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ON THE EXISTENCE OF RESOLVENTS [1)]

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Introduction. In [6] a measure-theoretic theorem is proved which gives a sufficient condition for a proper kernel V on a measure space $(E,\underline{\underline{E}})$ to be the kernel V_o defined by a sub-Markovian resolvent.

The theorem of Lion is an immediate corollary of this result. It states that, for E locally compact and σ-compact, if V satisfies the complete maximum principle and maps $\underline{\underline{C}}_c(E)$ into $\underline{\underline{C}}_o(E)$ then there is a sub-Markovian resolvent (V_λ) on $\underline{\underline{C}}_o(E)$ with $V = V_o$ on $\underline{\underline{C}}_c(E)$.

F. Hirsch in [2], by using different methods from those of Lion in [3], has shown that the theorem of Lion is valid for arbitrary locally compact spaces. The purpose of this note is to obtain a generalisation of theorem 2 in [6] which has as corollary 3·1 the above result of Hirsch. In addition, it has as corollary 3·3 the result of Mokobodzki and Sibony in [5] (without the restriction, imposed in [6], that E be σ-compact) and as corollary 3·2 an extension of Hirsch's result to kernels satisfying the domination principle.

The generalised theorem. Let E be a set and let $\underline{\underline{E}}$ be a σ-ring of subsets of E (i.e. $\underline{\underline{E}}$ is closed under countable unions and relative complements). A function $f : E \to \bar{R}^+$ is measurable if for all $\alpha > 0$, $\{f > \alpha\} \in \underline{\underline{E}}$. It will be said to be locally measurable if for each $X \in \underline{\underline{E}}$, $f|X$ is measurable in the usual

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sense with respect to the σ-field of sets of the form $A \cap X$, $A \in \underline{\underline{E}}$. A kernel on $(E,\underline{\underline{E}})$ is an additive, increasing, positively homogeneous map $V : \underline{\underline{E}}^+ \to \underline{\underline{E}}^+$ such that if $(f_n) \subset \underline{\underline{E}}^+$ is decreasing to f then (Vf_n) increases to Vf (here $\underline{\underline{E}}^+$ denotes the convex cone of non-negative measurable functions).

Let E be locally compact. In [1] the σ-ring $\underline{\underline{B}}_o$ of Baire sets is defined to be the smallest σ-ring containing $\underline{\underline{H}}$, the class of compact G_δ-subsets of E.

It is shown in [1] that, every $A \in \underline{\underline{B}}_o$ is a subset of a countable union of sets from $\underline{\underline{H}}$.

Remark. It is not hard to see that if $K_1, K_2 \in \underline{\underline{H}}$ then $K_1 \setminus K_2$ is a countable union of sets from $\underline{\underline{H}}$ i.e. is in $\underline{\underline{H}}_\sigma$. Consequently, the class of $\underline{\underline{H}}$-Borelian sets $\underline{\underline{B}}(\underline{\underline{H}})$ coincides with $\underline{\underline{B}}_o$($\underline{\underline{B}}(\underline{\underline{H}})$ is the smallest class containing $\underline{\underline{H}}$ and closed under countable unions and countable intersections).

A convex cone $\underline{\underline{C}} \subset \underline{\underline{C}}^+(E)$ is said to be adapted if for each $f \in \underline{\underline{C}}$ there exists $g \in \underline{\underline{C}}$ with $f \in o(g)$ i.e. for all $\epsilon > o$ there exists a compact $K = K(\epsilon)$ with $f(x) \leq \epsilon\, g(x)$ if $x \notin K$.

Proposition 1. Let E be locally compact and denote by M a positive linear map of $\underline{\underline{C}}_c(E)$ into $\underline{\underline{C}}(E)$ with $M(\underline{\underline{C}}_c^+(E)) \subset \underline{\underline{C}}$, where $\underline{\underline{C}} = \underline{\underline{C}}_o^+(E)$ or is an adapted cone. Then there is a unique kernel N on $(E,\underline{\underline{B}}_o)$ such that N agrees with M on $\underline{\underline{C}}_c(E)$.

The kernel N satisfies the principle of domination (respectively, the complete maximum principle) if for all φ, $\psi \in \underline{\underline{C}}^+(E)$,

$M\varphi \geq M\psi$ on $\{\psi > 0\}$ implies $M\varphi \geq M\psi$ and for all $x \in E$ there exists $\varphi \in C_c(E)$ with $M\varphi(x) \neq 0$ (respectively, $1 + M\varphi \geq M\psi$ on $\{\psi > 0\}$ implies $1 + M\varphi \geq M\psi$).

<u>Proof</u>: If $\underline{\underline{C}}$ is an adapted cone then each $f \in \underline{\underline{C}}$ vanishes outside of a countable union of compact sets. Hence, $f = \sum_n \varphi_n$, $(\varphi_n) \subset \underline{\underline{C}}_c^+(E)$. Consequently, $M\varphi \in \underline{\underline{B}}_o^+$ if $\varphi \in \underline{\underline{C}}_c^+(E)$.

Define the Radon measure μ_x by setting $< \mu_x, \varphi > = M\varphi(x)$, $\varphi \in \underline{\underline{C}}_c(E)$. If $f \in \underline{\underline{B}}_o^+$ define $N(x, f) = < \mu_x, f >$.

Let $K \in \underline{\underline{H}}$ and let $\underline{\underline{F}}$ be the σ-field of sets of the form $A \cap K$, $A \in \underline{\underline{B}}_o$. Denote by $\underline{\underline{M}}$ the vector space of differences of non-negative bounded Baire measurable functions. Then $\underline{\underline{F}}_b$ can be naturally identified with a subspace of $\underline{\underline{M}}$.

Consider $\underline{\underline{K}} = \{f \in \underline{\underline{F}}_b \mid Nf \in \underline{\underline{M}}\}$. This a subspace of $\underline{\underline{F}}_b$ closed under monotone limits and containing 1 (since $K \in \underline{\underline{H}}$). Assume $(f_n) \subset \underline{\underline{K}}$ and that f_n tends to f uniformly. There is a set $X \in \underline{\underline{B}}_o$ such that, for all n, $\{|Nf_n| > 0\} \subset X$ and the functions Nf_n are all measurable in the usual sense, when viewed on X, with respect to the σ-field on X induced by $\underline{\underline{B}}_o$. Hence, Nf_n (which equals $\lim_{n\to\infty} Nf_n$) is in $\underline{\underline{M}}$ since when viewed on X it is measurable. It follows from IT20 in [4] that $\underline{\underline{K}} = \underline{\underline{F}}_b$. Furthermore, if $f \in \underline{\underline{F}}_b^+$ then $Nf \in \underline{\underline{B}}_o^+$. This follows since $Nf \geq 0$ is locally measurable and $\{Nf > o\} \in \underline{\underline{B}}_o^+$.

If $f \in \underline{\underline{B}}_o^+$ then $\{f > 0\} \subset \bigcup_n K_n$, $(K_n) \subset \underline{\underline{H}}$, increasing. Hence, $Nf = \lim_{n\to\infty} N(f\, 1_{K_n})$ and so is in $\underline{\underline{B}}_o^+$.

The last two statements follow by the argument used to prove XT4 in [4].

Let N be a kernel on $(E,\underline{\underline{E}})$. A set $A \in \underline{\underline{E}}$ will be said to be <u>bounded</u> if $N1_A$ is finite and will be said to be <u>σ-bounded</u> if it is a countable union of bounded sets.

A locally measurable function u will said to be <u>supermedian</u> if, for all f and $g \in \underline{\underline{E}}^+$,

$$u + Nf \geq Ng \text{ on } \{g > 0\} \text{ implies } u + Nf \geq Ng.$$

A supermedian function u is said to <u>vanish at the boundary</u> if there exists an increasing sequence (A_n) of bounded sets with $\inf_n R^s_{\complement A_n} u = 0$ (note: it is no longer required as in [6] that $\bigcup_n A_n = E$).

<u>Proposition 2.</u> The following conditions are equivalent:

(1) every finite potential Nf, $f \in \underline{\underline{E}}^+$, vanishes at the boundary; and

(2) the potential $N1_A$ of every bounded set A vanishes at the boundary.

<u>Proof:</u> Let $f \in \underline{\underline{E}}^+$ have a finite potential Nf and let $\{f > 0\} = \bigcup_n B_n$, with each B_n bounded and (B_n) increasing. It can be assumed further that f is bounded on each set B_n. Let $N1_{B_n}$ vanish at the boundary relative to (A^n_m). Then, if $A_p = \bigcup_{n=1}^{p} \bigcup_{m=1}^{p} A^n_m$, each $N1_{B_n}$ vanishes at the boundary relative to (A_p).

Hence, each $N(f\,1_{B_n})$ vanishes at the boundary relative to (A_p) and so, by the lemma preceding proposition 1 in [6], Nf vanishes at the boundary relative to (A_p).

<u>Theorem 3.</u> Let N be a kernel on $(E,\underline{\underline{E}})$ that satisfies the domination principle and is such that the following conditions are satisfied:

(1) every set $A \in \underline{\underline{E}}$ is σ-bounded;

(2) every finite potential Nf, $f \in \underline{\underline{E}}^+$, vanishes at the boundary ; and

(3) if $A \in \underline{\underline{E}}$ there exists a finite supermedian function u which is strictly positive on A.

Then there exists a unique resolvent $(N_\lambda)_{\lambda>0}$ of kernels N_λ on $(E,\underline{\underline{E}})$ with $N_0 = N$. Further, this resolvent is sub-Markovian if N satisfies the complete maximum principle.

Proof: If $g \in \underline{\underline{E}}^+$ there exists a set $X \in \underline{\underline{E}}$ with the following properties:

(1) $\{g > 0\} \subset X$;

(2) $X = \bigcup_n A_n$, each A_n bounded; and

(3) for each n, $N1_{A_n}$ vanishes on $\complement X$ and vanishes at the boundary relative to (A_n). First note that if (B_n) is any sequence of bounded sets then there exist sequences (B_n') and (B_n'') of bounded sets such that (a) each $N1_{B_n}$ vanishes at the boundary relative to (B_n')(see the proof of proposition 2) and (b) for each n, $\{N1_{B_n} > 0\} \subset \bigcup_m B_m''$. Consequently, there exists a sequence (C_n) of bounded sets such that, for all n, $N1_{C_n}$ vanishes on $(\bigcup_n C_n)$ and vanishes at the boundary relative to some subsequence (depending on n) of (C_n). Further, (C_n) can be assumed to contain $\{g > 0\}$ in its union.

Let $X = \bigcup_n C_n$ and $A_n = \bigcup_{i=1}^n C_i$.

Let $\underline{\underline{X}}$ be the σ-field of sets of the form $A \cap X, A \in \underline{\underline{E}}$. If $f \in \underline{\underline{E}}^+$ and $\{f > 0\} \subset X$ then (3) implies $\{Nf > 0\} \subset X$. Hence, N induces a proper kernel R on $(X, \underline{\underline{X}})$ which satisfies the domination principle.

Let u_o be a finite supermedian function on E which is strictly positive on X. Define $Vf = (1/u_o)\, R(fu_o)$, $f \in \underline{\underline{X}}^+$.

Then the kernel V satisfies the hypotheses of theorem 2 in [6] since u supermedian implies u/u_o restricted to X is supermedian relative to V. Let $(V_\lambda)_{\lambda > o}$ be the sub-Markovian resolvent on $(X, \underline{\underline{X}})$ with $V = V_o$. Then, if $R_\lambda h = u_o\, V_\lambda\, (h/u_o)$, $h \in \underline{\underline{X}}^+$, $(R_\lambda)_{\lambda > o}$ is the resolvent of kernels on $(X, \underline{\underline{X}})$ with $R_o = R$. Hence, $Ng = (I + \lambda N)\, R_\lambda\, g$.

Denote by X' another set in E satisfying (1), (2) and (3) and let R' be the kernel induced on $(X', \underline{\underline{X}}')$ by N. Let $(R'_\lambda)_{\lambda > o}$ be the resolvent on $(X', \underline{\underline{X}}')$ with $R'_o = R'$. Since $Ng = (I + \lambda N)\, R'_\lambda g$ it then follows from XT 7 in [4] that, for all $\lambda > o$, $R_\lambda g = R'_\lambda g$.

Define $N_\lambda g$ to be $R_\lambda g$, where $(R_\lambda)_{\lambda > o}$ is the resolvent defined by a set $X \in \underline{\underline{E}}$ satisfying (1), (2) and (3) and the kernel R induced by N.

It follows that (i) each N_λ is a kernel, (ii) $(N_\lambda)_{\lambda > o}$ is a resolvent family and (iii) $N = N_o$.

<u>Application to locally compact spaces.</u> Let E be a locally compact and denote by $\underline{\underline{B}}_o$ the σ-ring of Baire subsets of E.

<u>Corollary 3.1.</u> (F. Hirsch [2]). Let V be a positive linear map of $\underline{\underline{C}}_c$ (E) into $\underline{\underline{C}}_o$ (E) such that for all φ, $\psi \in \underline{\underline{C}}_c^+$ (E)

$$1 + V\varphi \geq V\psi \text{ on } \{\psi > 0\} \text{ implies } 1 + V\varphi \geq V\psi.$$

Then there is a resolvent family $(V_\lambda)_{\lambda > o}$ of sub-Markovian operators V_λ on $\underline{\underline{C}}_o(E)$ with $V\varphi = \lim_{\lambda \to 0} V_\lambda \varphi$, for all $\varphi \in \underline{\underline{C}}_c(E)$.

<u>Proof</u>: From proposition 1 and theorem 3 it follows that there is a sub-Markovian resolvent $(V_\lambda)_{\lambda > o}$ of sub-Markovian kernels V_λ on $(E, \underline{\underline{B}}_o)$ with $V_o = V$. Note that by the lemma preceding proposition 1 in [6] every finite potential Vf vanishes at the boundary because $\{f > 0\} \subset \bigcup_n K_n$, $(K_n) \subset \underline{\underline{H}}$ and each $V 1_{K_n}$ clearly vanishes at the boundary.

Let $\varphi \in \underline{\underline{C}}_o^+$ (E). Then there exists an open set $X \in \underline{\underline{B}}_o$ such that (1) $\{\varphi > 0\} \subset X$; (2) $X = \bigcup_n K_n$, $(K_n) \subset \underline{\underline{H}}$; and (3) for each n, $V 1_{K_n}$ vanishes on X and vanishes at the boundary relative to (K_n). It suffices to note that the sets B_n, B_n' and B_n'' in the proof of the theorem can all be assumed to be open and relatively compact under the hypotheses of this corollary.

Consequently, there exists $a \in \underline{\underline{C}}^+(E)$ with (1) $X = \{a > 0\}$ and (2) Va bounded. The argument given in the remark following corollary 2·4 in [6] then implies $V_\lambda \varphi$ is continuous for all $\lambda > 0$. Hence, each V_λ leaves $\underline{\underline{C}}_o(E)$ invariant.

<u>Corollary 3.2.</u> Let M be a positive linear map of $\underline{\underline{C}}_c(E)$ into $\underline{\underline{C}}_o(E)$ such that, for all φ, $\psi \in \underline{\underline{C}}_c^+$ (E)

$$M\varphi \geq M\psi \text{ on } \{\psi > 0\} \text{ implies } M\varphi \geq M\psi.$$

Assume M is non-degenerate, i.e. for all $x \in E$ there exists $\varphi \in \underline{\underline{C}}_c(E)$ with $M\varphi(x) \neq 0$ and that there is a supermedian function u with $\overline{\{u < 1\}}$ compact. Then there is resolvent family $(M_\lambda)_{\lambda > 0}$ of unbounded operators M_λ on $\underline{\underline{C}}_o(E)$ such that $M\varphi = \lim_{\lambda \to 0} M_\lambda \varphi$, for all $\varphi \in \underline{\underline{C}}_c(E)$.

Proof: Let N be the kernel on $(E, \underline{\underline{B}}_o)$ determined by M. It satisfies the domination principle since M is non-degenerate. If $X \in \underline{\underline{B}}_o$ then there exists $(\varphi_n) \subset \underline{\underline{C}}_c^+(E)$ with $u = \sum_n M(\varphi_n) \in \underline{\underline{C}}^+(E)$ and $\{u > o\} \supset X$. The argument of proposition 3 in [6] shows that u is a supermedian function.

Because there exists a supermedian function u with $\overline{\{u < 1\}}$ compact, every finite potential Nf vanishes at the boundary and so there is a resolvent $(N_\lambda)_{\lambda > 0}$ of kernels N_λ on $(E, \underline{\underline{B}}_o)$ with $N_o = N$.

Because each $X \in \underline{\underline{E}}$ is contained in $\{u > o\}$, for some u continuous and supermedian, it follows that, if $\varphi \in \underline{\underline{C}}_c^+(E)$ and $X \in \underline{\underline{E}}$ is open and satisfies the conditions in the proof of corollary 3·1, then the kernel V on $(X, \underline{\underline{X}})$ induced by N and u maps $\underline{\underline{C}}_c(X)$ into $\underline{\underline{C}}_o(X)$. Hence $R_\lambda \varphi \in \underline{\underline{C}}_o(X)$ and so N_λ leaves $\underline{\underline{C}}_o(E)$ invariant. Define $M_\lambda \varphi = N_\lambda \varphi$, if $\varphi \in \underline{\underline{C}}_o(E)$.

Application to adapted cones

Let E be locally compact and denote by $\underline{\underline{C}} \subset \underline{\underline{C}}^+(E)$ an adapted cone. Let M be a positive linear map of $\underline{\underline{C}}_c(E)$ into $\underline{\underline{C}}(E)$ with $M(\underline{\underline{C}}_c^+(E)) \subset \underline{\underline{C}}$. Assume the following conditions satisfied:

A_2) for each $x \in E$ there exists $u \in \underline{\underline{C}}$, $u(x) > 0$; and

A_4) if $u \in \underline{\underline{C}}$ and $\varphi \in \underline{\underline{C}}_c^+(E)$ then

$$u \geq M\varphi \text{ whenever } u \geq M\varphi \text{ on } \{\varphi > o\}.$$

Corollary 3·3 (Mokobodzki-Sibony [5]). Let N be the kernel on $(E,\underline{\underline{B}}_o)$ determined by M. Then there is a resolvent $(N_\lambda)_{\lambda > o}$ of kernels on $(E,\underline{\underline{B}}_o)$ with $N_o = N$.

Proof: Let $\underline{\underline{C}}_\sigma$ denote the set of continuous functions u on E of the form $u = \sum_n u_n$, $(u_n) \subset \underline{\underline{C}}$. Each $u \in \underline{\underline{C}}_\sigma$ is supermedian (see proposition 3 in [6])and for each $X \in B_o$ condition A_2) implies that there exists $u \in \underline{\underline{C}}_\sigma$ with $\{u > 0\} \supset X$.

If $\varphi \in \underline{\underline{C}}_c^+(E)$ the fact that $\underline{\underline{C}}$ is adapted and contains only supermedian functions implies $N\varphi$ vanishes at the boundary. Hence, as in the proof of corollary 3·1, each finite potential Nf vanishes at the boundary. The result then follows immediately.

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