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RELAXATION IN INFINITE SPIN SYSTEMS

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We present in this paper some recent results of DOBRUŠIN and HOLLEY in the field of statistical mechanics.

I. Consider a finite or countable set I, the set of "lattice sites", and associate to each $i \in I$ a "spin" x_i equal to ± 1; a spin configuration is a family $x = (x_i)_{i \in I}$; the set S of all spin configurations will be provided with its usual product topology and product σ-field. For each i let X_i be mapping which assigns to $x \in S$ its i-th coordinate.

Further, for each i we are given a "local energy function" $U(.,i)$ on S which we assume to be of the form

$$U(x,i) = \sum_{i \in P} J_P (\prod_{j \in P} x_j) ,$$

where J_P is a real number for every finite subset of of I such that for all $i \in I$ $\sum_{i \in P} |J_P| < \infty$.

A probability measure μ on S will be called a Gibbs measure iff for every $i \in I$ and $x \in S$

$$(\exp(-U(x,i)) + \exp(-U(\tau_i x,i)))^{-1} \cdot \exp(-U(x,i))$$

is a version of the conditional μ-probability of $\{X_i = x_i\}$ under the hypothesis $X_j = x_j$ for all $j \neq i$. (Here τ_i is the inversion of the spin at site i : $y = \tau_i(x)$ iff $y_i = -x_i$ and $y_j = x_j$ for $j \neq i$.) Under the above assumptions, the existence of a Gibbs measure is always guaranteed; for the question of its uniqueness see [1] or [5].

II. In order to describe relaxation phenomena or time dependent behaviour of a spin system we will look for a stochastic process (if possible Markovian) with state space S which may serve as a model for the changes which undergoes a spin configuration under the influence of some thermal disorder. From the mathematical point of view there are many ways to define such a process; see for example [6].

Here we try to find a Markov process whose generator A is defined at least on all functions on S which depend only on a finite number of coordinates and is of the form

$$(*) \qquad (Af)(x) = \sum_{i\in I} \exp(U(x,i))\cdot(f(\tau_i x) - f(x)).$$

For finite I there is no problem; in the case of infinite I the construction of a suitable stochastic semigroup acting on $\mathcal{C}(S)$, the space of all continuous functions on S , and hence a Markov Process is given in [2]. Since that construction involves a nice combinatorial argument we will present it here in all essential steps.

Theorem 1. Let for each $i \in I$ the continuous function $V(.,i)$ on S be given, $V(.,i) \geq 0$. For finite $E \subset I$ let $(P_t^E)_{t\geq 0}$ be the semigroup on $\mathcal{C}(S)$ with generator A^E , where

$$(A^E f)(x) = \sum_{i\in E} V(x,i)(f(\tau_i x) - f(x)) , \; f \in \mathcal{C}(S).$$

Set $G(i,j) := \sup_x \left| V(\tau_j x,i) - V(x,i)\right|$, $i,j \in I$.

Then, if $\sup_i (\sum_j G(i,j)) < \infty$, there exists for each $f \in \mathcal{C}(S)$ and $t > 0$ the limit (in the uniform sense)

$$P_t f := \lim_{E\nearrow I} P_t^E f$$

and defines a Feller semigroup $(P_t)_{t\geq 0}$ on $\mathcal{C}(S)$. (The generator A of (P_t) is of the form $(*)$ if $V = \exp U$.)

The theorem is an easy consequence of the following

Lemma. Let C,D,E be finite subsets of I with $C \subset D \subset E$;

let $f = f(z)$ *be an element of* $\mathcal{C}(S)$ *depending only on* z_i, $i \in C$, *and satisfying* $0 \leq f \leq 1$;
let $x, y \in S$ *be fixed and such that* $x_i = y_i$ *for* $i \in D$;
let (Q_t) *and* (R_t) *be the semigroups on* $\mathcal{C}(S)$ *with generator, respectively,*

$$g \longmapsto \sum_{i \in E} V(.,i)(g \circ \tau_i - g), \quad g \longmapsto \sum_{i \in E} W(.,i)(g \circ \tau_i - g),$$

where all $W(.,i)$ *are continuous and positive and* $W(.,i) = V(.,i)$ *for* $i \in D$.
Then one has for all $t \geq 0$

$$\left| Q_t(x,f) - R_t(y,f) \right| \leq \sum_{i \in C} \sum_{j \notin D} (\exp tG)(i,j) .$$

The proof of the lemma is based on an idea of WASSERSTEIN ([7]), who introduces the following measure of a distance between two probability measures on the product space S : we say that s_i, $i \in I$, is a *bound for* μ *and* ν iff there exists a probability measure ρ on $S \times S$ having μ and ν as its projections on the first and second factor and such that

$$\rho(X_i \neq Y_i) \leq s_i \quad \text{for all} \quad i \in I .$$

(Here X_i, resp. Y_i, assigns to each $(x,y) \in S \times S$ the coordinate x_i, resp. y_i.)

It is easy to see that if s_i, $i \in I$, is a bound for μ and ν then we have for a function f on S with values between 0 and 1 and depending only on z_i, $i \in C$,

$$(+) \qquad |\langle \mu - \nu, f \rangle| \leq \sum_{i \in C} s_i .$$

So one tries to find a bound for the measures $Q_t(x,.)$ and $R_t(y,.)$ by constructing a suitable joint realization of the processes $(X(t))_{t \geq 0}$ and $(Y(t))_{t \geq 0}$ having (Q_t), resp. (R_t), as transition semigroup and x, resp. y, as initial point.

A realization of one process, say $(X(t))$, may be found the following way :

$X_i(.)$ remains unchanged equal to x_i for all $i \notin E$;
for each $i \in E$ we consider independently a Poisson point process $p_i = (p_i^1, p_i^2, \ldots)$ and construct the path $X(.)$ given p_i, $i \in E$, in a deterministic way, namely

$$X_i(0) = x_i \ , \ i \in E \ ;$$

$X_i(.)$ jumps at time t if and only if $v_i(t)$, defined as $\int_0^t V(X(s),i)ds$, is equal to some point p_i^k, $k \geq 1$, in the sample p_i, $i \in E$.

By the same procedure we construct the process $(Y(t))$ using functionals w_i instead of v_i, where $w_i(t) = \int_0^t W(Y(s),i)ds$.
The salient point is that we define both processes by means of the <u>same</u> set of auxiliary Poisson point processes p_i, $i \in E$.
Then one has for $i \in D$, $t > 0$:

$$P(X_i(t) \neq Y_i(t)) \leq P(\text{there is a point of } p_i \text{ between } v_i(t) \text{ and } w_i(t)) \leq$$

$$\leq \mathcal{E}\,|v_i(t) - w_i(t)| \quad \text{(by the strong Markov property)} \leq$$

$$\leq \int_0^t \mathcal{E}\,|V(X(s),i) - W(Y(s),i)|\, ds \leq \int_0^t \mathcal{E}\Big(\sum_{X_j(s) \neq Y_j(s)} G(i,j)\Big)ds =$$

$$= \int_0^t \sum_{j \in E} G(i,j) \cdot P(X_j(s) \neq Y_j(s))\, ds \ .$$

As is easy to check, $P(X_i(t) \neq Y_i(t))$, $i \in E$, is dominated by the solution of the integral equation

$$s_i(t) = s_i(0) + \int_0^t \sum_{j \in E} G(i,j) \cdot s_j(s)\, ds \quad , \ i \in E \ ,$$

where the initial condition is

$$s_i(0) = 0 \ \text{ for } \ i \in D \ , \quad s_i(0) = 1 \ \text{ for } \ i \in E \setminus D \ ;$$

that solution has the explicit form

$$s_i(t) = \sum_{j \in E \setminus D} (\exp tG)(i,j) \ , \ i \in E \ ,$$

which is the desired bound for the two measures $Q_t(x,.)$ and $R_t(y,.)$. In view of (+) the lemma is proven.

We remark that the lemma allows one to obtain a lower estimate for a "relaxation time" : if a spin system with the stochastic behaviour governed by the semigroup (P_t) is changed at time 0 arbitrarily outside some finite set D the local behaviour at some fixed $i \in D$ will not be influenced too heavily at a time $t \leq t_o$, where t_o is the smallest solution of (say)

$$\sum_{j \notin D} (\exp tG)(i,j) = \varepsilon \ , \text{ and } \varepsilon \text{ is small.}$$

III. In order to consider the semigroup (P_t) in a more concrete situation we will specify our assumptions :

let I be equal to Z^ν, the lattice of points with integer valued coordinates in ν-dimensional space, $\nu \geq 1$;

let a finite collection of finite sets K , $K \subset Z^\nu$ and $0 \in K$, be given and to each K a real number J_K ; put

$$U(x,i) = \sum_K J_K \Big(\prod_{j-i \in K} x_j \Big) \ , \ i \in Z \ ,$$

and as before $V(x,i) = \exp U(x,i)$, $x \in S$, $i \in Z^\nu$;

let (P_t) be the semigroup constructed in theorem 1.

We will say that a probability measure μ on S is <u>shift invariant</u>, iff for any finite $C \subset Z^\nu$ and $\alpha_j = \pm 1$, $j \in C$,

$$\mu(X_j = \alpha_j, \ j \in C) = \mu(X_{j+i} = \alpha_j \ , \ j \in C) \quad \text{for every } i \in Z^\nu .$$

Then the following theorems hold ([3],[4]):

Theorem 2. A shift invariant probability measure μ on S is Gibbs if and only if it is invariant under (P_t), i.e. if

$$\mu P_t = \mu \quad \text{for all} \quad t \geq 0 .$$

In that case the process $(X(t))_{t\geq 0}$ is reversible under the law P^{μ}, i.e. one has for $t > 0$, A and $B \subset S$

$$P^{\mu}(X(0) \in A,\ X(t) \in B) = P^{\mu}(X(0) \in B,\ X(t) \in A) .$$

Theorem 3. If there exists a unique shiftinvariant Gibbs measure $\bar{\mu}$ and if μ is any shift invariant probability measure on S, μP_t converges to $\bar{\mu}$ weakly, i.e.

$$\langle \bar{\mu}, f \rangle = \lim_{t\to\infty} \langle \mu P_t, f \rangle \quad \text{for all} \quad f \in \mathcal{C}(S) .$$

The last theorem says roughly speaking that in our model of a spin system at the end of the relaxation process described by (P_t) the system is found in the equilibrium state $\bar{\mu}$ regardless of what its initial state was.

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