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NOTE ON LAST EXIT DECOMPOSITION

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This will show how to derive the last exit decomposition in the boundary theory for Markov chains as set forth in [1] and [2]. The lectures given in Strasbourg were cut short toward the end as indicated by the last sentence on p.87 of [2].

One must now begin there by going to pp.161-2 of [1]. Here are the two formulas for the chain starting at a boundary atom a_0 :

$$\text{(I)}\quad P\{X(t)=j \;:\; t<\beta_{n+1} \left| \begin{matrix} Z_0,\dots,Z_{n-1},Z_n=a \\ \beta_0,\dots,\beta_{n-1},\beta_n=s \end{matrix} \right.\} = \rho_j^a(t-s) \;,\; 0\leqq s\leqq t \quad ;$$

$$\text{(II)}\quad P\{X(t)=j\} = \sum_{n=0}^{\infty} \sum_{a_1\dots a_n} \int_0^t \rho_j^{a_n}(t-s)\, d[F^{a_0 a_1} * \dots * F^{a_{n-1} a_n}](s) \quad .$$

The β_n's are the successive boundary switching time and may be assumed to be strictly increasing when all atoms are sticky, Z_n is the new atom switched to at β_n , ρ^a is the entrance law from a and before another switching, F^{ab} is the distribution of the switching time from a to b (namely that of the switching time when the chain starts at a, provided that the next switch is to b, see p.53 of [2]). Formulas (I) and (II) are easy consequences of the moderate Markov property applied at boundary hitting times.

Given any t, let n be determined by $\beta_n\leqq t<\beta_{n+1}$. Let H_n^b be defined as follows :

$$H_n^b(ds) = P\{\beta_n\in ds \;;\; X(\beta_n+)=b\}$$

(this depends also on the starting point a_0 of the chain, since P itself depends on it). Then (II) may be rewritten as

$$P\{X(t)=j\} = \sum_n \int_0^t H_n^b(ds)\rho_j^b(t-s) \;.$$

Now use (1) on p.62 of [2] :

$$\text{(III)}\qquad \rho_j^b(t) = \int_0^t \eta_j^b(t-s)E^b(ds)$$

to get the preceding probability to be

$$\sum_n \sum_b \int_0^t (H_n^b * E^b)(ds)\eta_j^b(t-s) \;.$$

This is it. What we did above is simply this : suppose the n-th switch is to b, then consider the last exit time from b before the next switch. The total time up to then has the distribution $\sum_n (H_n^b * E^b)$. But the post-last-exit-from b process has the entrance law η^b regardless of what has happened before, as indicated on pp. 75-6 of [1]. One can also see this in two steps : first up to the switching time to b, then between this and the last exit time from b, using (III) above.

This is the analogue of the strong Markov property for a last exit time, proved in a general setting by Pittenger-Shih . It is not even necessary for the analytic derivation in the above. Symbolically, all we do is to lump together the first three factors in the complete decomposition formula (p.72 of [2])

$$\ell_i(\lambda)[I-F(\lambda)]^{-1}E(\lambda)\eta_j(\lambda)$$

to get the last exit decomposition, just as we get the first entrance decomposition by lumping together the last three factors. The Laplace transforms are used merely for convenience and do not smear over anything, because there are explicit formulas like (I) and (II). The latter can be used to avoid the inversion of the matrix $I-F(\lambda)$ above, as shown by Lamb (Zeitschr. fur Wahrsch. vol.19 (213-224), 1971).

[1] Boundary theory for Markov chains II. Acta Math. vol.115, pp. 111-163 (1966).

[2] Lectures on Boundary Theory for Markov chains. Annals of Math. Studies 65, Princeton University Press, 1970.