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STOPPING SEQUENCES

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I. Introduction:

The following result of Skorochod is well-known:
Let μ and ν be probability distributions on R^1 such that ν has finite variance.
If $\int k d\mu \leq \int k d\nu$ for all convex functions k, then there exists a stopping-time τ in the process of Brownian motion (B_t) starting with initial measure μ, such that

1) The distribution of B_τ is ν
2) $\mathcal{E}\tau = \text{var}\,\nu - \text{var}\,\mu$.

This result follows from the usual "lemma of Skorochod" by means of the theorem of Hardy-Littlewood-Polya-..., which establish a decomposability-property for the ordered cone of all finite measures with finite variance ($\mu \dot{<} \nu$ iff $\int k d\mu \leq \int k d\nu$ for all convex k)

An algebraic version of this assertion is:
If $\mu_1+\mu_2 < \nu$, then there exist measures ν_1, ν_2 such that $\mu_1 < \nu_1$, $\mu_2 < \nu_2$ and $\nu_1+\nu_2 = \nu$.

In this paper we are not interested in Brownian motion, but rather in discrete-time Markoff-processes. P is a fixed positive contraction of the L^1 on a σ-finite measure-space $(E,\mathcal{B},\rho)$. We study stopping-times for processes with transition probabilities P and initial measure μ, such that $\mathcal{L}(X_\tau)$ is a given measure ν.

There is a theorem due to H. Rost [5] asserting:
Let μ and ν be finite measures on $(E,\mathcal{L})$ absolutely continuous with respect to ρ. There exists a ("randomized") stopping-time τ with $\mathcal{L}(X_\tau) = \nu$ while $\mathcal{L}(X_o) = \mu$ iff

$$\int f d\mu \geq \int f d\nu \quad \text{for all P-excessive functions } f.$$

Clearly this theorem is very satisfactory for transient P. Skorochod's lemma refers to Brownian motion, which is recurrent. The interesting question in the recurrent case is, whether there exist "short" stopping-times τ.

Skorochod asks for τ's with finite expectation.

H. Rost [6] has discribed shortness in the general case by conditions of uniform integrability of certain submartingales $(f(X_{t\wedge\tau}))_t$, where the f are P-defective functions.

Here we take into consideration the measure

$$\eta = \mathcal{L}(X_o;\tau>o) + \mathcal{L}(X_1;\tau>1) + \dots$$

called the total effect of τ.

(Notice $\|\eta\| = \Pr(\tau>o) + \Pr(\tau>1) + \dots = \mathcal{E}\tau$)

II. Stopping-Sequences.

Definition 1: Let $X_o, X_1, \ldots$ be a Markoff-process with initial distribution $\mathcal{L}(X_o) = \mu$ and let τ be a stopping-time. We call

$$\mathfrak{m} = (\mu; \mu_o, \mu_1, \ldots)$$

the stopping-sequence associated with τ, if $\mu_k = \mathcal{L}(X_k; \tau > k)$.

Definition 2: A sequence $\mathfrak{m} = (\mu; \mu_o, \mu_1, \ldots)$ of measures on $(E, \mathcal{B})$ is called a stopping sequence if

$$\mu \geq \mu_o \quad \text{and} \quad \mu_{k-1}P \geq \mu_k \quad \text{for } k = 1, 2, \ldots$$

Remark: The stopping-sequence associated with a stopping-time τ satisfies the conditions of definition 2. In fact:

$$\begin{aligned} \mu_{k-1}P - \mu_k &= \mathcal{L}(X_k; \tau \geq k) - \mathcal{L}(X_k; \tau > k) = \\ &= \mathcal{L}(X_k; \tau = k). \\ \mu - \mu_o &= \mathcal{L}(X_o; \tau = o). \end{aligned}$$

Notation: Let $\mathfrak{m} = (\mu; \mu_o, \mu_1, \ldots)$ be a stopping-sequence

a) μ is called the initial distribution: $M(\mathfrak{m})$

b) $\lambda_k = \mu_o + \ldots + \mu_{k-1}$ is called the effect till time k: $\Lambda_k(\mathfrak{m})$

$\eta = \mu_o + \mu_1 + \ldots$ is called the total effect: $H(\mathfrak{m})$

c) $\Gamma_o = \mu - \mu_o$ is called the residue at time 0: $\Gamma_o(\mathfrak{m})$

$\Gamma_k = \mu_{k-1}P - \mu_k$ is called the residue at time k: $\Gamma_k(\mathfrak{m})$ for $k = 1, 2, .$

d) $\rho = \Gamma_o + \ldots + \Gamma_k$ is called the residue till time k: $P_k(\mathfrak{m})$

$\nu = \Gamma_o + \Gamma_1 + \ldots$ is called the final distribution: $N(\mathfrak{m})$

Lemma: If $\mathfrak{m}$ is a stopping-sequence, then

$$\begin{aligned} \lambda_{k+1} + \rho_k &= \lambda_k P + \mu \\ \eta + \nu &= \eta P + \mu \end{aligned}$$

The second equation says in the usual terminology, that η is a solution of the Poisson-equation.

Notation: Let μ,ν be finite measures, η an arbitrary measure. We say, that μ,ν,η are in the relation

$\mu \underset{\eta}{\longmapsto} \nu$ iff there exists a stopping-sequence

with $M(\mathfrak{m}) = \mu$, $N(\mathfrak{m}) = \nu$ and $H(\mathfrak{m}) = \eta$.

Theorem 1:

a) $\mu' \underset{\eta'}{\longmapsto} \nu'$, $\mu'' \underset{\eta''}{\longmapsto} \nu'' \implies \mu'+\mu'' \underset{\eta'+\eta''}{\longmapsto} \nu'+\nu''$

b) $\mu'+\mu'' \underset{\eta}{\longmapsto} \nu \implies$ there exist decompositions $\nu = \nu'+\nu''$, $\mu = \mu'+\mu''$ such that $\mu' \underset{\eta'}{\longmapsto} \nu'$ and $\mu'' \underset{\eta''}{\longmapsto} \nu''$

c) $\mu \underset{\eta'}{\longmapsto} \nu^*$, $\nu^* \underset{\eta''}{\longmapsto} \nu \implies \mu \underset{\eta'+\eta''}{\longmapsto} \nu$.

The proof is easy, if one constructs "randomized stopping-times without memory" generating a stopping-sequence $\mathfrak{m}$ as follows:
Let $X_o, X_1, \ldots$ be a Markoff-process with initial distribution μ.
Let be $d_o, d_1, \ldots$ functions on $(E,\mathcal{B})$ with values in $[o,1]$ such that

$$\mu_o = \mu \cdot d_o \qquad \mu_k = (\mu_{k-1}P)\cdot d_k$$

Construct τ such, that the conditional probability for a particle arriving in x at time k to be stopped, is $1-d_k(x)$.

Remark: The following assertion about the relation $\mu \underset{\eta}{\longmapsto} \nu$ has not yet been proven in full generality:

$$\mu \underset{\eta}{\longmapsto} \nu,\ \eta' \leq \eta,\ \mu \underset{\eta'}{\longmapsto} \nu' \implies \nu' \underset{\eta-\eta'}{\longmapsto} \nu$$

III. Filling and Flooding.

We describe three devices to construct interesting stopping-sequences:

A. The filling-scheme for (μ,ν).

$$\mu_o = (\mu-\nu)^+ \qquad \nu_o = (\mu-\nu)^-$$

$$\mu_{k+1} = (\mu_k P-\nu_k)^+ \qquad \nu_{k+1} = (\mu_k P-\nu_k)^-$$

$\mathcal{W} = (\mu;\mu_o,\mu_1,\ldots)$ is called the filling-scheme for (μ,ν).

Remarks:

a) For the filling-scheme we have

$\Gamma_o+\Gamma_1+\ldots+\Gamma_k = \nu-\nu_k \geq o, \quad N(\mathcal{W}) \leq \nu.$

b) We say, that (μ,ν) is exact for filling if $\nu_k \downarrow o$

c) An alternative way to determine the filling-scheme uses a recursive definition of the λ_k:

$$\lambda_{k+1} = \lambda_k \vee (\lambda_k P+\mu-\nu)$$

d) If $\mathcal{W}$ is a filling-scheme with $M(\mathcal{W}) = \mu$, $N(\mathcal{W}) = \nu$ then $\lambda_{k+1}\wedge(\nu-\rho_k) = o$ for $k = o,1,2,\ldots$

B. The flooding-scheme for (μ,η).

$$\mu_o = \mu\wedge\eta \qquad \eta_o = \eta-\mu_o$$

$$\mu_{k+1} = \mu_k P\wedge\eta_k \qquad \eta_{k+1} = \eta_k-\mu_{k+1} = \eta-\mu_o-\ldots-\mu_{k+1}.$$

$\mathcal{W} = (\mu;\mu_o,\mu_1,\ldots)$ is called the flooding-scheme for (μ,η).

Remarks:

a) For the flooding-scheme we have

$$H(\mathcal{W}) \leq \eta$$

b) We say that (μ,η) is exact for flooding, if

$$H(\mathcal{W}) = \eta$$

c) An alternative way to determine the flooding-scheme uses a recursive definition of the λ_k

$$\lambda_{k+1} = (\lambda_k P+\mu)\wedge\eta .$$

d) If $\mathfrak{m}$ is a flooding-scheme with $M(\mathfrak{m}) = \mu$, $H(\mathfrak{m}) = \eta$

then $\quad \rho_k\wedge(\eta-\lambda_{k+1}) = o \quad$ for $k = o,1,2,\ldots$

C. Restricted flooding for $(\mathfrak{m},\eta')$.

Let $\mathfrak{m}$ be a stopping-sequence, η' a measure.

Let $d_o,d_1,\ldots$ be the densities

$$\mu_o = \mu\cdot d_o,\ \mu_k = \mu_{k-1}P\cdot d_k$$

$$\mu'_o = \mu_o\wedge\eta'$$

$$\mu'_{k+1} = (\mu'_k P)\cdot d_k\wedge(\eta'-\mu'_o-\ldots-\mu'_k)$$

$\mathfrak{m}' = (\mu;\mu'_o,\mu'_1,\ldots)$ is called the scheme of restricted flooding for $(\mathfrak{m},\eta')$

Grafting and cutting of branches.

Definition 3: Let $\mathfrak{m} = (\mu;\mu_o,\mu_1,\ldots)$ and

$\mathfrak{m}^* = (\mu^*;\mu^*_o,\mu^*_1,\ldots)$ be stopping-sequences with

$\mu^*_o \leq \Gamma_k(\mathfrak{m})$ for a certain k.

$\mathfrak{m}' = (\mu;\mu_o,\ldots,\mu_{k-1},\mu_k+\mu^*_o,\mu_{k+1}+\mu^*_1,\ldots)$

is then a stopping-sequence. We write

$$\mathfrak{m}' = \mathfrak{m} \underset{k}{\oplus} \mathfrak{m}^*$$

$$\mathfrak{m} = \mathfrak{m}' \underset{k}{\ominus} \mathfrak{m}^*$$

(Clearly $\mathfrak{m}$ is uniquely determined by $\mathfrak{m}'$ and $\mathfrak{m}^*$).

Definition 4: Let $\mathfrak{m}', \mathfrak{m}''$ be stopping-sequences.

a) We write $\mathfrak{m}' \underset{d}{>>} \mathfrak{m}''$ if there exist numbers k,l with $k \leq l$ and stopping-sequences $\mathfrak{m}, \mathfrak{m}^*$ such that

$$\mathfrak{m}' = \mathfrak{m} \underset{k}{\oplus} \mathfrak{m}^* , \qquad \mathfrak{m}'' = \mathfrak{m} \underset{l}{\oplus} \mathfrak{m}^*$$

b) If there exist stopping-sequences $\mathfrak{m}^1, \mathfrak{m}^2, \ldots, \mathfrak{m}^n$ such that

$\mathfrak{m}' \underset{d}{>>} \mathfrak{m}^1 \underset{d}{>>} \ldots \underset{d}{>>} \mathfrak{m}^n \underset{d}{>>} \mathfrak{m}''$, we write

$\mathfrak{m}' >> \mathfrak{m}''$

Notice, that >> is an ordering and

$\mathfrak{m}' >> \mathfrak{m}''$ implies

$$M(\mathfrak{m}') = M(\mathfrak{m}''),\ N(\mathfrak{m}') = N(\mathfrak{m}''),\ H(\mathfrak{m}') = H(\mathfrak{m}'')$$

$$\Lambda_k(\mathfrak{m}') \geq \Lambda_k(\mathfrak{m}'') \quad \text{for all } k.$$

Theorem 2: For every stopping-sequence $\mathfrak{m}$ for $\mu \underset{\eta}{\longmapsto} \nu$, there exist $\mathfrak{m}^o, \mathfrak{m}^1, \ldots$ such that

$$\mathfrak{m} = \mathfrak{m}^o << \mathfrak{m}^1 << \ \ldots \quad \text{and}$$

$$\Lambda_k(\mathfrak{m}^n) \nearrow \lambda_k \quad \text{for all } k$$

where λ_k are the effects of the flooding-scheme for (μ,η).

Remark: In the case where η is not σ-finite, it may happen that the final distribution of the flooding-scheme is strictly smaller than ν.

Corollary: If $\mu \underset{\eta}{\longmapsto} \nu$ with η σ-finite, then (μ,η) is exact for flooding. For every stopping-sequence $\mathfrak{m}$ for $\mu \underset{\eta}{\longmapsto} \nu$ we have

$$\Lambda_k(\mathfrak{m}) \leq \lambda_k \quad , \qquad k = 1,2,\ldots$$

where λ_k is the effect of the flooding-scheme. In particular if τ is a stopping-sequence generating $\mathfrak{m}$, τ^* a stopping-sequence generating the flooding-scheme, then

$$
\begin{aligned}
\mathcal{E}\tau &= \|\eta\| = \mathcal{E}\tau^* \\
\tfrac{1}{2}\mathcal{E}(\tau(\tau-1)) &= \Pr(\tau>1)+2\cdot\Pr(\tau>2)+\ldots = \\
&= \Pr(\tau>1)+\Pr(\tau>2)+\Pr(\tau>3)+\ldots \\
&\quad + \qquad\quad +\Pr(\tau>2)+\Pr(\tau>3)+\ldots \\
&\qquad\qquad\qquad\qquad\qquad +\Pr(\tau>3)+\ldots \\
&\qquad\qquad\qquad\qquad\qquad\qquad \ldots \\
&= \|\eta-\Lambda_1(m)\|+\|\eta-\Lambda_2(m)\|+\ldots \\
&\quad \|\eta-\lambda_1\|+\|\eta-\lambda_2\|+\ldots = \tfrac{1}{2}\mathcal{E}(\tau^*(\tau^*-1)).
\end{aligned}
$$

Remark: This corollary supports the conjecture in [1], that Root's stopping-devices for Brownian motion yields a stopping-time with minimal variance.
(Compare remark d) concerning the flooding scheme).
Moreover, it shows, that a stopping-time with minimal expectation and minimal variance makes also $\mathcal{E}(\varphi(\tau))$ minimal for every convex φ.

Announcement: An extremality property with respect to <<, a kind of converse to that one of the flooding-scheme can be proven for the filling-scheme.
A way to get a "good" stopping-time for (μ,ν) (if it exists), is the following:
Construct the filling scheme for (μ,ν). Let its total effect by η.
Construct then the flooding-scheme for (μ,η) and associate a stopping-time.

Cutting short a stopping sequence.

Definition 5: We call the stopping-sequence $\mathcal{M}'$ shorter than the stopping-sequence $\mathcal{M}$ and write $\mathcal{M}' \sqsubset \mathcal{M}$ if the densities $d_k(\mu_o = \mu \cdot d_o,\ \mu_k = \mu_{k-1} P \cdot d_k)$ satisfy

$$\mu_o' \le \mu \cdot d_o, \quad \mu_k' \le \mu_{k-1}' P \cdot d_k$$

Lemma: If $\mathcal{M}' \sqsubset \mathcal{M}$, then there exist stopping-sequences $\mathcal{M}^o, \mathcal{M}^1, \dots$ such that

$$\mathcal{M} = (\mathcal{M}' \underset{o}{\oplus} \mathcal{M}^o) \underset{1}{\oplus} \mathcal{M}^1 \underset{2}{\oplus} \dots$$

Remark: Restricted flooding for $(\mathcal{M}, \eta')$ yields a stopping-sequence $\mathcal{M}'$ with $\mathcal{M}' \sqsubset \mathcal{M}$.

Theorem 3: Let $\mathcal{M}$ be a stopping-sequence for $\mu \underset{\eta}{\longmapsto} \nu$ and let η' be a measure with $\eta' \le \eta$ and

$$\eta' + \nu = \eta' P + \mu$$

then there exists a stopping-sequence $\mathcal{M}'$ for $\mu \underset{\eta'}{\longmapsto} \nu$ with

$$\mathcal{M}' \sqsubset \mathcal{M}$$

In order to prove this theorem one shows, that the residues ρ_k' for the restricted flooding-scheme stay below ν. This can be done by induction. The assertion $\rho_k' \le \nu$ is equivalent with

$$\lambda_{k+1}' \le \lambda_k' P + (\mu - \nu).$$

Announcement: By iterated use of theorem 3 one can get a decomposition of "memoryless" stopping-times, which generalizes Neveu's investigation [4] , which concerns bijective measure preserving transformations rather than just positive contractions P of a measure-space.

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