline{a}},dc)P^0_{t-s}(c,\cdot),$$

where $\underline{\underline{a}} = (a_1,\cdots,a_n)$.
This equation was treated by H.Tanaka[11,12] and T.Ueno[13,14]. Tanaka constructed a Markov process on a large state space $\underline{\underline{S}} = \bigcup_{n=1}^{\infty} S^n$ such that: Let H_t be the transition probability of the Markov process and $\hat{f}$ be the measure on $\underline{\underline{S}}$ whose restriction on S^n is the n-fold product of f. Then the solution of (1) is given by

$$u_t(B) = \int_{\underline{\underline{S}}} \hat{f}(d\underline{\underline{a}})H_t(\underline{\underline{a}},B), \quad B\subset S.$$

Moreover he characterized the Markov process in terms of the following convolution property: Let $\underline{\underline{\phi}} = (\phi_1,\phi_2,\cdots)$ and $\underline{\underline{\psi}} = (\psi_1,\psi_2,\cdots)$ be defined on $\underline{\underline{S}}$ where ϕ_n is the restriction of $\underline{\underline{\phi}}$ on S^n. Define a convolution $\underline{\underline{\phi}}*\underline{\underline{\psi}}$ by

$$(\underline{\underline{\phi}}*\underline{\underline{\psi}})_n = \text{sym.} \sum_{\substack{i+j=n\\ i,j\geq 1}} \phi_i(a_1,\cdots,a_i)\psi_j(a_{i+1},\cdots,a_n).$$

Then the transition probability satisfies

$$H_t(\underline{\underline{\phi}}*\underline{\underline{\psi}}) = (H_t\underline{\underline{\phi}})*(H_t\underline{\underline{\psi}}).$$

Let's call it Tanaka's collision property. He also proved that if we put

$$P^f(t,a,B) = \int_{\underline{\underline{S}}} \hat{f}(d\underline{\underline{b}})H_t(a\cdot\underline{\underline{b}},B), \quad a\in S,\ B\subset S,$$[1]

then it is a temporally inhomogeneous transition probability of McKean's non-constant transition mechanism, i.e.

$$P^f(t+s,a,B) = \int P^f(t,a,db)P^{u_t}(s,b,B),$$

$$u_t(B) = \int f(da)P^f(t,a,B).$$

2. Duality between Boltzmann's and S-equations

To have the duality, we need assumptions on existence of density functions with respect to a finite(or σ-finite) measure $\mu(da)$ on S.

Assumption.

(A.1) There exists a density function $\bar{\pi}_n(a_1,\cdots,a_n;b)$ with respect to μ such that

$$\bar{\pi}_n(a_1,\cdots,a_n;db) = \bar{\pi}_n(a_1,\cdots,a_n;b)\mu(db).$$

Set

$$q_n'(b) = \int_{S^n} \hat{\mu}(d\underline{\underline{a}})\bar{q}_n(\underline{\underline{a}})\bar{\pi}_n(\underline{\underline{a}},b)$$

and

$$c(b) = \sum_{n=2}^{\infty} q_n'(b) < \infty.$$

(A.2) There exists a transition density function $P_t^0(a,b)$ with respect to μ, and

$$P_t^0(a,db) = P_t^0(a,b)\mu(db), \text{ and}$$

$$\bar{P}_t^0(a,db) = \mu(db)P_t^0(b,a)$$

are transition probabilities of $\exp(-c_t)$ -subprocesses of right continuous Markov processes in duality with respect to μ.

Put

$$q_n(b) = q_n'(b)/c(b), \quad \text{if } c(b) > 0,$$
$$= 0, \quad \text{if } c(b) = 0.$$

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1 $\underline{\underline{a}}\cdot\underline{\underline{b}} = (a_1,\cdots,a_n,b_1,\cdots,b_m)$ when $\underline{\underline{a}}=(a_1,\cdots,a_n)$ and $\underline{\underline{b}}=(b_1,\cdots,b_m)$.

$$\pi_n(b,d\underline{\underline{a}}) = \bar{q}_n(\underline{\underline{a}})\bar{\pi}_n(\underline{\underline{a}},b)\hat{\mu}(d\underline{\underline{a}})/q_n'(b), \qquad \text{if } q_n'(b)>0,$$

$$= \text{any probability measure on } S^n, \text{ if } q_n'(b)=0,$$

where $\underline{\underline{a}} = (a_1,\cdots a_n) \in S^n$.

Then we can write the Boltzmann's equation (1) in terms of density functions, i.e. if $f(da) = f(a)\mu(da)$ $(\|f\| \le 1)$, then $u_t(da)$ has a density function $u_t(a)$ with respect to μ, and it satisfies

$$u_t(a) = P_t^0 f(a) + \int_0^t ds\ P_s^0\{c \sum_{n=2}^{\infty} q_n \int_{S^n} \pi_n(\cdot,d\underline{\underline{b}})\hat{u}(\underline{\underline{b}})\}(a),$$ [1]

Clearly this is the so-called S-equation for a branching Markov process determined by $\{P_t^0, q_n, \pi_n\}$, $q_0 = q_1 = 0$ in the present case (cf. [2,8]), and the solution $u_t(a)$ of the S-equation is given in terms of the branching Markov process $(X_t, P_{\underline{\underline{a}}}, \underline{\underline{a}} \in \underline{\underline{S}}^\partial)$ on $\underline{\underline{S}}^\partial = \bigcup_{n=0}^{\infty} S^n$, $S^0 = \{\partial\}$, an extra point;

$$u_t(a) = E_a[\hat{f}(X_t)], \quad a \in S,$$

(cf. [2,8]).

PROPOSITION 1. <u>There exists a transition density</u> $T_t(\underline{\underline{a}},\underline{\underline{b}})$ <u>of the branching Markov process with respect to</u> $\hat{\mu}$, <u>and it satisfies the branching property in density form</u>:

$$T_t(\nu_1 * \nu_2, \cdot) = T_t(\nu_1,\cdot) * T_t(\nu_2,\cdot).$$ [2]

PROOF. Put

$$T_t^0(\underline{\underline{a}},\underline{\underline{b}}) = \prod_{i=1}^{m} P_t^0(a_i,b_i), \text{ when } \underline{\underline{a}} \text{ and } \underline{\underline{b}} \text{ are in } S^m,$$

and define

$$T_t^n(\underline{\underline{a}},\underline{\underline{b}}) = E_{\underline{\underline{a}}}[\, T_{t-\tau}^{n-1}(X_\tau,\underline{\underline{b}})], \quad n \geq 1,$$

where τ is the first branching time of the process X_t.

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1 $\hat{u}(\underline{\underline{b}}) = \prod_{j=1}^{n} u(b_j)$, when $\underline{\underline{b}} = (b_1,\cdots,b_n)$

2 $\int_{\underline{\underline{S}}^\partial} \nu_1 * \nu_2(d\underline{\underline{a}})F(\underline{\underline{a}}) = \int_{\underline{\underline{S}}^\partial}\int_{\underline{\underline{S}}^\partial} \nu_1(d\underline{\underline{a}}_1)\nu_2(d\underline{\underline{a}}_2)F(\underline{\underline{a}}_1 \cdot \underline{\underline{a}}_2)$.

Then

$$T_t(\underline{\underline{a}},\underline{\underline{b}}) = \sum_{n=0}^{\infty} T_t^n(\underline{\underline{a}},\underline{\underline{b}})$$

provides a desired transition density. For the branching property refer to ([8], where the property is proved not in the density form).

Put

$$H_t(\underline{\underline{a}},d\underline{\underline{b}}) = \hat{\mu}(d\underline{\underline{b}})T_t(\underline{\underline{b}},\underline{\underline{a}}).$$

Then if $\hat{\mu}$ is an excessive measure for T_t, H_t is a transition probability.

PROPOSITION 2. H_t <u>satisfies Tanaka's collision property</u>.

PROOF. Because of the branching property of $T_t(\underline{\underline{a}},\underline{\underline{b}})$

$$H_t(\phi*\psi)(\underline{\underline{a}}) = \int\phi*\psi\hat{\mu}(d\underline{\underline{b}})T_t(\underline{\underline{b}},\underline{\underline{a}}) = \int(\phi\hat{\mu})*(\psi\hat{\mu})(d\underline{\underline{b}})T_t(\underline{\underline{b}},\underline{\underline{a}})$$

$$= (\phi\hat{\mu}T_t)*(\psi\hat{\mu}T_t)(\underline{\underline{a}}) = H_t\phi*H_t\psi(\underline{\underline{a}}).$$ [1]

3. Excessive measures

This is a preparatory remark on excessive measures of Markov processes. Let T_t be the transition probability of a right continuous Markov process on a locally compact Hausdorf space with a countable open base.

PROPOSITION 3. <u>Let μ be a measure which is finite on every compact sets. Then the following statements are equivalent</u>
(i) μ <u>is an excessive measure for the Markov process</u>,
(ii) $\mu T_t f \leq \mu f$ <u>for every</u> $f \in C_k^+$,
(iii) $\mu(\alpha G_\alpha f) \leq \mu f$ <u>for every</u> $\alpha > 0$ <u>and</u> $f \in C_k^+$,
(iv) $\mu Au \leq 0$ <u>for every</u> u <u>in</u> $\{G_\alpha f;\ \alpha > 0,\ f \in C_k^+\}$,
<u>where C_k^+ is the space of non-negative continuous functions with compact supports</u>, G_α <u>is the resolvent of</u> T_t, <u>and</u> A <u>is the generator of</u> T_t.

PROOF. Clearly (i) $\leftrightarrow$ (ii) $\rightarrow$ (iii). (iii) $\rightarrow$ (ii) is proved in Nagasawa-Sato ([6] lemma 3.3). Since $AG_\alpha f = \alpha G_\alpha f - f$, (iii) $\leftrightarrow$ (iv) is clear.

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1 When $q_0 \neq 0$, this convolution property must be modified as we see in the later section.

4. Multiplicative excessive measures of CGW

In the theory of branching Markov processes the state space is $\underline{\underline{S}}^{\partial} = \bigcup_{n=0}^{\infty} S^n$. However, since ∂ is a trap, there is no excessive measure on $\underline{\underline{S}}^{\partial}$ in general. Therefore we exclude ∂ and consider the process on $\underline{\underline{S}} = \bigcup_{n=1}^{\infty} S^n$.

Let us call an excessive measure of the form $\hat{\mu}$ to be <u>multiplicative</u>. We will first prove the existence of m-excessive measures for continuous parameter Galton-Watson processes (abbreviated as CGW). We don't assume $q_0 = 0$ in this section ($q_0 = 0$ for the dual of collision processes). The existence of the unique invariant measure for CGW is proved in Harris([1] , p.111) under the assumption $q_0 > 0$, but the invariant measure is not multiplicative except when $q_0 = q_2 = 1/2$.

The CGW process is a Markov chain on $Z^+=\{0,1,2,\cdots\}$ satisfying

(i) $\quad P_n[t < \tau] = \exp(-nct)$,

(ii) $\quad P_n[X_\tau = m] = q_{m-n+1}, \quad P_0[X_t = 0] = 1,$

where c is a non-negative constant, τ is the first jumping time, and $q_n \geqq 0$, $\sum_{n=0}^{\infty} q_n = 1$ $(q_1 = 0)$.

As mentioned above we exclude $\{0\}$ from the state space in the following. Since the generator of the transition semi-group of the CGW process is

$$Af(n) = cn\{ \sum_{m=n-1}^{\infty} q_{m-n+1}f_m - f_n\}, \quad n = 1,2,\cdots,$$

and since $\hat{\mu}$ is excessive if and only if $\hat{\mu}Af \leq 0$, we have

LEMMA 1. <u>Take</u> $\mu > 0$. $\hat{\mu} = \{\mu^n; n = 1,2,\cdots\}$ <u>is m-excessive measure for</u> CGW <u>process if and only if</u>

$$(2) \quad \sum_{k=0}^{m} (m+1-k)q_k\mu^{-k} - m\mu^{-1} \leqq 0, \quad \underline{\text{for}}\ m = 1,2,3,\cdots.$$

PROOF. For $g = G_\alpha f$, $f \in C_k^+$, we have

$$0 \geqq \sum_{n=1}^{\infty} \mu^n Ag(n) = \sum_{n=1}^{\infty} \mu^n cn\{ \sum_{m=n-1}^{\infty} q_{m-n+1}g_m - g_n\}$$

$$= c \sum_{m=1}^{\infty} \{ \sum_{n=1}^{m+1} nq_{m-n+1}\mu^{n} - m\mu^{n} \} g_m .$$

Since we can find $f \in C_k^+$ such that $g_m > 0$ for $m = 1,2,3,\cdots$, we have

$$\sum_{n=1}^{m+1} nq_{m-n+1}\mu^{m} - m\mu^{n} \leqq 0,$$

multiplying μ^{-m-1} and putting $m-n+1 = k$,

$$\sum_{k=0}^{m} (m+1-k)q_k\mu^{-k} - m\mu^{-1} \leqq 0.$$

LEMMA 2. $\hat{\mu}$ _is_ m-_excessive measure if and only if_

(a) $$2q_0 \leqq \mu^{-1}$$

(b) $$h(\mu^{-1}) \leqq \mu^{-1},$$

where $h(u) = \sum_{n=o}^{\infty} q_n u^n$ _is the probability generating function of_ q_n.

REMARK. The lemma implies $q \leqq \mu^{-1} \leqq r$, where q and r are non-negative roots of $h(u) - u = 0$.

PROOF OF LEMMA 2. If $\hat{\mu}$ is excessive, Lemma 1 implies, putting $m=1$,

$$2q_0 \leqq \mu^{-1},$$

and

$$\sum_{k=0}^{\infty} a_k^m q_k \mu^{-k} - m\mu^{-1}/(m+1) \leqq 0,$$

where

$$a_k^m = 1 - k/(m+1), \quad k \leqq m,$$
$$= 0, \quad k > m.$$

Since a_k^m increases to 1 when m tends to infinity, we have

$$h(\mu^{-1}) - \mu^{-1} \leqq 0.$$

Conversely, suppose (a) and (b) are satisfied. Then (2) follows by induction; assuming (2) for $m \geqq 1$,

$$\sum_{k=0}^{m+1} (m+2-k)q_k\mu^{-k} - (m+1)\mu^{-1}$$

$$= \sum_{k=0}^{m} (m+1-k)q_k\mu^{-k} + \sum_{k=0}^{m+1} q_k\mu^{-k} - m\mu^{-1} - \mu^{-1}$$

$$\leqq \sum_{k=0}^{m+1} q_k\mu^{-k} - \mu^{-1} \leqq h(\mu^{-1}) - \mu^{-1} \leqq 0.$$

THEOREM 1. <u>M-excessive measures for a CGW process exist</u>

(i) <u>if critical</u> ($h'(1) = 1$) <u>when and only when</u> $q_0 = q_2 = 1/2$ <u>and</u> $\mu = 1$,

(ii) <u>if supercritical</u> ($h'(1) > 1$) <u>when and only when</u> $q_0 \leqq 1/2$ <u>and</u> $1 \leqq \mu \leqq 1/2q_0$,

(iii) <u>if subcritical</u> ($h'(1) < 1$) <u>when and only when</u> $1/2 \leqq q_0 \leqq r/2$ <u>and</u> $1/r \leqq \mu \leqq 1/2q_0$,

<u>where</u> $0 \leqq q \leqq r$ <u>are two roots of</u> $h(u) - u = 0$.

PROOF. (i) When the process is critical, $q = r = 1$. Therefore we have $\mu = 1$ by (b) of lemma 2. Suppose $2q_0 < 1$, then $\Sigma_{n=2}^{\infty} q_n > 1/2$, and

$$\Sigma_{n=2}^{\infty} nq_n \geqq 2\Sigma_{n=2}^{\infty} q_n > 2\cdot(1/2) = 1,$$

which contradicts to $h'(1) = \Sigma_{n=2}^{\infty} nq_n = 1$. Thus we have $2q_0 = 1$ by (a) of lemma 2. Because $2q_0 = 1$, we have

$$\Sigma_{n=2}^{\infty} nq_n \geqq 2\Sigma_{n=2}^{\infty} q_n = 1.$$

However, since the equality must be held, we have $q_3 = q_4 = \cdots = 0$. Hence, $q_0 = q_2 = 1/2$.

(ii) When $0 \leqq u \leqq 1$, we have

$$h(u) - u \leqq q_0 + (1-q_0)u^2 - u. \tag{3}$$

If the process is supercritical, $0 \leqq q < r = 1$. By lemma 2 and the remark, we have $2q_0 \leqq 1$. Then by (3)

$$h(2q_0) - 2q_0 \leqq -q_0(1-2q_0)^2 \leqq 0.$$

This implies $q \leqq 2q_0$ and hence $2q_0 \leqq \mu^{-1} \leqq 1$.

(iii) When subcritical, $q = 1 < r$. Let us prove $1 \leqq 2q_0$. Suppose $2q_0 < 1$, then by (3) $h(2q_0) - 2q_0 < 0$. This meens the existence of a root q such that $q < 2q_0 < 1$, contradicting to subcriticality. If $1 \leqq 2q_0 \leqq r$, then $2q_0 \leqq \mu^{-1} \leqq r$ provides $\hat{\mu}$, completing the proof.

From the theorem we find an interesting fact. For example, let us suppose $q_0 + q_3 = 1$ ($q_3 \neq 0$). Then there is no m-excessive measure if $1/2 < q_0 < (1+\sqrt{5})/4$. When $q_0 + q_2 = 1$ ($q_0 \neq 0$), there is at least one m-excessive measure. When $q_0 = 0$ (this is the case if the CGW process is the dual of collision processes) the CGW process is supercritical and $\mu \geqq 1$ provides m-excessive measure(cf.[9]).

It is quite natural to have the following question: If $q_0 \neq 0$ and if m-excessive measure $\hat{\mu}$ exists, what is the $\hat{\mu}$-dual markov process? Let's prove that even in this case, the dual process has a property similar to Tanaka's collision property. Let H_t be the dual transition probability with respect to $\hat{\mu}$ defined by

$$H_t(n,m) = \mu^m T_t(m,n)\mu^{-n}, \quad n,m \geqq 1$$

$$H_t(0,0) = 1, \quad H_t(0,m) = 0, \ m \geqq 1,$$

where $T_t(n,m)$ is the transition probability of the CGW process. Defining a convolution $f*g$ of $f = (f_1,f_2,\cdots)$ and $g = (g_1,g_2,\cdots)$ by

$$(f*g)(n) = f_n + f_{n-1}g_1 + \cdots + f_1 g_{n-1} + g_n, \quad n \geqq 1,$$

we have

PROPOSITION 4. *For* $n = 1,2,3,\cdots$

(4) $H_t(f*g)(n) = (H_t f * H_t g)(n) + H_t f(n) H_t^0 g(0) + H_t^0 f(0) H_t g(n)$,

where $H_t^0(0,m) = \mu^m T_t(m,0)$.

REMARK. When $q_0 = 0$, $T_t(m,0) = 0$. Therefore (4) reduces to Tanaka's collision property

PROOF OF PROPOSITION. Let $n \geqq 1$. By the definition of H_t

$$H_t(f*g)(n) = \Sigma_{m=0}^{\infty}(f\hat{\mu}*g\hat{\mu})(m)T_t(m,n)\mu^{-n},$$

where a term for $m=0$ is formally added because $T_t(0,n) = 0$, then by the branching property of T_t,

$$= \Sigma_{\substack{n_1+n_2=n \\ n_1,n_2\geqq 0}} (f\hat{\mu}T_t)(n_1)(g\hat{\mu}T_t)(n_2)\mu^{-n} .$$

Because

$$f\hat{\mu}T_t(n) = H_t f(n)\mu^n, \quad n \geqq 1,$$

$$= H_t^0 f(0), \quad n = 0,$$

we get (4).

REMARK. Since the generator B of the $\hat{\mu}$-dual Markov process of the CGW process is given by

$$Bf(m) = c\{\Sigma_{n=1}^{m+1} n\mu^{n-m}q_{m+1-n}f_n - mf_m\}, \quad m \geqq 1,$$

we have

PROPOSITION 5. <u>The necessary and sufficient condition for $\hat{\lambda}$ to be</u> m-<u>excessive measure for the</u> $\hat{\mu}$-<u>dual of the CGW process is</u>

$$\mu q \leqq \lambda \leqq \mu r.$$

PROOF. Because

$$\Sigma_{m=1}^{\infty} \lambda^m Bf(m) = c\ \Sigma_{n=1}^{\infty} nf_n\lambda^n\{\Sigma_{k=0}^{\infty}(\lambda/\mu)^{k-1}q_k - 1\},$$

$\hat{\lambda}$ is m-excessive measure if and only if

$$\Sigma_{k=0}^{\infty}q_k(\lambda/\mu)^k - \lambda/\mu \leqq 0,$$

thus we have

$$q \leqq \lambda/\mu \leqq r.$$

5. General cases

Let us extend the result in the previoûs section to wider cass of branching Markov processes. Let S be a state space of one particle as usual. Given a right continuous Markov process on S with a transition probability P_t, bounded non-negative measurable function c, $q_n \geq 0$ with $\Sigma q_n(x) = 1$, and probability kernels $\pi_n(x, d\underline{y})$ defined on $S \times S^n$, $n = 0, 1, 2, \cdots$, we can construct a branching Markov process $(X_t, P_{\underline{a}})$ on $\underline{S}^\partial$ (cf. [2,8]).

Let $\underline{P}_t$ and $\underline{P}_t^0$ be direct products of P_t and the transition probability P_t^0 of $\exp(-c_t)$-subprocess, respectively. Then the transition probability $\underline{T}_t$ of the branching Markov process satisfies for $\underline{x} = (x_1, x_2, \cdots, x_n)$ and for a symmetric bounded measurable function $F = (F_n)$ on $\underline{S}^\partial$,

$$(5) \quad \underline{T}_t F(\underline{x}) = \underline{P}_t^0 F(\underline{x}) + \int_0^t dr \int_{S^n} \underline{P}_r^0(\underline{x}, d\underline{y}) \Phi(\underline{y}, \underline{T}_{t-r} F),$$

where

$$\Phi(\underline{y}, F) = \sum_{k=1}^{n} c(y_k) \sum_{m=1}^{\infty} q_m(y_k) \int_{S^m} \pi_m(y_k, d\underline{z}) F(y_1, \cdots, y_{k-1}, \underline{z}, y_{k+1}, \cdots, y_n).$$

This is verified through construction of the process(cf. [2,8]). Therefore we have

$$(\underline{T}_t F(\underline{x}) - F(\underline{x}))/t = (\underline{P}_t^0 F(\underline{x}) - F(\underline{x}))/t + 1/t \cdot \int_0^t dr \int_{S^n} \underline{P}_r^0(\underline{x}, d\underline{y}) \Phi(\underline{y}, \underline{T}_{t-r} F).$$

The second term is bounded by the uniform norm of F, because $\underline{P}_r^0(\underline{x}, d\underline{y}) \Sigma_{k=1}^n c(y_k)$ is a probability measure on S^n, and the rest part of the integrand on S^n is bounded by

$$\| F \| \sum_{m=0}^{\infty} q_m(y_k) \int_{S^m} \pi_m(y_k, d\underline{z}) \leq \| F \| \sum_{m=0}^{\infty} q_m(y_k) = \| F \|,$$

and it converges to $\Phi(\underline{x}, F)$, $(t \downarrow 0)$. Therefore if F belongs to the domain of the weak generator $\underline{G}^0$ of $\underline{P}_t^0$, then so does to the domain of the weak generator $\underline{A}$ of $\underline{T}_t$, and vice versa. Therefore we have $D(\underline{A}) = D(\underline{G}^0)$ and $\underline{A}f = \underline{G}^0 F + \Phi(\cdot, F)$. Thus we have

PROPOSITION 6. Let $\underline{\underline{G}}$ be the weak generator of $\underline{\underline{P}}_t$, then $D(\underline{\underline{A}}) = D(\underline{\underline{G}}^0) \subset D(\underline{\underline{G}})$ and for symmetric $F \in D(\underline{\underline{A}})$ and $\underline{\underline{x}}$ in S^n

$$\underline{\underline{A}}F(\underline{\underline{x}}) = \underline{\underline{G}}F(\underline{\underline{x}}) + \Psi(\underline{\underline{x}},F)$$

where $\Psi(\underline{\underline{x}},F) = \Phi(\underline{\underline{x}},F) - \Sigma_{k=1}^{n} c(x_k)F(\underline{\underline{x}})$.

We assume in the following that there is a P_t-invariant measure dx on S.

Assumption B. (i) There exists a density function $\pi_n(x,\underline{\underline{z}})$ with respect to $\widehat{d\underline{\underline{z}}}$;

$$\pi_n(x,d\underline{\underline{z}}) = \pi_n(x,\underline{\underline{z}})\,\widehat{d\underline{\underline{z}}} :$$

(ii)
$$\underline{q}_0 = \int_S q_0(x)dx < \infty,$$

$$\underline{q}_k = \sup_{\underline{\underline{z}} \in S^k} \int_S dx q_k(x)\pi_k(x,\underline{\underline{z}}) < \infty, \quad k = 1,2,3,\cdots.$$

(iii)
$$c(x) \equiv 1.$$

Put

$$\underline{h}(u) = \sum_{k=0}^{\infty} \underline{q}_k u^k.$$

THEOREM 2. If there are non-negative solutions $0 \leqq \eta_1 \leqq \eta_2$ (at most two) of $\underline{h}(u) - u = 0$, then $\widehat{\mu d\underline{\underline{x}}}$ (μ is a positive constant) is m-excessive measure for the branching Markov process determined by $\{P_t, c=1, q_n, \pi_n\}$, when

$$\eta_1 \leqq 2\underline{q}_0 \leqq \eta_2$$

and

$$1/\eta_2 \leqq \mu \leqq 1/2\underline{q}_0.$$

PROOF. Because μdx is an invariant measure of P_t,

$$\int \widehat{\mu d\underline{\underline{x}}}\ \underline{\underline{G}}F = 0$$

for non-negative $F \in D(\underline{\underline{G}})$. Therefore $\widehat{\mu d\underline{\underline{x}}}$ is excessive for $\underline{\underline{T}}_t$ if and only if

$$\int \widehat{\mu d\underline{\underline{x}}}\ \Psi(\underline{\underline{x}},F) \leqq 0.$$

This is equivalent to

$$\sum_{k=0}^{m} (m+1-k) \sup_{\underline{\underline{z}} \in S^k} \int \mu dx\, q_k(x)\pi_k(x,\underline{\underline{z}})/\mu^k - m \leqq 0, \quad m = 1,2,3,\cdots.$$

Therefore we get the theorem by the same arguments as in the case of CGW processes.

EXAMPLE 1. When $\underline{q}_0 = 0$, $\underline{h}(u) = u$ has two solutions $u=0$ and $\eta > 0$, where η is the solution of

$$\Sigma_{n=2}^{\infty} \underline{q}_n u^{n-1} = 1.$$

If q_n is constant, then $\underline{q}_n = q_n$ and $\eta = 1$. Thus $\mu \geqq 1$ gives an m-excessive measure $\widehat{\mu d\underline{\underline{x}}}$. Therefore the duality described in §2 and §3 is justified in this case.

EXAMPLE 2. When dx is P_t-invariant <u>probability</u> measure, q_n = constant, and $\pi_n(x,\underline{\underline{z}}) \equiv 1$, $n=1,2,\cdots$, then $\underline{h}(u) = h(u) = \Sigma q_n u^n$. Therefore we can state the same conclusion as for CGW processes.

EXAMPLE 3. (due to K.Uchiyama) Take n-dimensional Brownian motion. The Lebesgue measure dx is the invariant measure for the motion. Assume $\{q_n,\pi_n\}$ satisfy that q_0 is bounded by 1 and belongs to L^1, and

$$q_k(x) = a_k(1-q_0(x)), \quad \Sigma_{k=2}^{\infty} a_k = 1,\ a_k \geqq 0;$$

$$\pi_k(x,\underline{\underline{z}}) = p_k(x-z_1)\times\cdots\times p_k(x-z_k), \text{ for } \underline{\underline{z}} = (z_1,\cdots,z_k),$$

where p_k is a probability density function which belongs to L^k. Then a sufficient condition for existence of m-excessive measure $\widehat{\mu dx}$ is

$$2\underline{q}_0 \leqq 1/\mu, \quad a_0 + \sum_{k=2}^{\infty} a_k \| p_k \|^k (1/\mu)^k \leqq 1/\mu.$$

where $\| p_k \|$ is L^k norm. This follows from

$$\sup_{\underline{\underline{z}}} \int q_k(x)\, dx\, \pi_k(x,\underline{\underline{z}}) \leqq a_k \| p_k \|^k.$$

For example, take 1-dimensional Brownian motion and

$$q_0(x) = \exp(-ax^2), \quad q_2(x) = 1 - q_0(x),$$

$$\pi_2(x,\underline{\underline{z}}) = (2\pi b)^{-1/2} \exp\{-1/2b\cdot((x-z_1)^2+(x-z_2)^2)\}.$$

Then $\underline{q}_0 = \sqrt{\pi/a}$, $\bar{q}_2 \leqq 1/2\sqrt{\pi b}$, and a sufficient condition for $\widehat{\mu dx}$ to be excessive is given by

$$2\sqrt{\pi/a} \leqq 1/\mu \leqq \sqrt{\pi b} + \sqrt{\pi b - 2\pi\sqrt{b/a}}$$

i,e, $ab \geqq 4$.

APPENDIX. When $q_2 = 1$, the resolvent of the CGW process is

$$G(n,m) = 1/m, \quad 1 \leqq n \leqq m,$$
$$= 0, \quad \text{otherwise.}$$

In the case, $\mu \geqq 1$ gives an m-excessive measure $\hat{\mu}$. Let's find which initial distribution ν gives $\hat{\mu}$ as a potential such that $\nu G = \hat{\mu}$. If $\nu G = \hat{\mu}$, we have

$$\Sigma_{n=1}^{m} \nu_n = m\mu^m,$$

and

$$\nu_n = n\mu^n - (n-1)\mu^{n-1}, \quad n = 1,2,3,\cdots,$$

is the one we need. Let's take this initial distribution . Then the reversed process of the CGW process from an L-time (cf.[7]) is a collision process with the $\hat{\mu}$-dual transition probability.

Since the resolvent of the $\hat{\mu}$-dual process is given by

$$\bar{G}(n,m) = \mu^{m-n}/n, \quad 1 \leqq m \leqq n,$$
$$= 0, \quad \text{otherwise,}$$

and since $0 \leqq \lambda \leqq \mu$ gives m-excessive measure $\hat{\lambda}$ for the $\hat{\mu}$-dual, the initial distribution ν which gives $\hat{\lambda}$ as a potential ($\nu\bar{G}=\hat{\lambda}$) is given by

$$\nu_m = m\lambda^m(1 - \lambda/\mu).$$

As a special case if we take $\mu = 1$ then $\nu_n = 1$ $(n \geqq 1)$ gives $\nu\bar{G}(m) = 1$ $(m \geqq 1)$, but $\hat{1}$ is not a potential for the $\hat{1}$-dual process. The author does not know when m-excessive measure is given as a potential in general.

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