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REMARKS ON THE HYPOTHESES OF DUALITY

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Let E be a locally compact topological space with a countable base and let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a strong Markov process on E, right continuous with left limits. Let $\{U^p(x,.)\}_{p \geq 0}$ be the resolvent operator of X; we shall assume X is transient in the sense that $U^0(x,.)$ is a Radon measure for each x in E. (If this is not the case we will work instead with the α-process formed by exponential killing with parameter α.)

It was shown in [11] that if X satisfies hypothesis (L), then X has a left continuous moderate Markov dual process $\hat{X}$,* i.e., the resolvent $\{\hat{U}^p(x,.)\}$ of $\hat{X}$ is in duality with $\{U^p(x,.)\}$ in the sense of [2], p. 253. Thus although we are not necessarily operating in the classic context of duality--there need not exist a cofine topology--we do have an excessive reference measure ξ and a function $u^\alpha(x,y)$ defined on ExE for each $\alpha \geq 0$, with the properties that

(i) $x \mapsto u^\alpha(x,y)$ is α-excessive for the resolvent $\{U^p\}$, $\forall y \in E$ and $\forall \alpha$;

(ii) $y \mapsto u^\alpha(x,y)$ is α-excessive for the resolvent $\{\hat{U}^p\}$, $\forall x \in E$ and $\forall \alpha$;

(iii) $U^\alpha f(x) = \int u^\alpha(x,y) f(y)\, \xi(dy)$ and $\hat{U}^\alpha f(y) = \int u^\alpha(x,y) f(x)\, \xi(dx)$ for all $\alpha \geq 0$, $f \in b\mathcal{E}^*$ (the bounded, universally measurable functions).

The purpose of this note is to point out that several results which are well-known under classical duality hypotheses remain valid (sometimes with minor changes) under weaker hypotheses, in some cases hypothesis (L) alone. During the preparation of this note we have become aware of two other papers, [1] and [4], which treat (among other problems) some of the topics discussed here. The present work is in much the same spirit

(*) $\hat{X}$ need not be left continuous at ζ, but we will consistently abuse the language by calling such a process left continuous.

as [4], although we make no appeal to the theory of Ray compactification; the results here follow rather easily from those of [1], and our presentation should thus be regarded as a natural extension of [1]. The results of our §1 also overlap, while remaining less interesting than, the results of §III of [1]. We plead guilty to Azéma's charge of having sought to "generalize abusively" the two characterization theorems of Meyer to the non-duality case, offering in defense only the remark that, to a certain extent anyway, it seems to work.

1. Potentials of Additive Functionals

Let A be an additive functional of X which is a.s. finite; here "additive functional" means that A charges neither 0 nor the lifetime ζ. Following Revuz ([8]) we associate with A the measure ν_A, where

$$\nu_A(f) = \lim_{t\downarrow 0} 1/t\, E^{\xi}\left\{\int_0^t f(X_s)\, dA_s\right\}$$

for $f \in b\mathcal{E}_+$ (the set of non-negative, bounded, measurable--in the Borel sets of E--functions). Under the classical duality hypotheses, Revuz shows that if A is natural and σ-integrable (i.e., $E = \bigcup_n E_n$ with $E_n \in \mathcal{E}$, $\nu_A(1_{E_n})<\infty$) and if the α-potential of A is finite ξ-a.e., then

$$(1.1)\quad E^x \int_0^\infty e^{-\alpha t}\, dA_t = \int_E u^\alpha(x,y)\, \nu_A(dy)$$

From (1.1) one deduces easily that for $f \in \mathcal{E}_+$,

$$(1.2)\quad E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dA_t = \int_E u^\alpha(x,y) f(y)\, \nu_A(dy) .$$

Our first observation is that (1.2) remains valid for continuous additive functionals under only hypothesis (L). (It may fail for natural additive functionals in the absence of a cofine topology).

Lemma 1.1 Let X satisfy hypothesis (L). Then (1.2) is valid provided A is continuous.

Proof. The proof is essentially that of Revuz ([8], p. 517). The key observation here is that for $f \in b\mathcal{E}_+$, the function $s \to \hat{U}^\alpha f(\hat{X}_s)$ is left continuous with right limits on $(0,\zeta)$ a.s. $\hat{P}^\xi$ ([11], p. 139), which implies, using the reversal operator ([12], § 5) that a.s. P^ξ, $s \to \hat{U}^\alpha f(X_s)$ is right continuous with left limits on $(0,\zeta)$. Thus the function $[\hat{U}^\alpha f(X_s)]_-$, which is left continuous, differs, for almost all ω, from $\hat{U}^\alpha f(X_s)$ on a countable s-set, which is not charged by the continuous additive functional A. Applying the argument of Revuz ([8], p. 517) we have

$$(1.3)\quad \int \hat{U}^\alpha f(x)\, \nu_A(dx) = \lim_{\beta\uparrow\infty} \beta E^\xi \int_0^\infty e^{-\beta s}[\hat{U}^\alpha f(X_s)]_-\, dA_s$$

$$= \lim_{\beta\uparrow\infty} E^\xi \lim_{n\uparrow\infty}\left\{\sum_{k\geq 0} \beta\, e^{-\beta k/n}[\hat{U}^\alpha f(X_{k/n})]_-(A_{(k+1)/n} - A_{k/n})\right\}$$

The proof is completed as in [8] once it is verified that $P^\xi\{\hat{U}^\alpha f(X_s) \neq [\hat{U}^\alpha f(X_s)]_-\} = 0$ for any $s>0$. By Fubini, this probability is zero for a.e. (Lebesgue) s; let s_0 be such an s, and let $t>0$ be arbitrary. We then have

$$P^\xi\{\hat{U}^\alpha f(X_{s_0+t}) \neq [\hat{U}^\alpha f(X_{s_0+t})]_-\} = P^\xi\{\hat{U}^\alpha f(X_{s_0})\circ\theta_t \neq [\hat{U}^\alpha f(X_{s_0})]_-\circ\theta_t\}$$

$$\leq P^{\xi P_t}\{\hat{U}^\alpha f(X_{s_0}) \neq [\hat{U}^\alpha f(X_{s_0})]_-\} = 0$$

since ξ is excessive, Q.E.D.

At this point we introduce two exceptional sets which play a large role in what follows.

Definition 1.1 $\hat{H} = \{y \in E: \hat{U}(y,E) = 0\}$

One verifies easily that $y \in \hat{H} \to u(x,y) = 0$ for all $x \in E$, and that H is polar for $\hat{X}$, and hence, by the reversal operator, for X.

Definition 1.2 $\hat{B} = \{x \in E: \hat{P}_0(x,.) \neq \delta_x\}$, where $\hat{P}_0(x,.) = \lim_{t\downarrow 0} \hat{P}_t(x,.)$.

The argument of Prop. 4.2 of [11] shows that $\hat{B}$ is semipolar. Clearly $\hat{H}$

(the set of points which co-branch to Δ) is a subset of $\hat{B}$ (the co-branch points).

Lemma 1.2 Let μ_1 and μ_2 be measures such that $U\mu_1 = U\mu_2 < \infty$ a.e. ξ, and such that μ_1 does not charge $\hat{H}$ and μ_2 does not charge semipolars. Then $\mu_1 = \mu_2$.

Proof. The argument of [2] , VI (1.15) adapted to our case shows, first, that $U\nu = 0 \rightarrow \nu = 0$ if ν does not charge $\hat{H}$, and, second, that $\mu_1\hat{P}_0 = \mu_2\hat{P}_0$, i.e., that $\mu_1|_{\hat{B}^c} = \mu_2|_{\hat{B}^c}$. Thus $U(\mu_1 1_{\hat{B}}) = U(\mu_2 1_{\hat{B}}) = 0$ (since μ_2 does not charge semipolars) and by the first property, $\mu_1 1_{\hat{B}} = 0$. Thus $\mu_1 = \mu_2$.

We come now to the main results of this section.

Theorem 1.1 Let X satisfy hypothesis (L), and let μ be a σ-finite measure. $U\mu$ is a regular potential if and only if

(i) $U\mu$ is everywhere finite, and

(ii) A semipolar $\rightarrow \mu(A \cap \hat{H}^c) = 0$.

Proof. If $U\mu$ is a regular potential then $U\mu$ is finite and there exists a continuous additive functional A of X such that $U\mu(x) = E^x(A_\infty)$; by Lemma 1.1 the latter expression equals $U\nu_A$. Since ν_A does not charge semipolars, it follows by Lemma 1.2 that, off $\hat{H}$, $\mu = \nu_A$ and hence that μ does not charge the intersection of any semipolar with $\hat{H}^c$.

Conversely, suppose that $U\mu < \infty$ and that the trace of μ on $\hat{H}^c$ does not charge semipolars. The result will then follow easily from the following analogue of Doob's classical theorem, which we state in a general form for use also in Theorem 1.2.

Proposition 1.1 Let $\hat{X}$ be a left continuous moderate Markov process in duality with a right continuous strong Markov process. Let $\{f_n\}$ be a series of $\hat{X}$-excessive functions decreasing to the supermedian function f, and let $\bar{f}$ be the excessive regularization of f.

Then $\{\bar{f} < f\}$ is semipolar. Further, if $f = 0$ except on a set of potential zero, then $f = 0$ except on a polar set.

Proof. Following [11], p. 147, we define

$$\tilde{f}(x) = \limsup_{\substack{y \to x \\ y \neq x}} f(y)$$

where the limit is taken in the essentially fine topology of X (the strong Markov dual). By [11], $\{f(x) \neq \tilde{f}(x)\}$ is of potential zero, and $\{f(x) > \tilde{f}(x)\}$ is $\hat{X}$-polar, and thus by reversal X-polar. (It is here that the hypothesis $f_n \downarrow f$ intervenes; the result quoted from [11] concerns an excessive, not simply supermedian, f, and since we have not shown any continuity properties of the function $s \to f(\hat{X}_s)$, the hypothesis $f_n \downarrow f$ is needed to guarantee $\hat{P}_T f \leq f$ for a stopping time T.)

Thus if $f = 0$ except on a set of potential zero, we have $0 = \bar{f} \leq f \leq \tilde{f}$ off a polar set. But $t \to \tilde{f}(X_t)$ is right continuous P^{ξ}-a.s. on $(0,\zeta)$, and by the hypothesis on f and Fubini's theorem, $t \to \tilde{f}(X_t)$ is P^{ξ}-a.s. equal to zero on a t-set of full Lebesgue measure; hence $\{\tilde{f} > 0\}$ is polar, which implies $\{f > 0\}$ is polar, proving the second claim of the proposition.

To prove the first assertion, note first that if $\{T_n\}$ is a sequence of stopping times with values in the dyadic rationals and such that $T_n \uparrow\uparrow T$, we have that $\lim_n \hat{E}^x\left[\bar{f}(\hat{X}_{T_n})\right]$ exists for all x such that $\hat{E}^x\{\bar{f}(\hat{X}_0)\} < \infty$ (this is because $\bar{f}(\hat{X}_t)$ is a supermartingale). A modification of the argument of Proposition 6(b) of [7] then applies, since $\hat{X}$ is predictable, to give that

$s \to \bar{f}(\hat{X}_s)$ has left limits. Hence

(1.4) $\hat{P}^{\xi}\{[t: \bar{f}(\hat{X}_t) \neq [\bar{f}(\hat{X}_t)]_-] \text{ is countable}\} = 1.$

But by Lemma C.1 of [11], and the fact that $\{\bar{f} < f\}$ is of potential zero, it follows that

(1.5) $\hat{P}^{\xi}\{[t: \bar{f}(\hat{X}_t) \neq \tilde{f}(\hat{X}_t)] \text{ is of Lebesgue measure zero}\} = 1.$

Since $t \to \tilde{f}(\hat{X}_t)$ and $t \to [\bar{f}(\hat{X}_t)]_-$ are both left continuous it follows that

(1.6) $\hat{P}^{\xi}\{\exists t: [\bar{f}(\hat{X}_t)]_- \neq \tilde{f}(\hat{X}_t)\} = 0$;

by (1.4) and (1.5) we then have

(1.7) $\hat{P}^{\xi}\{[t: \bar{f}(\hat{X}_t) \neq \tilde{f}(\hat{X}_t)] \text{ is countable}\} = 1.$

But since $\bar{f} \leq f \leq \tilde{f}$ off a polar this implies that $\{\bar{f} < f\}$ is semipolar, proving the proposition.

Returning now to the proof of Theorem 1.1, let $\{T_n\}$ be a sequence of stopping times increasing to T a.s. P^x. Since $z \to u(z,y)$ is excessive, it follows that for each fixed y and z, $P_{T_n} u(z,y)$ decreases with n. If $f \in b\mathcal{E}^*_+$ is of compact support, then $P_T Uf(x) - P_{T_n} Uf(x) = E^x \int_{T_n}^{T} f(X_t)\, dt \to 0$ when $n \uparrow \infty$. Thus $P_{T_n} u(x,.) \downarrow P_T u(x,.)$ a.e. Let $g^x(.) = \lim_n P_{T_n} u(x,.)$; then $g^x(.) =$ $= P_T u(x,.)$ a.e. and thus $\bar{g}^x(.) = P_T u(x,.)$ a.e. and hence identically since both are coexcessive. Since $\{\bar{g}^x < g^x\}$ is semipolar by Prop. 1.1, μ does not charge $\hat{H}^c \cap \{\bar{g}^x < g^x\}$ by hypothesis; but $u(.,y) = 0$ for $y \in \hat{H}$, so that $P_{T_n} U\mu(x)$ $= \int P_{T_n} u(x,y)\mu(dy) \to \int P_T u(x,y)\mu(dy) = P_T U\mu(x)$ for each x, i.e., $U\mu$ is a regular potential.

<u>Corollary 1.1</u> There is a 1-1 correspondence between the continuous additive functionals A of X with finite potential and the measures μ such that $U\mu < \infty$ and the trace of μ on $\hat{H}^c$ does not charge semipolars. In fact, any such μ is of the form ν_A for a continuous additive functional A.

The companion theorem to Theorem 1.1 concerns "natural" additive functionals, except that for processes not necessarily standard the notion of natural additive functional should be replaced by that of additive functionals not charging any totally inaccessible stopping time.

Theorem 1.2 Let X be quasi-left continuous and satisfy hypothesis (1), and let μ be a σ-finite measure. $U\mu$ is a natural potential if and only if

(i) $U\mu < \infty$; and

(ii) if A is polar, then $\mu(A \cap \hat{B}^c) = 0$.

Proof. If $U\mu$ is a natural potential then $U\mu$ is finite by definition. If the trace of μ on $\hat{B}^c$ charges a polar set A, then it must charge a compact polar subset K of B^c; let ν be the restriction of μ to K. Then $U\nu \leq U\mu$ and $U\nu$ is a natural potential.

Let $x \notin K$ and let $\{G_n\}$ be a sequence of open sets containing K such that $\lim_n T_{G_n} \geq \zeta$ P^x-a.s. (It is only for this choice of G_n that we require quasi-left continuity.)

Lemma 1.3 $P_{G_n} U\nu(x) = U\nu(x)$.

Admitting for the moment the truth of Lemma 1.3, the first half of the proof of Theorem 1.2 is easily finished. By the definition of natural potential, $P_{G_n} U\nu(x) \to 0$ as $n \uparrow \infty$. Thus $U\nu = 0$ off the polar set K, so $U\nu$ is identically zero; by Lemma 1.2 ν vanishes off H, and in particular $\mu(A \cap \hat{B}^c) = 0$.

Proof of Lemma 1.3. The proof in the case of two strong Markov processes in duality is immediate from Hunt's "switching formula" ([2], VI, 1.16). Although this formula is not valid in the absence of a cofine topology, a weaker form of it will suffice for our purposes.

Let $\hat{T}_{G_n}$ be the first hitting time of G_n by the process $\hat{X}$; it is verified without difficulty that T_{G_n} and $\hat{T}_{G_n}$ are dual terminal times in the sense of [12]. It then follows from (4.12) of [12] that for $f,g \in b\mathcal{E}_+$

$$(1.8) \qquad \langle g, E^{\cdot}\int_0^{T_{G_n}} f(X_t)\, dt\rangle_{\xi} = \langle f, \hat{E}^{\cdot}\int_0^{\hat{T}_{G_n}} g(\hat{X}_t)\, dt\rangle_{\xi}$$

But the fundamental duality formula for X and $\hat{X}$, in conjunction with (1.8), gives

$$(1.9) \qquad \langle g, P_{G_n} Uf\rangle_{\xi} = \langle f, \hat{E}^{\cdot}\int_{\hat{T}_{G_n}}^{\infty} g(\hat{X}_t)\, dt\rangle_{\xi}$$

This being true for all $f \in b\mathcal{E}_+$, we have then

$$(1.10) \qquad \iint P_{G_n}(x,dy)u(y,w)g(x)\,\xi(dx) = \hat{E}^{w}\int_{\hat{T}_{G_n}}^{\infty} g(\hat{X}_t)dt \qquad \xi\text{-a.e.}$$

and therefore identically since both sides are coexcessive. But for $w \in G_n$, $\hat{P}^w(\hat{T}_{G_n} = 0) = 1$ if $w \in \hat{B}^c$; since ν lives on $\hat{B}^c$ we therefore have $P_{G_n}U\nu(x) = U\nu(x)$, Q.E.D.

To prove the other half of Theorem 1.2, suppose that μ is a measure the trace of which on $\hat{B}^c$ does not charge polars, and such that $U\mu < \infty$. If $\{T_n\}$ is an increasing sequence of stopping times with $\lim_n T_n \geq \zeta$, the argument used in the proof of Theorem 1.1 shows that $P_{T_n}u(x,.)$ decreases a.e. to zero. Let $f^x(.) = \lim_n P_{T_n}u(x,.)$ for x fixed. Now consider the measure $\mu\hat{P}_0$; it is easily seen (from p. 131 of [11], for example) that $\hat{P}_0(x,\hat{B}) = 0$ for all $x \in E$. Also, $\mu\hat{P}_0$ and μ agree on $\hat{B}^c$. Since $\mu\hat{P}_0$ does not charge $\hat{B}$, and $f^x(.) = 0$ off a polar set (Prop. 1.1), it follows that

$$(1.11) \qquad \int f^x\, d(\mu\hat{P}_0) = \int_{\hat{B}^c} f^x\, d\mu = 0.$$

But by the argument of Lemma 1.2, $\int g\, d\mu = \int g\, d(\mu\hat{P}_0)$ for all coexcessive g; since f is the decreasing limit of coexcessive functions it follows that $\int f \, d\mu = \int f \, d(\mu \hat{P}_0) = 0$. Hence $\lim_n P_{T_n} U\mu = 0$ and $U\mu$ is a natural potential.

Remark 1. Note that the quasi-left continuity of X was not used in the proof of the second half of the theorem.

Remark 2. In the absence of a cofine topology it is of course possible that two non-equivalent natural additive functionals could give rise to the same measure ν_A (though we suspect that this is not possible if $\nu_A(\hat{B}) = 0$). Thus we cannot hope to set up a 1-1 correspondence as in Corollary 1.

To complete this section we will consider the case of a measure μ for which $U\mu$ is not necessarily finite. We will say that a point $x \in E$ is coregular for a nearly Borel set A if $\hat{P}^x(\hat{T}_A = 0) = 1$. (This is of course the usual definition, except that in our case we cannot assert that the probability in question takes only the values zero and one.) Let ${}^rA$ denote the set of points coregular for A.

Lemma 1.4 $A - {}^rA$ is semipolar.

Proof. Using the construction outlined in [2], p. 55-57, there exists a decreasing sequence of open sets $\{G_n\}$ containing A such that $\hat{D}_{G_n} \uparrow \hat{D}_A$ $\hat{P}^{\xi}$-a.s. on $\{\hat{D}_A < \infty\}$, where $\hat{D}_B = \inf \{t \geq 0: \hat{X}_t \in B\}$. Let $f_n(x) = \hat{E}^x(e^{-\hat{D}_{G_n}}; \hat{D}_{G_n} < \hat{\zeta})$. Then $f_n(x) \downarrow f(x)$, and by the above,

(1.12) $f(x) = \hat{E}^x(e^{-\hat{D}_A}; \hat{D}_A < \hat{\zeta})$ ξ-a.e.

Let $g(x) = \hat{E}^x(e^{-\hat{D}_A}; \hat{D}_A < \hat{\zeta})$. Then the excessive regularization $\bar{g}(x)$ of $g(x)$ is clearly equal to $\hat{E}^x(e^{-\hat{T}_A}; \hat{T}_A < \hat{\zeta})$ and equals $g(x)$ ξ-a.e.; denoting by $\bar{f}$ the excessive regularization of f, it follows that $\bar{f} = \bar{g}$ ξ-a.e. and hence everywhere since both are 1-coexcessive. But $f_n(x) = 1$ on A for each n (except possibly on the semipolar set $\hat{B}$) so that $f(x) = 1$ on $A \cap \hat{B}^c$. But $\{\bar{f} < f\}$ is semipolar by Prop. 1.1, so that $\{\bar{f} < 1\}$ is semipolar, Q.E.D.

Corollary 1.2 If μ does not charge semipolar sets and B is a nearly Borel set carrying μ, then $P_B U\mu = U\mu$.

Proof. This follows from Lemma 1.4 and the proof of Lemma 1.3. (A different proof of this result is given in [4].)

Corollary 1.3 If μ has support in a compact set K and does not charge semipolars, then $\sup \{U^{\alpha}\mu(x): x \in E\} = \sup \{U^{\alpha}\mu(x): x \in K\}$.

Proof. By Corollary 1.2 we have $P_K^{\alpha} U^{\alpha}\mu = U^{\alpha}\mu$ and the result follows since for each x the measure $P_K^{\alpha}(x,.)$ lives on K.

We then have, as in [4] or [10],

Proposition 1.2 Let λ be a σ-finite measure which does not charge semipolars. There exists an equivalent finite measure μ with a bounded 1-potential.

Constructing the additive functional which corresponds to μ in Prop. 1.2, we can extend Theorem 1.1 to say that a σ-finite measure which does not charge semipolars corresponds to a continuous additive functional with a bounded 1-potential.

2. Two Applications

Our first application is a simple consequence of Lemma 1.1. Let X be a process such that each point x is regular for itself and such that the local time $L_t^x(\omega)$ at x has the property that $(t,x) \longrightarrow L_t^x(\omega)$ is a.s. continuous (some conditions guaranteeing this will be found in [6]). Then we have the following generalization of a result well-known under duality hypotheses ([2], p. 294):

Proposition 2.1 Let X satisfy hypothesis (L). Let $h(x) = u^1(x,x)$, and suppose that L_t^x is normalized for each x to make $U_L^1 x\, 1_E(x) = 1$. If A is a continuous additive functional of X, there exists a measure ν such that

$$A_t = \int L_t^x \, \nu(dx) \qquad \text{for all t a.s.}$$

Proof. Suppose first that A has finite 1-potential. Let $\nu = h\,\nu_A$, where ν_A is the Revuz measure associated with A. Let $B_t = \int L_t^x\,\nu(dx)$; clearly B is a continuous additive functional. We have

$$E^x \int_0^\infty e^{-t}\,dB_t = E^x \int_0^\infty e^{-t}\,d\left[\int L_t^y\,\nu(dy)\right] = E^x\left\{\int_E h(y)\,\nu_A\,(dy) \int_0^\infty e^{-t}\,dL_t^y\right\}$$

$$= \int_E h(y)\left[u^1(x,y)/u^1(y,y)\right]\nu_A(dy) = \int_E u^1(x,y)\,\nu_A(dy)\ .$$

But by Lemma 1.1, the 1-potential of A_t is also equal to $\int_E u^1(x,y)\,\nu_A(dy)$; thus A and B are indistinguishable. The passage to general A is completed by the observation ([5]) that A can be represented as a sum of continuous additive functionals with finite 1-potential.

Corollary 2.1 For any Borel set D, $\int_D u^1(x,x)\,L_t^x\,\xi\,(dx)$

$$= \int_0^t 1_D(X_s)ds \quad \text{for all } t \text{ a.s.}$$

Our second application concerns the bounded maximum principle (see [3], and [9] for some later comments). Let μ be a σ-finite measure. Following [3] we say that a process satisfies the bounded maximum principle if:

(2.1) Whenever μ has compact support K,

$$U^\alpha\mu \text{ bounded} \longrightarrow \sup_{x\in E} U^\alpha\mu(x) = \sup_{x\in K} U^\alpha\mu(x).$$

In [3] it is proved under strong hypotheses (including duality) that (2.1) is equivalent to

(2.2) All semipolar sets are polar.

These hypotheses were weakened slightly in [9] but even there it was necessary to assume, in addition to duality, that the function $U^\alpha(.,E)$ was lower semicontinuous for some $\alpha > 0$. We shall prove a version of this theorem here requiring only that X be special standard and satisfy hypothesis (L) as well as hypothesis (B), given below:

(B) Let A and C be Borel sets with A a neighborhood of C. Then

$$T_C \circ \theta_{T_A} = 0 \text{ a.s. on } \{T_A = T_C < \infty\}.$$

(It is known from [1] that hypothesis (B) is equivalent to the quasi-left continuity of the right-continuous version of $\hat{X}$).

As remarked in [9], the critical step in establishing the equivalence of (2.1) and (2.2) is (in our context) this result:

<u>Proposition 2.2</u> Let X be special standard and satisfy hypothesis (L). If $U^\alpha\mu$ is bounded, the trace of μ on $\hat{B}^c$ does not charge polars.

<u>Proof.</u> Tracing through the proof of this result in [9], we see that the essential point is the proof that $E^x(Z\, 1_{(\zeta = \zeta_A)}) = 0$, where $Z(\omega) = \lim_{t\uparrow\zeta} U\mu(X_t(\omega))$ and ζ_A is the accessible part of ζ. Letting $D_n = \{x: \alpha U^\alpha(x,E) \leq 1/n\}$, then $\lim_n P_{D_n} U\mu(x) = E^x(Z\, 1_{(\zeta = \zeta_A)})$ and $T_{D_n} \uparrow \zeta$ a.s.; letting $\hat{T}_{D_n}$ be the terminal time dual to T_{D_n} ([12], § 4) we easily verify that $\hat{T}_{D_n}$ is an increasing sequence. Calling $\hat{T}$ the limit, we have $\hat{P}^\xi\{\hat{T} < \hat{\zeta}\} = P^\xi\{\lim_n T_{D_n} < \zeta\} = 0$ so that $\hat{T} \geq \hat{\zeta}$ a.s. (the lower semicontinuity was invoked in [9] only to establish this point). We then have, for h integrable,

$$\infty > \langle h, P_{D_n} U\mu \rangle_\xi = \hat{E}^{\hat{\mu}} \int_{\hat{T}_{D_n}}^{\hat{\zeta}} h(\hat{X}_t)\, dt$$

where the equality results from the argument used in Lemma 1.3. Hence $P_{D_n} U\mu \longrightarrow 0$ a.e. ξ, therefore everywhere since $U\mu$ is bounded, and we have that $E^x(Z\, 1_{(\zeta = \zeta_A)}) = 0$. As in [9] this implies that $U\mu$ is a natural potential; Theorem 1.2 then says that μ does not charge $A \cap \hat{B}^c$ if A is polar.

Here finally is the theorem alluded to above; not surprisingly, the statement is modified somewhat to account for the presence of cobranch points. The appropriate analogues of (2.1) and (2.2) for our purposes are

(2.3) For all μ with compact support $K \subset \hat{B}^c$,

$$U^\alpha\mu \text{ bounded} \mapsto \sup_{x \in E} U^\alpha\mu(x) = \sup_{x \in K} U^\alpha\mu(x)$$

(2.4) All semipolar sets contained in $\hat{B}^c$ are polar.

<u>Theorem 2.1</u> Let X be special standard and satisfy hypotheses (L) and (B). Then $\forall \alpha > 0$, (2.3) and (2.4) are equivalent.

<u>Proof.</u> First assume (2.4) is true. Let μ be a measure with compact support $K \subset \hat{B}^c$ and such that $U^\alpha\mu$ is bounded. According to Prop. 2.2, μ does not charge the intersection of any polar set with $\hat{B}^c$. Thus by (2.4) μ does not charge any semipolar set; by Corollary 1.3, $\sup_{x \in E} U^\alpha\mu(x) = \sup_{x \in K} U^\alpha\mu(x)$.

Now assume that (2.3) holds. An examination of the proof of the corresponding result in [3] (Theorem 5.3) reveals that the duality hypotheses intervene only in the establishment of hypothesis (B), which we have assumed, and in Lemma 5.6, which for our purposes can be weakened as follows:

<u>Lemma 2.1</u> Assume (2.3) If $K \subset \hat{B}^c$ is compact and thin, then

$$\sup_{x \in E} E^x(e^{-\alpha T_K}) = \sup_{x \in K} E^x(e^{-\alpha T_K})$$

<u>Proof.</u> Let $\{h_n\}$ be a sequence of bounded nonnegative functions such that $U^\alpha h_n \uparrow 1$; and let $\mu_n = h_n \xi$. Then (letting $\phi^\alpha_K = E^\cdot(e^{-\alpha T_K})$)

$$\phi^\alpha_K = P^\alpha_K 1 \geqslant P^\alpha_K U^\alpha \mu_n$$

Now by formula (1.9), we have

(2.5) $$\langle g, P^\alpha_K U^\alpha h_n \rangle_\xi \qquad \langle h_n, \hat{E}^\cdot \int_{\hat{T}_K}^\infty e^{-\alpha t} g(\hat{X}_t)dt \rangle_\xi$$

If we can show that $\{(t,\omega): \hat{X}_t(\omega) \in K\}$ is closed from the right a.s., it will follow, since $\hat{X}$ is predictable, that $\hat{T}_K$ is a predictable stopping time, and we can take advantage of the moderate Markov property of $\hat{X}$. But since X is quasi-left continuous, it follows ([11], Prop. 5.5) that

$$P^{\xi}\left\{\exists\, t<\zeta : X_{t-}\in K \text{ and } X_{t-}\neq X_t\right\} = 0.$$

By reversal we then have

$$\hat{P}^{\xi}\left\{\exists\, t<\zeta : \hat{X}_{t+}\in K \text{ and } \hat{X}_t\neq \hat{X}_{t+}\right\} = 0,$$

so that $\hat{T}_K$ is predictable. Thus (2.5) becomes

$$(2.6)\qquad \langle g,\, P_K^{\alpha}U^{\alpha}h_n\rangle_{\xi} = \langle \hat{P}_K^{\alpha}U^{\alpha}g,\, h_n\rangle_{\xi}$$

and we thus have

$$(2.7)\qquad P_K^{\alpha}U^{\alpha}\mu_n = U\,(\mu_n\hat{P}_K^{\alpha})$$

The proof is then completed as in [3]. Let $M = \sup_{x\in K}\Phi_K^{\alpha}(x)$. $\mu_n\hat{P}_K^{\alpha}$ is carried by K and $U^{\alpha}(\mu_n\hat{P}_K^{\alpha}) \leq \Phi_K^{\alpha}$. Since $\Phi_K^{\alpha}\leq M$ on K, (2.3) implies that $P_K^{\alpha}U^{\alpha}\mu_n = U^{\alpha}(\mu_n\hat{P}_K^{\alpha}) \leq M$. But $P_K^{\alpha}U^{\alpha}\mu_n \uparrow \Phi_K^{\alpha}$ and so $\sup_{x\in E}\Phi_K^{\alpha}\leq M$, proving the lemma.

The proof of Theorem 2.1 is then completed exactly as in [3], except for the restriction to $\hat{B}^c$.

In closing, we remark that there are undoubtedly many other results in probabilistic potential theory, known under strong duality hypotheses, which remain "moralement" valid in the kind of greater generality contemplated here (for example, Proposition 7.3 of [3] can be proved using hypotheses (L) and (B) and the lower semicontinuity of excessive functions). There is, however, a class of "deeper" results where the cofine topology seems to be indispensable (VI, 2.11 of [2], the "Riesz decomposition theorem", is of this type) -- these are the true "duality theorems" of probabilistic potential theory.

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