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SKOROKHOD STOPPING VIA POTENTIAL THEORY

by David Heath *

I. Introduction. We present here a potential-theoretic viewpoint of the construction of Skorokhod given in [2] for the proof of the following result:

If μ is a probability measure on $\mathbb{R}$ with $\int x^2 \, d\mu < \infty$ and $\int x \, d\mu = 0$ then there exists a (randomized) stopping time T for Brownian motion (X_t , $t \geq 0$) starting at 0 such that the distribution of X_T is μ and $E(T) = \int x^2 \, d\mu$.

The construction of [2] consists essentially of finding a monotone collection ($I(s)$, $s \in [0,1]$) of intervals in $\mathbb{R}$ such that the required stopping time may be defined as

$$T = \inf \{ t > 0 : X_t \notin I(S) \}$$

where S is a random variable independent of the Brownian motion, with distribution uniform on [0,1]. (Actually, in [2] the intervals are parameterized differently so that S has distribution μ.) Here is a proof which differs slightly from that of Skorokhod. To simplify the notation we restrict our attention (as did [2]) to the case in which μ has a continuous distribution function.

It is easy to see that under this hypothesis one can find a family of intervals ($I(s)$, $s \in [0,1]$) of the form $I(s) = (x_1(s), x_2(s))$ for which

a) $\mu(I(s)) = s$

and b) $\int_{I(s)} x \, d\mu = 0.$

Clearly x_1 and x_2 are strictly monotone functions, and they are essentially unique. Define T as above, and let ν be the distribution of X_T. We

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wish to show that $\nu = \mu$. It is obvious that conditions a) and b) remain true if μ is replaced by ν.

Let f be any bounded continuous function on $\mathbb{R}$ and let σ be any probability measure satisfying conditions a) and b). Define $J_s = \int_{I(s)} f\, d\sigma$. Clearly J is absolutely continuous; moreover, except on the countable set $\{ s: x_1 \text{ or } x_2 \text{ is not continuous at } s\}$, J is differentiable and one can easily check that

$$J'_s = \frac{-x_1(s)}{x_2(s)-x_1(s)} f(x_2(s)) + \frac{x_2(s)}{x_2(s)-x_1(s)} f(x_1(s)) .$$

Since $J_0 = 0$ and $J_1 = \int f\, d\sigma$, we see that

$$\int f\, d\sigma = \int_0^1 J'_s \, ds = \int_0^1 P_{I'(s)} f(0) \, ds$$

where $I'(s)$ is the complement of $I(s)$ and $P_{I'(s)}$ is the corresponding hitting operator for Brownian motion. From this we conclude that $\int f\, d\sigma$ depends only upon the family $(I(s))$ and hence $\int f\, d\nu = \int f\, d\mu$, so that $\nu = \mu$. The equation for E(T) is easily obtained.

II. Another Definition Of $(I(s))$. The proof given above relies on the geometry of $\mathbb{R}$ both for selecting the intervals and for identifying ν with μ. We seek now another characterization of these intervals. Suppose now that μ has support in some bounded interval E and let $U\mu$ be the potential of μ in E given by

$$U\mu(x) = -2 \int_0^x F(y)\, dy + h(x)$$

where F is the distribution function of μ and h is a harmonic (i.e. linear) function chosen to make $U\mu$ vanish at the endpoints of E. Fix $s \in [0,1]$ and let $x_1 = x_1(s)$ and $x_2 = x_2(s)$. Define

$$t_i(x) = U\mu(x_i) + (x-x_i)\,(U\mu)'(x_i) \qquad \text{for } i=1,2.$$

It is easy to check that because of the choice of $I(s)$, $t_1(0) = t_2(0)$. Further, one can check that the function g defined by

$$g(x) = \begin{cases} U\mu(x) & \text{if } x\notin(x_1,x_2) \\ t_1(x) & \text{if } x\varepsilon(x_1,0) \\ t_2(x) & \text{if } x\varepsilon[0,x_2) \end{cases}$$

is the potential of a probability measure with mass s at 0, mean 0, and agreeing with μ on $(-\infty,x_1)$ (x_2,∞). Notice that we have then

$$g = R(U\mu - sU\varepsilon_0) + sU\varepsilon_0 ,$$

where ε_0 is the probability measure assigning mass one to $\{0\}$ and Rf is the infimum of all supermedian functions dominating f. Also,

$$I(s) = \{\ x:\ R(U\mu - sU\varepsilon_0) > U\mu - sU\varepsilon_0\ \}.$$

The purpose of the next two sections is to show that the above structure exists in at least slightly more generality.

III. A Theorem Of Mokobodzki. The result of this section is due to G. Mokobodzki (private communication) and is more general than we shall use. We suppose given a (sub-) Markovian semigroup; 'excessive' and 'supermedian' are with respect to this semigroup. As above, if f is any function we let Rf be the infimum of all supermedian functions dominating f and $R_A f = R(fI_A)$.

LEMMA. Suppose Rf is everywhere finite and $A = \{\ f>(1-\varepsilon)Rf\ \}$ for some ε in $(0,1)$. Then $R_A Rf = Rf$.

PROOF. Set $g = R_A Rf$. Both on A and off A it is clear that $f \leq (1-\varepsilon)Rf + \varepsilon g$; hence this supermedian function dominates Rf. Since $g \leq Rf$, it is also dominated by Rf; hence $(1-\varepsilon)Rf + \varepsilon g = Rf$, so $g=Rf$. □

Now let a and b be excessive functions with a everywhere finite. For $t\varepsilon[0,1]$, let $\bar{t} = 1-t$ and define:

$$v_t = a - \bar{t}b\,, \qquad V_t = Rv_t,$$
$$A_t^\varepsilon = \{\, v_t > V_t - \varepsilon\bar{t}a \,\}, \quad h_t^\varepsilon = R_{A_t^\varepsilon}\, b\,.$$

Clearly $V_t \leqslant a$, so $A_t^\varepsilon \supseteq \{\, v_t > (1-\varepsilon\bar{t})\, V_t \,\}$; thus according to the lemma $R_{A_t^\varepsilon} V_t = V_t$. Moreover, for $\lambda\varepsilon[0,1]$ we have $v_{\lambda s+\bar{\lambda}t} = \lambda v_s + \bar{\lambda} v_t$; the subadditivity of R then shows that $V_\cdot$ is convex.

Fix x and consider the graphs of $V_t(x)$ (convex) and $v_t(x)+\varepsilon\bar{t}a(x)$ (linear). These functions are equal at t=1; hence $\{\, t\colon V_t(x)<v_t(x)+\varepsilon\bar{t}a(x)\}$ is of the form $[C,1]$. This means that A_t^ε increases with t. As $\varepsilon \searrow 0$, $A_t^\varepsilon \searrow$, so $h_t^\varepsilon \searrow$; call its limit h_t. We then have the following:

THEOREM. $a - R(a-b) = \int_0^1 h_t\, dt$.

PROOF. Since $v_t = v_s + (t-s)b$, the subadditivity of R gives (for s<t) $V_t \leqslant V_s + (t-s)b$; apply $R_{A_t^\varepsilon}$ to get $V_t \leqslant V_s + (t-s)\, h_t^\varepsilon$.

In the other direction, on A_s^ε we have $v_s > V_s - \varepsilon\bar{t}a$, so that on A_s^ε, $V_t \geqslant v_t = v_s + (t-s)b > V_s - \varepsilon\bar{t}a + (t-s)b$. Thus $(V_t + \varepsilon\bar{t}a)\, I_{A_s^\varepsilon} \geqslant V_s\, I_{A_s^\varepsilon} + (t-s)\, I_{A_s^\varepsilon}\, b$. Since $R_{A_s^\varepsilon}$ is additive on supermedian functions, we obtain

$$V_t + \varepsilon\bar{t}a \geqslant R_{A_s^\varepsilon}(\, V_t + \varepsilon\bar{t}a) \geqslant R_{A_s^\varepsilon}\, V_s + (t-s)\; R_{A_s^\varepsilon}\, b \;=\; V_s + (t-s)\; h_s^\varepsilon\,.$$

Combining these, we obtain

$$h_s^\varepsilon - \frac{\varepsilon\bar{t}a}{(t-s)} \leqslant \frac{V_t - V_s}{(t-s)} \leqslant h_t^\varepsilon\,.$$

After letting $\varepsilon \searrow 0$, we see that the right derivative of $V_\cdot$ lies between $h_\cdot$ and $h_{\cdot+}$. Therefore $V_1 - V_0 = \int_0^1 h_t\, dt$ as desired. □

IV. Skorokhod Stopping In $\mathbb{R}^N$. Let E be a bounded ball about 0 in $\mathbb{R}^1$ or $\mathbb{R}^2$ or $E = \mathbb{R}^N$ for N>2. Let $(X_t, t\geqslant 0)$ be standard Brownian motion in $\mathbb{R}^N$ starting at 0 and killed when it leaves E, and let u and U be the associated potential kernel, as in Blumenthal and Getoor [1], p. 253. Let ε_0 be as before.

THEOREM. Let μ be a probability measure with support in E satisfying

$U\mu \leq U\varepsilon_0$ and $U\mu(0) < \infty$. There is then a monotone collection $(A(s), s\varepsilon[0,1])$ of subsets of E such that if S is independent of $(X_t, t\geq 0)$ with distribution uniform on $[0,1]$ and $T = \inf\{t>0: X_t \in A(S)\}$, then X_T has distribution μ.

PROOF. According to exercise (1.27) of Chapter VI of [1], $(X_t, t\geq 0)$ is in duality with itself relative to Lebesgue measure restricted to E. Set $a = U\mu$ and $b = U\varepsilon_0$; because of the regularity of Brownian motion we can set $\varepsilon=0$ in the proof of the previous theorem. Letting $A(s)=A_s^{0+}$ we obtain $a = \int_0^1 R_{A(s)}b\, ds$. It follows from Theorem (6.12) of Chapter II of [1] that $R_{A(t)}b = P_{A(t)}b$ almost everywhere (Lebesgue measure) for each $t\varepsilon[0,1]$. Using Fubini's theorem we can then conclude that $\int_0^1 R_{A(s)}b\, ds = \int_0^1 P_{A(s)}b\, ds$ almost everywhere. Since $P_{A(s)}b$ is a monotone function of s, its integral can be expressed as the limit of an increasing sequence of excessive functions and is therefore excessive. Since this integral is almost everywhere equal to a, an excessive function, by Proposition (1.3) of Chapter VI of [1], they must be equal, i.e., $a = \int_0^1 P_{A(s)}b\, ds$.

Suppose now that T is defined as in the statement of the theorem, and let ν be the distribution of X_T. Clearly $\nu(\cdot) = \int_0^1 P_{A(s)}(0,\cdot)\, ds$. Now $U\nu(x) = \int_0^1 \int_E u(x,z)\, P_{A(s)}(0,dz)\, ds = \int_0^1 P_{A(s)}u(x,0)\, ds = \int_0^1 P_{A(s)}b(x)\, ds = a(x)$, where the second equality follows from Theorem (1.16) of Chapter VI of [1]. Thus μ and ν have the same potential; by Proposition (1.15) of Chapter VI of [1], they must be equal. □

REFERENCES

[1] R. M. BLUMENTHAL and R. K. GETOOR Markov Processes and Potential Theory. Academic Press (1968).

[2] A. V. SKOROKHOD Studies In The Theory Of Random Processes. Addison-Wesley (1965).