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A SEMI-MARKOVIAN MODEL FOR THE BROWNIAN MOTION

M.G. Mürmann

Introduction

In the last few years several authors studied stochastic models for an infinite particle system and its dynamic (for a first survey see Spitzer [7]). Within this scope Spitzer [6] and Holley [4] founded a new theory of the Brownian motion. (In this article Brownian motion means the physical phenomenon). Holley constructs a stochastic model for the motion of a heavy particle colliding with the light particles of a one-dimensional gas starting in a Poisson process, each of them independently given a velocity from a common distribution. He proves a convergence theorem stating that for suitable velocity distributions the process converges in distribution to the Ornstein-Uhlenbeck process, if the mass of the heavy particle, the density of the Poisson process and the velocities of the light particles simultaneously tend to infinity in a suitable normalisation. In higher dimensions new methods were necessary, since point collisions will in general not be possible. A first attempt has been made by Hennion [3] for a two-dimensional system. He represents the heavy particle by a disk and derives a convergence theorem, analogous to that of Holley.

In this article we shall present a construction of an approximating stochastic model for the velocity of a heavy particle interacting with the light particles of a gas of arbitrary dimension via a potential of compact support. For two reasons the process is not a Markov process: firstly because of the positive interaction times of the single particles and secondly because of the particles that formerly interacted with the heavy particle. The main purpose of this article is to study the semi-Markovian character of the non-Markovian behaviour resulting from the first reason. The influence of the second stating will be neglected.

The idea is the following: For suitable potentials there exist times at which no light particle interacts with the heavy particle. The existence of these times is combined with a queuing problem, which will be treated separately in section 1.2. At these times the future does not depend on the past - presuming the just-mentioned neglection. So the process falls into interaction blocks with the property that the blocks are separated by times with the Markov property. This is meant by the semi-Markovian character of the process. But in contrast to a semi-Markov process the paths are not constant within the blocks.

In the first section we develop a stochastic model for the distribution of the interaction blocks in dependence of the initial velocity of the heavy particle. The whole process, which consists of connected blocks, will be treated in the third section where we construct an abstract version of processes of this type and indicate methods for the study of these processes. A more detailed introduction to the abstract model will be given in the beginning of that section. The second section, which serves as a preparation for the third one, introduces the space of all paths of finite but variable length, which are right continuous and have left limits, in a locally compact state space with a denumerable base - representing the space of the blocks - and the space of all those paths with a marked time representing the initial block with time zero. We shall give these spaces a topology of the Skorohod type rendering them Polish spaces. Since these spaces are interesting not only in this context, we discuss the topology in more detail than will be necessary for the sequel.

We got the suggestion of constructing processes which fall into interaction blocks as a general model for the Brownian motion by a model of von Waldenfels [8] for the pressure broadening of spectral lines. But his assumptions and methods are quite different from ours, with the exception of section 1.3, where we partially follow his lines. The author wants to thank Professor von Waldenfels for his advice and encouragement, which were invaluable for this article.

1. The distribution of the interaction blocks

1.1 The physical situation

Consider in the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$ a system of particles of mass m starting in a Poisson process with density one, each of them indeoendently given a velocity from a probability measure ν with finite first moment, or in other words: starting in the phase space $\mathbb{R}^\ell \times \mathbb{R}^\ell$ in a $\lambda \otimes \nu$-Poisson process, where λ is the Lebesgue measure on $\mathbb{R}^\ell$. The particles will move with constant velocity. Then the system remains in a $\lambda \otimes \nu$-Poisson process at each time.

Now suppose that a heavy particle of mass $M \gg m$ moves in this system interacting with the light particles via a given potential ϕ. We assume that ϕ is continuously differentiable and has compact support. We allow ϕ to take the value $+\infty$. Fix $R > 0$ with the property: $\phi(x) = 0$ for $|x| \geqslant R$. We call the closed ball around the heavy particle with radius R the interaction region, though we do not claim that R is minimal. As we will derive in this section a stochastic model for the velocity process of the heavy particle from one time, in which no light particle is in the interaction region, until the first time there is again no particle in it, while in the meantime there has been at least one particle in it, - called an interaction block -, we suppose that at time zero the heavy particle has a given velocity V_o and no light particle is in the interaction region. This initial situation is the same as described by Hennion [3].

Two things have to be considered: the distribution of the parameters of the interacting particles, i.e., the times they enter the interaction region and their phase coordinates at these times, and the velocity process of the heavy particle in dependence of the parameters. We did not succeed in deriving an exact model for the described situation, but for $M >> m$ it seems to be a good approximation to it.

As was mentioned above the initial situation is the same as in Hennion [3], and it remains the same until the first time a light particle enters the interaction region. So we can adopt his results concerning the distribution of the interaction parameters of the first interacting particle and the invariance property concerning the system of the other particles at this time. The dimension is not important for that. So let τ_1 be the time, $(x_1(\tau_1), v_1(\tau_1))$ the phase coordinates - the position relative to the heavy particle - of the first interacting particle, then τ_1 and $(x_1(\tau_1), v_1(\tau_1))$ are independently distributed. Let $C_{\ell-1}$ be the volume of the $(\ell-1)$-dimensional unit ball $(C_o = 1)$, set for

$$w \in \mathbb{R}^{\ell},\ w \neq 0 :\ S_{R,w} = \{x \in \mathbb{R}^{\ell},\ |x| = R,\ \langle x,w\rangle \leqslant 0\}$$

and $\sigma_{R,w}$ the $(\ell-1)$-dimensional surface measure of $S_{R,w}$ conceived as a measure on $\mathbb{R}^{\ell}$. Then τ_1 has an exponential distribution with parameter $c(V_o) = C_{\ell-1}R^{\ell-1} \int |V_o-v|\,d\nu(v)$. In order to avoid the case that $c(V_o) = 0$ for one V_o, we exclude that ν is the Dirac-measure of some point. The distribution of $(x_1(\tau_1), v_1(\tau_1))$ is given by

$$P(x_1(\tau_1) \in A,\ v_1(\tau_1) \in B) = \frac{1}{c(V_o)} \int_B \left(- \frac{\langle x, v-V_o\rangle}{R|v-V_o|}\,\sigma_{R,v-V_o}(A)\right)|v-V_o|\,d\nu(v)$$

for A, B Borel sets of $\mathbb{R}^{\ell}$.

Furthermore, the distribution of the system of the other particles is the same as in the initial situation - the positions always relative to the heavy particle. But from now on the situation is different, as the velocity of the heavy particle is no longer constant. But as for the distribution of the following interacting particles within one block, we shall illustrate below that for $M \gg m$ we can approximate it by the distribution we should get if V were constant. So if $\tau_1+\tau_2$ is the arrival time of the second interacting particle and $(x_2(\tau_1+\tau_2),\ v_2(\tau_1+\tau_2))$ are its initial phase coordinates, then $(\tau_2, x_2(\tau_1+\tau_2),\ v_2(\tau_1+\tau_2))$ has the same distribution as $(\tau_1, x_1(\tau_1), v_1(\tau_1))$ with the difference that τ_2 is restricted to the time interval, in which the first particle is in the interaction region - the time it leaves it not included, otherwise we stop the interaction block after the interaction of the first particle; etc. Before we construct a probability space for the parameters of the interacting particles of one block and discuss, under which assumptions only finite particles are within one block with probability one, we turn over to the motion of the heavy particle, as we need the interaction times for that.

Let $V(t)$ be the velocity process of the heavy particle. For $0 \leqslant t < \tau_1$ we have $V(t) = V_0$. For $t \geqslant \tau_1$ we get $V(t)$ from the equations of motion regarded separately between the arrivals of the single particles. As we got the position parameters relative to the heavy particle, we write the equations in terms of the motion of the centre of gravity and the motions of the light particles with respect to the heavy particles. So for t between $\tau_1+\ldots+\tau_n$ and $\tau_1+\ldots+\tau_{n+1}$ resp. the end of the block we set:

v_i = velocity of the i-th particle $(i = 1, \ldots, n)$

x_i = position of the i-th particle relative to the heavy particle

$$w_i = \dot{x}_i = v_i - V$$

$$V_g^{(n)} = \frac{M}{M+n\cdot m} V + \sum_{i=1}^{n} \frac{m}{M+n\cdot m} v_i$$

Then we get the equations:

$$\dot{V}_g^{(n)} = 0$$

$$\mu \dot{w}_i = - \operatorname{grad} \phi(x_i) - \gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} \operatorname{grad} \phi(x_j)$$

with the reduced mass $\mu \left(\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M}\right)$ and $\gamma = \frac{m}{M+m}$

For $M \gg m$ we have $\gamma \ll 1$ and may neglect the term $- \gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} \operatorname{grad} \phi(x_j)$. We have to do this not because more-particle-problems are not solvable - this is not important to us, since we shall not give explicit solutions -, but because otherwise there arise too great difficulties concerning the interaction times. The neglection of this term means that all particles interact independently of the others, and as a consequence the interaction time of the particles is not influenced by the other particles. If we regard this equation also for the particles outside the interaction region and neglect this term, too, then we shall get $\mu\dot{w} = 0$, which corresponds to our approximation concerning the distribution of the parameters.

The initial conditions $V(\tau_1+\ldots+\tau_n)$, $x_i(\tau_1+\ldots+\tau_n)$, $w_i(\tau_1+\ldots+\tau_n)$ $(i=1,\ldots,n-1)$ are given by the final values of the preceding equation of motion, the initial conditions $x_n(\tau_1+\ldots+\tau_n)$, $v_n(\tau_1+\ldots+\tau_n)$ are given by the parameters of the n-th particle, but remark that in accordance with our approximation we have to put

$$w_n(\tau_1+\ldots+\tau_n) = v_n(\tau_1+\ldots+\tau_n) - V_0.$$

From them we can derive the initial conditions $V_g^{(n)}(\tau_1+\ldots+\tau_n)$, $x_i(\tau_1+\ldots+\tau_n)$, $w_i(\tau_1+\ldots+\tau_n)$ $(i = 1,\ldots,n)$. Since we do not regard the position process of the heavy particle, the initial position of the centre of gravity can be chosen arbitrarily.

From the solution of the differential equations with the given initial conditions, we get $V(t)$ by transforming back:

$$V(t) = V_g^{(n)}(t) - \sum_{i=1}^{n} \frac{m}{M+n\cdot m} w_i(t)$$

By this we get the process V in dependence of the interaction parameters.

1.2 A queuing problem

Before dealing with the distribution of the interaction parameters, we want to treat, apart from the following, a queuing problem combined with the finiteness of particle numbers within one block. For its description let us regard the facts we need for the problem. Particles enter the interaction region, their interarrival times being independently distributed with an exponential distribution with parameter $c = c(V_0)$. The independence of the interactions of the single particles and the fact that the interaction of each particle is governed by the same differential equation having the same distribution of initial values imply that the times the single particles stay in the interaction region are independent of their arrival times and are mutually independently distributed with a common distribution on $\mathbb{R}^+$, provided that with probability one no particle has an infinite interaction time. Let $F = F_{V_0}$ be their distribution function.

The described situation corresponds to a queue with exponentially distributed interarrival times, service times with distribution dF and an infinite number of counters, i.e., the service of a customer begins directly at his arrival. We ask for the existence of times when no counter is occupied. Let $Y(t)$ be the time after which all customers being present at time t are served. $Y(t)$ is a Markov process. For if $Y(t)$ is known, $Y(t+s)$ only depends on the arrival and service times of the customers arriving at the time interval $(t,t+s)$ and not of those arrived before the time t, since the time after which they are all served is known. If $Y(t) = 0$, then it remains 0 until the arrival of the next customer and then jumps to his service time. If $Y(t) = x>0$, then it decreases with slope -1 until the arrival of the next customer resp. until it takes the value 0, if in the meantime no customer arrived. In the first case let $x-\xi$ be the value of Y at that time. If the service time of the arriving customer is less or equal than $x-\xi$, then the process continues as before. If not, the process jumps to the service time of the customer. Now let $W_n(x)$ be the probability that during the time of the arrival and departure of the next n customers the process Y did not take the value 0 if $Y(t) = x$. Of course, for the derivation of W_n the knowledge of the process Y is not sufficient, since customers with a service time less or equal than the value of Y at their arrival do not influence the process Y.

Corresponding to our problem we are looking for a criterion where W_n converges to 0.

Of course, we have $W_0(x) = 1-e^{-cx}$.

For $n\geq 1$, W_n can be derived recursively. The event that during the arrival and the departure of the next n customers Y does not take the value 0 implies that during the remaining service times of the preceding customers one customer arrives. At his arrival the situation is the same with n replaced by $n-1$. So we get for $n\geq 1$

$$W_n(x) = \int_0^x ce^{-c\xi}\,[F(x-\xi)W_{n-1}(x-\xi) + \int_{(x-\xi)+}^{\infty} W_{n-1}(\zeta)dF(\zeta)]\,d\xi$$

If we set $W_{-1} \equiv 1$, then the recursion formula is also valid for $n=0$. There follows by induction that for $n\geq 0$, $W_n\leq W_{n-1}$ holds.

Let $W = \lim_{n\to\infty} W_n$. Then by monotone convergence we get for W

the equation:

$$(*)\quad W(x) = \int_0^x ce^{-c\xi}\left[F(x-\xi)W(x-\xi) + \int_{(x-\xi)+}^{+\infty} W(\zeta)\,dF(\zeta)\right]d\xi \quad (x \geq 0)$$

Furthermore, W is the maximal solution of (*), which is less or equal than 1. For let $\overline{W} \leq 1$ be another solution. Then by induction we get $\overline{W} \leq W_{n-1}$, which implies $\overline{W} \leq W$. So we have to study the solutions of (*).

<u>Lemma 1:</u> There exist non-negative bounded solutions of (*), which are different from the trivial solution $W \equiv 0$, iff $\int_0^\infty e^{-c\int_0^\zeta (1-F(\xi))d\xi}\,d\zeta < \infty$. In this case they are given by

$$W(x) = a\int_0^x e^{-c\int_0^\zeta (1-F(\xi))d\xi}\,d\zeta \text{ with } 0<a = W'(0) = c\int_0^\infty W(\zeta)\,dF(\zeta).$$

<u>Theorem 1:</u> $\lim_{n\to\infty} W_n = 0 \iff \int_0^\infty e^{-c\int_0^\zeta (1-F(\xi))d\xi}\,d\zeta = +\infty.$

This is especially fulfilled if F has a finite first moment.
In the opposite case we have

$$\lim_{n\to\infty} W_n(x) = \left(\int_0^\infty e^{-c\int_0^\zeta (1-F(\xi))d\xi}\,d\zeta\right)^{-1} \int_0^x e^{-c\int_0^\zeta (1-F(\xi))d\xi}\,d\zeta$$

<u>Proof:</u> The right side of the equation (*) is a convolution. We transform it.

$$W(x) = \int_0^x ce^{-c(x-\xi)}\left[F(\xi)W(\xi) + \int_{\xi+}^\infty W(\zeta)\,dF(\zeta)\right]$$

$$= \int_0^x ce^{-c(x-\xi)}\left[W(\xi) + \int_\xi^\infty \left(W(\zeta) - W(\xi)\right)dF(\zeta)\right]d\xi$$

This shows us that W is differentiable and that (*) is equivalent to:

$$W'(x) = c\int_x^\infty \left(W(\zeta) - W(x)\right)dF(\zeta) \quad \text{with} \quad W(0) = 0.$$

A consequence of this equation is $W'(0) = c\int_0^\infty W(\zeta)\,dF(\zeta)$. Partial integration yields

$$W'(x) = c\int_x^\infty (1-F(\zeta))W'(\zeta)\,d\zeta$$

Differentiating this equation we get a differential equation for W' with the solutions:

$$W'(x) = a\cdot e^{-c\int_0^x (1-F(\xi))d\xi} \quad \text{with arbitrary } a = W'(0).$$

But for the inverse conclusion we have to add the condition $W'(x) \to 0$ for $x \to +\infty$. For $a \neq 0$ the given solution fulfills this, iff the first moment of F is infinite since

$$\int_0^\infty (1-F(\xi))d\xi = \int_0^\infty \xi\, dF(\xi).$$

In this case we get the final solution: $W(x) = a\int_0^x e^{-c\int_0^\zeta (1-F(\xi))d\xi}\, d\zeta$

This solution is bounded for $a \neq 0$, iff $\int_0^\infty e^{-c\int_0^\zeta (1-F(\xi))d\xi}\, d\zeta < \infty$.

Since this condition implies that the first moment of F is infinite the lemma is proved.

Since $\lim W_n$ is the maximal solution below one, the first assertion of the Theorem is clear. If $\int_0^\infty e^{-c\int_0^\zeta (1-F(\xi))d\xi}\, d\zeta < \infty$,

then the solutions $W(x) = a\int_0^x e^{-c\int_0^\zeta (1-F(\xi))d\xi}\, d\zeta$ are monotonically increasing and depend in an increasing manner on a. So $\lim W_n$ is the solution of $(*)$ with $W(x) \to 1$ for $x \to \infty$ which is the stated one.

1.3 The distribution of the interaction parameters

As was described in 1.1, for given V_0, the stochastic behaviour of the interaction block only depends on the distribution of the interaction parameters of the particles. So they will form our basic sample space. We shall state the parameters as the interarrival times of the particles and their phase coordinates where it is convenient to choose $(x_i(\tau_1+\ldots+\tau_i),\ w_i(\tau_1+\ldots+\tau_i))$ as phase parameters of the i-th particle, since these are the initial conditions for the differential equation $\mu \dot{w}_i = -\operatorname{grad}\phi(x_i)$. So the space of the interaction parameters of one particle is $\mathbb{R}^+ \times Z$ with $Z = \mathbb{R}^\ell \times \mathbb{R}^\ell$, and our sample space is the space of all finite sequences in $\mathbb{R}^+ \times Z$, i.e. the topological sum $\Sigma = \bigcup_{k=1}^\infty (\mathbb{R}^+ \times Z)^k$.

In order to derive the probability measure on Σ, which corresponds to our assumptions, we start from the space $\mathfrak{Z} = (\mathbb{R}^+ \times Z)^{\mathbb{N}}$, where $\mathbb{N}$ is the set of the natural numbers. Let $p_{V_o}^{(1)}$ be the exponential distribution on $\mathbb{R}^+$ with parameter $c(V_o)$, and let $p_{V_o}^{(2)}$ be the distribution on Z derived from the distribution for the first particle as defined in 1.1 by a velocity space translation by V_o. On $\mathfrak{Z}$ we have the probability measure $p_{V_o} = (p_{V_o}^{(1)} \otimes p_{V_o}^{(2)})^{\otimes \mathbb{N}}$, from which we get the desired probability measure on Σ by stopping after the interaction parameters of the interaction block.

For this purpose we have to study the solutions of $\mu \dot{w} = -\text{grad}\,\phi(x)$ in dependence on the initial values in Z, especially with respect to their sojourn time in the interaction region. Though we supposed ϕ to be continuously differentiable, it may not be regular enough for our purposes. Here we cannot give explicit conditions under which ϕ is admissible, which may also depend on the measure ν. We shall only discuss the conditions arising from the probabilistic treatment. First we have to exclude with probability one singular solutions of the differential equation, since they are not unique and cause infinite interaction times. So we suppose that for each $V_o \in \mathbb{R}^\ell$ the initial values in Z, which yield singular solutions, are $p_{V_o}^{(2)}$-negligible. This is for example satisfied, if $\text{grad}\,\phi \neq 0$ in the interior of the interaction region. Now $p_{V_o}^{(2)}$-almost every $z \in Z$ is assigned an interaction time $\theta(z)$. θ is $p_{V_o}^{(2)}$-measurable and the image of $p_{V_o}^{(2)}$ under θ gives us the distribution of the interaction times. Let F_{V_o} be its distribution function.

Concerning the problem under which conditions the interaction block is terminated after a finite number of interacting particles with probability one, we now have the situation described in 1.2. So we have to presume that ϕ and ν yield distribution functions F_{V_o} with the derived condition. In most cases it will amount to the finiteness of the first moment. Here we shall only specify, for which parameters long interaction times are possible. First they occur for small velocities, but their influence is in general cancelled by the factor $|v|$ in the measure $p_{V_o}^{(2)}$ (see 1.1). Furthermore, they are possible for initial conditions near those which yield singular solutions, and possibly for great velocities, if ϕ is unbounded. A possibility of avoiding this problem would be to truncate the single interactions after a fixed time $\Theta > 0$.

We are now able to finish the derivation of the probability measure.

For $k \geq 1$ let $\mathfrak{Z}_k \subset \mathfrak{Z}$ be the set of interaction parameters, which cause a stopping of the block after the k-th particle, i.e., the set of $(\tau_i, z_i)_{i \geq 1} \in \mathfrak{Z}$ with the property

$$\min\{j:\ \tau_1+\ldots+\tau_j \geq \max(\tau_1+\theta(z_1),\tau_1+\tau_2+\theta(z_2),\ldots,\tau_1+\ldots+\tau_{j-1}+\theta(z_{j-1}))\}=k+1$$

$\mathfrak{Z}_k$ is a p_{V_o}-measurable subset of $\mathfrak{Z}$ for each $V_o \in \mathbb{R}^\ell$. On $\mathfrak{Z}_k$ we define the p_{V_o}-measurable mapping α_k:

$$\alpha_k : \mathfrak{Z}_k \longrightarrow \Sigma$$

$$\alpha_k \left((\tau_i, z_i)_{i \geq 1} \right) = (\tau_i, z_i)_{i=1}^k$$

Since the $\mathfrak{Z}_k$ are disjoint, we can combine them getting a p_{V_o}-measurable mapping α:

$$\alpha : \mathfrak{Z}_f := \bigcup_{k=1}^{\infty} \mathfrak{Z}_k \longrightarrow \Sigma$$

$$\alpha \mid \mathfrak{Z}_k = \alpha_k$$

The assumptions concerning the finite numbers of particles in one interaction block just mean $p_{V_o}(\mathfrak{Z}_f) = 1$. We now get our desired probability measure P_{V_o} on Σ as the image of the probability measure p_{V_o} restricted to the measurable sets of $\mathfrak{Z}_f$ under the mapping α.

The process of one interaction block can now easily be defined. The lifetime of one block is given by a P_{V_o}-measurable mapping ζ :

$$\zeta : \Sigma \longrightarrow \mathbb{R}^+$$

$$\zeta\left((\tau_i, z_i)_{i=1}^k \right) = \max\{ \tau_1 + \theta(z_1), \ldots, \tau_1 + \ldots + \tau_k + \theta(z_k) \}.$$

Finally we get the paths $V(t)$ $(0 \leq t < \zeta)$ by the prescription of 1.1. The coordinate mappings $V(t)$ are P_{V_o}-measurable on Σ, because of the continuous dependence of the solutions of the differential equations from those initial values, which yield regular solutions.

We still remark that the definition of p_{V_o} easily implies that for a Borel set $A \subset \mathfrak{Z}$ the mapping $V_o \mapsto p_{V_o}(A)$ is measurable, and hence for a Borel set $B \subset \Sigma$ the mapping $V_o \mapsto P_{V_o}(B)$ is measurable.

In the third section we shall start with the final results in order to give a canonical representation of the distribution of the interaction blocks and then shall join them to the velocity process. We shall do this in an abstract model. For that purpose we have to introduce a space for the blocks, which will be done in the next section.

2 The spaces $D_f^+(E)$ and $\check{D}_f^+(E)$

Let E be a compact metrisable space and $I \subset \mathbb{R}$ be an interval. Then by $D_I(E)$ we denote the set of all mappings $\omega: I \to E$, which are right continuous and have left limits. For $I = \mathbb{R}^+$ we set $D_{\mathbb{R}^+}(E) =: D^+(E)$. On $D_I(E)$ one can define a metric, under which $D_I(E)$ becomes a Polish space (see Maisonneuve [5]). We shall extend the definitions and results to the case where the domain is not a fixed, but a variable finite interval. We set:

> $D_f^+(E)$ is the set of all mappings $\omega : [0,\zeta) \longrightarrow E$ which are right continuous and have left limits, even in ζ, where $0 < \zeta = \zeta(\omega) < \infty$ depends on ω and is called the lifetime of ω.

We want to define on $D_f^+(E)$ a metric, under which $D_f^+(E)$ is separable and complete.

For $0 < \zeta, \zeta' < \infty$ we denote by $\mathcal{T}_\zeta^{\zeta'}$ the set of all mappings $\tau: [0,\zeta) \longrightarrow [0,\zeta')$ which are strictly increasing and surjective.

For $\tau \in \mathcal{T}_\zeta^{\zeta'}$ we set $|\tau| := \sup\limits_{0 \leqslant t < \zeta} \{|\tau(t)-t| + \sup\limits_{\substack{0 \leqslant s < \zeta \\ s \neq t}} |\log \dfrac{\tau(t)-\tau(s)}{t-s}|\} \leqslant \infty$.

We have the following properties:

<u>Lemma 2:</u> Let $0 < \zeta, \zeta', \zeta'' < \infty$.

i) $\tau \in \mathcal{T}_\zeta^{\zeta'} \Rightarrow \tau^{-1} \in \mathcal{T}_{\zeta'}^{\zeta}, \ |\tau| = |\tau^{-1}|$

ii) $\tau \in \mathcal{T}_\zeta^{\zeta'}, \ \sigma \in \mathcal{T}_{\zeta'}^{\zeta''} \Rightarrow \sigma \circ \tau \in \mathcal{T}_\zeta^{\zeta''}, \ |\sigma \circ \tau| \leqslant |\sigma| + |\tau|$

iii) There exists a $\tau \in \mathcal{T}_\zeta^{\zeta'}$ with $|\tau| < \infty$

iv) $\tau \in \mathcal{T}_\zeta^{\zeta'} \Rightarrow |\tau| \geqslant |\zeta - \zeta'| + |\log \frac{\zeta'}{\zeta}|$

<u>Proof:</u> i) and ii) are trivial. For iii) we may take $\tau(t) = \frac{\zeta'}{\zeta} \cdot t$.
iv) In the sup of the definition of $|\tau|$ we take $t \uparrow \zeta$ and $s=0$.

Now we are able to define the metric on $D_f^+(E)$. For this we fix a metric ρ on E, which induces the topology of E.
On $D_f^+(E)$ we define the metric:

$$d(\omega,\omega') := \inf_{\tau \in \mathcal{T}_{\zeta(\omega)}^{\zeta(\omega')}} \{|\tau| + \sup_{0 \leqslant t < \zeta(\omega)} \rho(\omega(t), \omega'(\tau(t)))\} \qquad (\omega, \omega' \in D_f^+(E)).$$

The fact that d is finite follows from Lemma 2iii). The axioms of a metric can easily be derived from Lemma 2i) and ii).

<u>Theorem 2:</u> $D_f^+(E)$ is separable and complete under d.

Proof: Concerning the separability the set of all step functions with values of a fixed denumerable dense subset of E and rational discontinuity points and lifetimes form a denumerable dense subset of $D_f^+(E)$. Because of the finiteness of the lifetimes we do not need the exponential factor occurring in the definition of d in [5] for it.

The proof of the completeness is just the same as in the case of a fixed interval with two additional remarks. We have to verify that all compositions are possible, which is trivial, and that the domain of the limit cannot degenerate to a single point. But the log-term in Lemma 2iv) shows that this possibility is excluded.

Now let E be a locally compact space with denumerable base. We define $D_f^+(E)$ in the same way as we did in the compact case. Let $E' = E \cup \{\Delta\}$ be the Alexandrov compactification of E. E' is a compact metrisable space. We can imbed $D_f^+(E)$ in a canonical way in $D_f^+(E')$ and supply $D_f^+(E)$ with the relative topology. If E is compact itself, then Δ is an isolated point and so the definitions of $D_f^+(E)$ and its topology are consistent. We return to the general case.

Proposition 1: $D_f^+(E)$ is an open subset of $D_f^+(E')$.

Proof: Let $\omega \in D_f^+(E)$. Then it is clear that

$$\delta = \inf_{0 \leqslant t < \zeta(\omega)} \rho(\omega(t), \Delta) > 0$$

and $$\{\omega' : d(\omega, \omega') < \delta\} \subset D_f^+(E).$$

This proposition implies that $D_f^+(E)$ is a Polish space.

We finish the discussion of $D_f^+(E)$ by characterizing the topology of $D_f^+(E)$ by continuous mappings, which also implies a characterization of the σ-algebra of the Borel sets.

Lemma 2iv) shows that ζ is continuous. Let π_0 and π_ζ be the continuous mappings:

$$\pi_0 : D_f^+(E) \longrightarrow E \quad \text{and} \quad \pi_\zeta : D_f^+(E) \longrightarrow E$$
$$\omega \longmapsto \omega(0) \qquad\qquad \omega \longmapsto \omega(\zeta-)$$

Finally, by continuing each mapping $\omega \in D_f^+(E)$ on $\mathbb{R}^+$ by giving it the value Δ for $t \geqslant \zeta(\omega)$ we get a mapping I from $D_f^+(E)$ into $D^+(E')$.

Proposition 2: i) The mappings ζ, π_0 and π_ζ are continuous.
ii) I is an injective continuous mapping and $I(D_f^+(E))$ is a Borel subset of $D^+(E')$.

<u>Proof of ii):</u> The injectivity of I is clear. For the proof of the continuity let $(\omega_n)_{n\geq 1}$ be a sequence in $D_f^+(E)$ and $\omega \in D_f^+(E)$ with $d(\omega_n,\omega) \to 0$, which means the existence of $\tau_n \in \mathcal{T}^{\zeta(\omega_n)}_{\zeta(\omega)}$ with $|\tau_n| \to 0$ and $\rho(\omega_n(\tau_n(t)),\omega(t)) \longrightarrow 0$ uniformly on $[0, \zeta(\omega))$.

By $\mathcal{T}^+$ we denote the set of all strictly increasing surjective mappings from $\mathbb{R}^+$ to $\mathbb{R}^+$. For $\tilde{\tau} \in \mathcal{T}^+$ we set

$$|\tilde{\tau}| := \sup_{t\geq 0} \{|\tilde{\tau}(t)-t| + \sup_{\substack{s\geq 0\\ s\neq t}} |\log \frac{\tilde{\tau}(t)-\tilde{\tau}(s)}{t-s}|\} \leq +\infty .$$

Let $\tilde{\tau}_n \in \mathcal{T}^+$ be defined by

$$\tilde{\tau}_n(t) = \begin{cases} \tau_n(t) & 0\leq t< \zeta(\omega) \\ \zeta(\omega_n)+t-\zeta(\omega) & t\geq \zeta(\omega) \end{cases}$$

Then $\tilde{\tau}_n \in \mathcal{T}^+$ with $|\tilde{\tau}_n| = |\tau_n|$ and we have $\rho((I(\omega_n)\circ\tilde{\tau}_n)(t),I(\omega)(t))\to 0$ uniformly on $\mathbb{R}^+$. This implies the convergence of $I(\omega_n)$ to $I(\omega)$ in the topology of $D^+(E')$. So I is continuous.

Because of the right continuity we have

$$I(D_f^+(E)) = \bigcap_{n\geq 1} \bigcup_{k\geq 0} [\{ \tilde{\omega} \in D^+(E') : \inf_{t\leq \frac{k}{2^n}, t \text{ rat}} \rho(\tilde{\omega}(t), \Delta) > 0\} \cap$$

$$\cap \{ \tilde{\omega} \in D^+(E') : \tilde{\omega}(t) = \Delta, t > \frac{k+1}{2^n}, t \text{ rat}\}] \setminus \{[\Delta]\}$$

$[\Delta]$ is the constant mapping $\tilde{\omega} \equiv \Delta$.

Since for each $t \geq 0$ the coordinate mapping $\tilde{\omega} \longmapsto \tilde{\omega}(t)$ is measurable on $D^+(E')$, $I(D_f^+(E))$ is a Borel subset of $D^+(E')$.

<u>Proposition 3:</u> The topology of $D_f^+(E)$ is the weakest one, for which I, ζ and π_ζ are continuous.

<u>Corollary 1:</u> The σ-algebra of the Borel sets of $D_f^+(E)$ is generated by the coordinate mappings $X_t : \omega \longmapsto \omega(t)$ $(t\geq 0)$ extended by setting $X_t(\omega) = \Delta$ for $t \geq \zeta(\omega)$.

<u>Proof:</u> For the proof of Proposition 3 it remains to show:

If $(\omega_n)_{n\geq 1}$ is a sequence in $D_f^+(E)$ and $\omega \in D_f^+(E)$ with $I(\omega_n) \longrightarrow I(\omega)$, $\zeta(\omega_n) \longrightarrow \zeta(\omega)$ and $\pi_\zeta(\omega_n) \longrightarrow \pi_\zeta(\omega_n)$, then $\omega_n \longrightarrow \omega$.

Let $\tilde{\tau}_n \in \mathcal{T}^+$ with $|\tilde{\tau}_n| \to 0$ and $\rho((I(\omega_n)\circ \tau_n)(t),I(\omega)(t))\to 0$ locally uniformly. Since $\zeta(\omega)$ is a discontinuity point of $I(\omega)$, there exists n_0 with $\tilde{\tau}_n(\zeta(\omega)) = \zeta(\omega_n)$ for $n \geq n_0$. This conclusion is somewhat easier to prove than a similar one occurring in the proof of Theorem 3i). So we omit it here and refer to that proof.

For $n \geqslant n_o$ let τ_n be the restriction of $\tilde{\tau}_n$ to the interval $[0, \zeta(\omega))$. Then $\tau_n \in \mathcal{J}^{\zeta(\omega_n)}_{\zeta(\omega)}$ with $|\tau_n| \leqslant |\tilde{\tau}_n|$ and $\rho(\omega_n(\tau_n(t)), \omega(t)) \to 0$ uniformly on $[0, \zeta(\omega))$, which just means $\omega_n \to \omega$.

The Corollary follows from the fact that the corresponding assertion is true for $D^+(E)$ and that ζ and π_ζ are measurable with respect to the σ-algebra generated by the coordinate mappings.

If we regard on $D^+_f(E)$ the weakest topology for which I is continuous, then Proposition 2ii) implies that $D^+_f(E)$ is in this topology a Suslin space. By Corollary 1 the two topologies induce the same σ-algebra of Borel sets. But one can easily show that in Proposition 2 none of the mappings can be omitted. So the topology defined above is strictly finer and more adapted to $D^+_f(E)$.

Next we define the space of mappings $\omega \in D^+_f(E)$, whereby one point of the interval $[0,\zeta(\omega))$ is marked, which divides ω into two parts. So we set:

$$\check{D}^+_f(E) := \{(\omega,u) : \omega \in D^+_f(E),\ 0 \leqslant u < \zeta(\omega)\}.$$

$\check{D}^+_f(E)$ will not be regarded as a subspace of $D^+_f(E) \times \mathbb{R}^+$, since we shall supply it with a finer topology than the corresponding relative topology. But for the definition of the final topology we also need this relative topology, which in the following we shortly call the relative topology without referring to $D^+_f(E) \times \mathbb{R}^+$. The reason for the construction of a finer topology is again that we shall render mappings continuous, which are in a natural way linked with $\check{D}^+_f(E)$, namely the restrictions to the two parts of the mappings, where the restriction to the first part is not defined on the whole space. We set

$$\Pi_1 : \{(\omega, u) \in \check{D}^+_f(E) : u > 0\} \to D^+_f(E)$$

$$\zeta(\Pi_1(\omega, u)) = u$$

$$\Pi_1(\omega, u) = \omega|_{[0, u)}$$

$$\Pi_2 : \check{D}^+_f(E) \to D^+_f(E)$$

$$\zeta(\Pi_2(\omega, u)) = \zeta(\omega) - u$$

$$\Pi_2(\omega, u)(t) = \omega(t+u) \qquad (0 \leqslant t < \zeta(\Pi_2(\omega, u))$$

Since Π_1 is not defined on the whole space we cannot directly define a topology on $\check{D}^+_f(E)$ by Π_1 and Π_2. We have to proceed in a different way. For that purpose we define the mappings:

$\check{\pi}: \check{D}^+_f(E) \to E$ $\qquad$ $\check{\pi}_-: \check{D}^+_f(E) \to E$

$\check{\pi}(\omega, u) = \omega(u)$ $\qquad$ $\check{\pi}_-(\omega, u) = \omega(u-)$ for $u > 0$; $\check{\pi}_-(\omega,0)=\omega(0)$.

Now, for topologies, which are finer than the relative topology, the continuity of Π_1 and Π_2 and the continuity of $\check{\pi}$ and $\check{\pi}_-$ are equivalent. If Π_1 and Π_2 are continuous, then so are $\check{\pi} = \pi_0 \circ \Pi_2$ and $\check{\pi}_-|_{\{(\omega, u) \in \check{D}_f^+(E):\, u>0\}} = \pi_\zeta \circ \Pi_1$. As for the continuity of $\check{\pi}_-$ in the points $(\omega, 0)$ we do not need the continuity of Π_1 and Π_2. It is a consequence of the following Theorem 3ii). The converse of the equivalence assertion will be stated in Corollary 2.

So we supply $\check{D}_f^+(E)$ with the weakest topology which is finer than the relative topology and under which $\check{\pi}$ and $\check{\pi}_-$ are continuous. Then $\check{D}_f^+(E)$ is a Polish space, too.

The definition of the topology of $\check{D}_f^+(E)$ can directly be transformed to a definition by means of convergent sequences. We shall show the equivalence to a formally different notion of convergence, which yields the continuity of Π_1 and Π_2.

<u>Theorem 3:</u> Let $(\omega_n, u_n)_{n \geq 1}$ be a sequence in $\check{D}_f^+(E)$ and $(\omega, u) \in \check{D}_f^+(E)$.

i) If $u > 0$, then $(\omega_n, u_n) \to (\omega, u)$, iff there exist $\tau_n \in \mathcal{T}_{\zeta(\omega)}^{\zeta(\omega_n)}$ with $\tau_n(u) = u_n$, $|\tau_n| \to 0$ and $\rho(\omega_n(\tau_n(t)), \omega(t)) \to 0$ uniformly on $[0, \zeta(\omega))$.

ii) If $u = 0$, then $(\omega_n, u_n) \to (\omega, 0)$, iff $\omega_n \to \omega$ and $u_n \to 0$. In this case there exists $\tau_n \in \mathcal{T}_{\zeta(\omega)}^{\zeta(\omega_n)-u_n}$ with $|\tau_n| \to 0$ and $\rho(\omega_n(\tau_n(t)+u_n), \omega(t)) \to 0$ uniformly on $[0, \zeta(\omega))$.

<u>Corollary 2:</u> Π_1 and Π_2 are continuous.

<u>Proof:</u> i) One direction of the equivalence is trivial, namely, that the stated conditions imply the convergence.

So assume $(\omega_n, u_n) \to (\omega, u)$ with $u > 0$.

We have to distinguish the cases, where u is a discontinuity or a continuity point of ω.

First case: u is a discontinuity point of ω.
Let $\tau_n \in \mathcal{T}_{\zeta(\omega)}^{\zeta(\omega_n)}$ with $|\tau_n| \to 0$ and $\rho(\omega_n(\tau_n(t)), \omega(t)) \to 0$ uniformly on $[0, \zeta(\omega))$. We shall show that for n great enough $\tau_n(u) = u_n$ holds.

Let $\eta := \rho(\omega(u-), \omega(u)) > 0$.

There exists $\delta > 0$ with $[u - \delta, u + \delta) \subset [0, \zeta(\omega))$ and

$$\rho(\omega(t)), \omega(u)) < \tfrac{\eta}{4} \quad \text{for} \quad u \leq t < u + \delta$$

$$\rho(\omega(t)), \omega(u-)) < \tfrac{\eta}{4} \quad \text{for} \quad u - \delta \leq t < u$$

Now choose n_o such that for $n \geqslant n_o$ the following inequalities hold:

$$|\tau_n| < \frac{\delta}{2}$$
$$|u_n - u| < \frac{\delta}{2}$$
$$\rho(\omega_n(u_n), \omega(u)) < \frac{\eta}{2}$$
$$\rho(\omega_n(u_n-), \omega(u-)) < \frac{\eta}{2}$$
$$\rho(\omega_n(\tau_n(t)), \omega(t)) < \frac{\eta}{4} \quad \text{on } [0, \zeta(\omega))$$

This n_o satisfies $\tau_n(u) = u_n$ for $n \geqslant n_o$, which we shall show now. Let $n \geqslant n_o$. For $u - \delta \leqslant t < u + \delta$ we have:

$$\rho(\omega_n(\tau_n(t)), \omega(u-)) \leqslant \rho(\omega_n(\tau_n(t)), \omega(t)) + \rho(\omega(t), \omega(u-)) < \frac{\eta}{2}$$

and for $u \leqslant t < u + \delta$ we have

$$\rho(\omega_n(\tau_n(t)), \omega(u)) \leqslant \rho(\omega_n(\tau_n(t)), \omega(t)) + \rho(\omega(t), \omega(u)) < \frac{\eta}{2}$$

Since $\rho(\omega(u), \omega(u-)) = \eta$, $\rho(x, \omega(u-)) < \frac{\eta}{2}$ will imply $\rho(x, \omega(u)) > \frac{\eta}{2}$ and $\rho(x, \omega(u)) < \frac{\eta}{2}$ will imply $\rho(x, \omega(u-)) > \frac{\eta}{2}$. For $u - \delta \leqslant t < u + \delta$ we thus conclude:

$$u - \delta \leqslant t < u \iff \rho(\omega_n(\tau_n(t)), \omega(u-)) < \frac{\eta}{2}$$
$$u \leqslant t < u + \delta \iff \rho(\omega_n(\tau_n(t)), \omega(u)) < \frac{\eta}{2}$$

Now let $v_n = \tau_n^{-1}(u_n)$. Then we have:

$$|v_n - u| \leqslant |v_n - u_n| + |u_n - u| < \delta$$
$$\rho(\omega_n(\tau_n(v_n)), \omega(u)) < \frac{\eta}{2} \implies u \leqslant v_n < u + \delta$$
$$\rho(\omega_n(\tau_n(v_n)-), \omega(u-)) < \frac{\eta}{2} \implies u - \delta \leqslant v_n \leqslant u$$

So $v_n = u$ qed.

Second case: u is a continuity point of ω.

Let again $\tau_n \in \mathcal{T}_{\zeta(\omega)}^{\zeta(\omega_n)}$ with $|\tau_n| \to 0$ and $\rho(\omega_n(\tau_n(t)), \omega(t)) \to 0$ uniformly on $[0, \zeta(\omega))$. In this case we cannot expect $\tau_n(u) = u_n$ for n great enough. We have to take advantage of the continuity of ω in u to change τ_n in a neighborhood of u yielding that u is mapped into u_n without changing the convergence properties. For this purpose we only need the convergence of (ω_n, u_n) to (ω, u) in the relative topology. The change of τ_n will be performed by composing τ_n with suitable $\sigma_n \in \mathcal{T}_{\zeta(\omega)}^{\zeta(\omega)}$.

The inequality $|\tau_n^{-1}(u_n) - u| \leqslant |\tau_n^{-1}(u_n) - u_n| + |u_n - u| \leqslant |\tau_n^{-1}| + |u_n - u|$ implies the convergence $\tau_n^{-1}(u_n) \to u$. So we can choose $\delta_n > 0$ with $\delta_n \to 0$ and $\frac{|\tau_n^{-1}(u_n) - u|}{\delta_n} \to 0$. Furthermore, we may assume $[u - \delta_n, u + \delta_n) \subset [0, \zeta(\omega))$ and $\frac{|\tau_n^{-1}(u_n) - u|}{\delta_n} < 1$ for all n.

We define $\sigma_n \in \mathcal{T}_{\zeta(\omega)}^{\zeta(\omega)}$ by:

$$\sigma_n(t) = \begin{cases} t & \text{for } 0 \leqslant t < u-\delta_n \text{ and } u+\delta_n \leqslant t < \zeta(\omega) \\ \tau_n^{-1}(u_n) + \dfrac{\tau_n^{-1}(u_n)-(u-\delta_n)}{\delta_n}(t-u) & \text{for } u-\delta_n \leqslant t < u \\ \tau_n^{-1}(u_n) + \dfrac{u+\delta_n-\tau_n^{-1}(u_n)}{\delta_n}(t-u) & \text{for } u \leqslant t < u+\delta_n \end{cases}$$

We thus get $\sigma_n \in \mathcal{T}_{\zeta(\omega)}^{\zeta(\omega)}$ with $\sigma_n(u) = \tau_n^{-1}(u_n)$ and $|\sigma_n| \rightarrow 0$. The last fact follows from the easy estimations:

$$|\sigma_n(t)-t| \leqslant |\tau_n^{-1}(u_n)-u| \quad \text{for } 0 \leqslant t < \zeta(\omega)$$

$$\log\left|\frac{\sigma_n(t)-\sigma_n(s)}{t-s}\right| \leqslant \max\left(\log\left|1+\frac{\tau_n^{-1}(u_n)-u}{\delta_n}\right|, \log\left|1-\frac{\tau_n^{-1}(u_n)-u}{\delta_n}\right|\right)$$

$$\leqslant \frac{\dfrac{|\tau_n^{-1}(u_n)-u|}{\delta_n}}{1-\dfrac{|\tau_n^{-1}(u_n)-u|}{\delta_n}} \quad \text{for } 0 \leqslant t,\ s < \zeta(\omega),\ t \neq s$$

Set $\tau'_n := \tau_n \circ \sigma_n$. Then $\tau'_n \in \mathcal{T}_{\zeta(\omega)}^{\zeta(\omega_n)}$ with $\tau'_n(u) = u_n$ and $|\tau'_n| \rightarrow 0$. We must still show the uniform convergence

$\rho(\omega_n(\tau'_n(t), \omega(t)) \longrightarrow 0$ on $[0, \zeta(\omega))$.

So

$$\rho(\omega_n(\tau'_n(t)), \omega(t)) \leqslant \rho(\omega_n(\tau_n(\sigma_n(t))), \omega(\sigma_n(t))) + \rho(\omega(\sigma_n(t)), \omega(t))$$

The uniform convergence $\rho(\omega_n(\tau_n(s)), \omega(s)) \rightarrow 0$ on $[0,\zeta(\omega))$ implies the uniform convergence of the first term.

For the convergence of the second term let $\varepsilon > 0$. There exists $\delta > 0$ with $\rho(\omega(t), \omega(u)) < \frac{\varepsilon}{2}$ for $|t-u| < \delta$. Let n_o with $\delta_n \leqslant \delta$ for $n \geqslant n_o$. Then for $n \geqslant n_o$ we have $\rho(\omega(\sigma_n(t)), \omega(t)) < \varepsilon$ on $[0, \zeta(\omega))$.

This is trivial for $|t-u| \geqslant \delta_n$ since then $\sigma_n(t) = t$. For $|t-u| < \delta_n$ also $|\sigma_n(t)-u| < \delta_n$ holds, which implies

$$\rho(\omega(\sigma_n(t)), \omega(t)) \leqslant \rho(\omega(\sigma_n(t)), \omega(u)) + \rho(\omega(u), \omega(t)) < \varepsilon.$$

ii) The convergence $\omega_n \rightarrow \omega$ and $u_n \rightarrow 0$ is formally weaker than the convergence $(\omega_n, u_n) \rightarrow (\omega, 0)$.

So assume $\omega_n \longrightarrow \omega$ and $u_n \longrightarrow 0$.

The proof of the convergence $(\omega_n, u_n) \longrightarrow (\omega, 0)$ and of the additional assertion in ii) can be reduced to the continuity case of the proof of i), while defining $(\tilde\omega, \tilde u)$ by $\zeta(\tilde\omega) = \zeta(\omega) + 1$, $\tilde u = u + 1 = 1$ and

$$\tilde\omega(t) = \begin{cases} \omega(0) & 0 \leqslant t \leqslant 1 \\ \omega(t-1) & 1 \leqslant t < \zeta(\tilde\omega) \end{cases}$$

and defining $(\tilde\omega_n, \tilde u_n)$ in the same way.

Then we have $\tilde\omega_n \to \tilde\omega$, $\tilde u_n \to 1$ and 1 is a continuity point of $\tilde\omega$. Remark that in the continuity case of the proof of i) we only needed this convergence. So there exists $\tilde\tau_n \in \mathcal{T}^{\zeta(\tilde\omega_n)}_{\zeta(\tilde\omega)}$ with $\tilde\tau_n(1) = u_n+1$ and $\rho(\tilde\omega_n(\tilde\tau_n(t)), \tilde\omega(t)) \to 0$ uniformly on $[0, \zeta(\tilde\omega))$.

This implies

$$\omega_n(u_n) = \tilde\omega_n(\tilde u_n) \to \tilde\omega(\tilde u) = \omega(0)$$

$$\omega_n(u_n-) = \tilde\omega_n(u_n-) \to \tilde\omega(\tilde u-) = \omega(0)$$

from which the convergence $(\omega_n, u_n) \to (\omega, 0)$ follows.

Finally define $\tau_n \in \mathcal{T}^{\zeta(\omega_n)-u_n}_{\zeta(\omega)}$ by $\tau_n(t) = \tilde\tau_n(t+1)-(u_n+1)$. Then $|\tau_n| \leqslant |\tilde\tau_n| + u_n \to 0$ and

$\rho(\omega_n(\tau_n(t)+u_n), \omega(t)) = \rho(\tilde\omega(\tilde\tau_n(t+1)), \tilde\omega(t+1)) \to 0$ uniformly on $[0, \zeta(\omega))$, which completes the proof.

The Corollary is a direct consequence of the Theorem.

3. The abstract model

In section 1 we constructed a model for the distribution of the interaction blocks by assigning to each $V_0 \in \mathbb{R}^\ell$ a probability measure P_{V_0} on Σ and to each $V_0 \in \mathbb{R}^\ell$ and each $(\tau_i, z_i)_{i=1}^k \in \Sigma$ a path of finite length, which we may now regard as an element of $D_f^+(\mathbb{R}^\ell)$. The measurability properties stated at the end of section 1 together with Corollary 1 make possible a canonical representation of the distribution of the interaction blocks whereby each $V_0 \in \mathbb{R}^\ell$ is assigned a probability measure on $D_f^+(\mathbb{R}^\ell)$, which we call $R(V_0, \cdot)$. For each Borel set $A \subset D_f^+(\mathbb{R}^\ell)$ the mapping $V_0 \mapsto R(V_0, A)$ is measurable. So R has the properties of a kernel.

From the distribution of the interaction blocks described by the kernel R, we proceed to the construction of a model for the whole velocity process by joining the interaction blocks. Each block will be given the distribution belonging to the initial velocity which results from the preceding block. The transition from one block to the initial value of the following is formally described by the kernel

$S(\omega, B) = 1_B(\pi_\zeta(\omega))$ $(\omega \in D_f^+(\mathbb{R}^\ell)$, $B \subset \mathbb{R}^\ell$ Borel set$)$. So the interaction blocks form a Markov chain with state space $D_f^+(\mathbb{R}^\ell)$ and transition operator $Q = SR$. The initial block containing the time zero, plays a special rôle and will be represented by an element of $\check{D}_f^+(\mathbb{R}^\ell)$, whereby the marked time represents the time zero. Its distribution will be given by a probability measure on $\check{D}_f^+(\mathbb{R}^\ell)$.

We shall perform the described construction in an abstract model replacing the state space $\mathbb{R}^\ell$ with an arbitrary locally compact space E with a denumerable base and the special kernels R and S with arbitrary ones. Since their composition $Q = SR$ is the crucial kernel for the process, we shall proceed from it and get R and S by disintegration under corresponding conditions.

Thus, our abstract model consists of processes which are divided into blocks with the property that at the beginning of a block the future does not depend on the past. What we get is a generalization of semi-Markov processes, for which the constancy within one block is not required. After having constructed these processes, we shall indicate how methods and results originating from the theory of semi-Markov processes can be applied to the study of the described processes. For this purpose we associate to each one a semi-Markov process by setting the process within one block constant to the initial value. This associated semi-Markov process itself contains much information about the original process, namely its global behaviour, which does not depend on the fluctuations within the blocks. Furthermore, the initial values of the blocks form an imbedded Markov chain. Results concerning the associated semi-Markov process and the imbedded Markov chain can directly be adopted from the theory of semi-Markov processes. But both - the associated semi-Markov process and the imbedded Markov chain - will also be useful for further results. This will be shown for the finite-dimensional distributions and the ergodic behaviour.

3.1 Construction

As we have just given an intuitive survey of the construction of our abstract model, we can now restrict ourselves to its technical performance.

Let E be a locally compact space with a denumerable base and $E' = E \cup \{\Delta\}$ be its Alexandrov compactification. As the canonical sample space, on which we shall define the process, we have $\Omega = \check{D}_f^+(E) \times D_f^+(E)^{\mathbb{N}}$ supplied with the product topology and the σ-algebra $\mathcal{F}$ of its Borel sets. For $n = 0,1,2,\ldots$ let $\mathcal{F}_n$ be the σ-algebra of Borel sets depending only on the zeroth block - the block with the marked time - and the following n blocks.

In order to join the blocks we first define the times $T_n (n \geqslant 0)$, which separate them:

$$T_n : \Omega \longrightarrow \mathbb{R}$$
$$T_o((\omega_o, u), (\omega_j)_{j \geqslant 1}) = -u$$
$$T_n((\omega_o, u), (\omega_j)_{j \geqslant 1}) = -u + \zeta(\omega_o) + \ldots + \zeta(\omega_{n-1}) \quad (n \geqslant 1)$$

The mappings T_n are continuous. Furthermore, T_o is $\mathcal{F}_o$-measurable, and for $n \geqslant 1$ T_n is $\mathcal{F}_{n-1}$-measurable.

The connection of the blocks can now easily be defined by the coordinate mappings $X_t : \Omega \longrightarrow E' \; (t \in \mathbb{R}^+)$:

$$X_t((\omega_o, u), (\omega_j)_{j \geqslant 1}) = \begin{cases} \omega_j(t - T_j) & \text{for } T_j \leqslant t < T_{j+1} = T_j + \zeta(\omega_j) \\ \Delta & \text{for } t \geqslant \lim_{j \to \infty} T_j, \text{ if it is finite.} \end{cases}$$

It is clear that the mappings X_t are measurable. The corresponding paths are right continuous and - except in the time $\lim_{j \to \infty} T_j$, if it is finite - have left limits.

Now let Q be a Markovian kernel on $D_f^+(E)$ and μ be a probability measure on $\check{D}_f^+(E)$. The assumptions that the distribution of (ω_o, u) is given by μ and that $(\omega_j ; j \geqslant 0)$ is a Markov chain with transition kernel Q, define a unique probability measure P on $(\Omega, \mathcal{F})$, analogous to the construction of a Markov chain with the exception that the zeroth element is supplied with an additional random variable.

Thus we have got a stochastic process $(\Omega, \mathcal{F}, P, (X_t)_{t \in R^+})$, which is, however, too general for our purposes. We shall claim that Q is of a special form.

For a fixed $\omega \in D_f^+(E)$ the image of the probability measure $Q(\omega, \cdot)$ under the mapping π_o is a probability measure on E, which we call $S(\omega, \cdot)$. Since for a Borel set $B \subset E$ the mapping $\omega \mapsto S(\omega, B) = Q(\omega, \pi_o^{-1}(B))$ is measurable, S has the properties of a transition kernel with the difference that it leads from one space to the other. For a fixed $\omega \in D_f^+(E)$ we can disintegrate the measure $Q(\omega, \cdot)$ with respect to the mapping π_o (see Bourbaki [1], 2.7). Our additional assumption is now that these disintegrations can be performed independently of ω. We thus get a mapping R:

$R(x, A)$ defined for $x \in E$, $A \subset D_f^+(E)$ Borel set with the properties:

i) for each $x \in E$ the mapping $A \mapsto R(x, A)$ is a probability measure on $D_f^+(E)$
ii) for each Borel set $A \subset D_f^+(E)$ the mapping $x \mapsto R(x, A)$ is measurable
iii) for each $x \in E$ we have $R(x, \pi_o^{-1}(x)) = 1$

iv) for each $\omega \in D_f^+(E)$ and for each Borel set $A \subset D_f^+(E)$ we have

$$Q(\omega, A) = \int S(\omega, dx) R(x, A).$$

So R is the same type of transition kernel as S with the rôle of the two spaces interchanged. iv) can be written shortly $Q = SR$. Later on we shall see that the kernel $K = RS$ is also important to us.

This finishes the construction of our abstract model. Remark that the times T_n are no random times in the usual sense, since they are not definable on the paths, because each path can arbitrarily be divided into blocks. The knowledge of the times T_n requires the information contained in the sample space Ω. This shows us that for the notion of our abstract model μ and Q are indispensable.

As already remarked we shall associate a semi-Markov process and an imbedded Markov chain to the constructed process. For it we need the just mentioned Markovian kernel $K = RS$ and the semi-Markovian kernel $\tilde{K}$ (see Çinlar [2]) uniquely defined by

$$\tilde{K}(x, A \times B) = \int R(x, d\omega) 1_B(\zeta(\omega)) S(\omega, A) \quad (x \in E,\ A \subset E \text{ and } B \subset \mathbb{R}^+ \text{ Borel sets})$$

$\tilde{K}$ induces a linear positive contraction $K^{(1)}$ on the Banach space $L^{(1)}$ of real-valued bounded measurable functions on $E \times \mathbb{R}^+$ supplied with the sup-norm:

$$(K^{(1)}f)(x,t) = \int \tilde{K}(x, d(y,\zeta)) f(y, t-\zeta) 1_{\mathbb{R}^+}(t-\zeta).$$

$K^{(1)}$ will appear in the formulae for the finite-dimensional distributions. There we shall also need the more-dimensional analoga $K^{(r)}$, the linear positive contractions on the Banach spaces $L^{(r)}$ of real-valued bounded measurable functions on $E \times (\mathbb{R}^+)^r$ supplied with the sup-norm:

$$(K^{(r)}f)(x, t_1, \ldots, t_r) = \int \tilde{K}(x, d(y,\zeta)) f(y, t_1-\zeta, \ldots, t_r-\zeta) 1_{\mathbb{R}^+}(t_1-\zeta) \ldots 1_{\mathbb{R}^+}(t_r-\zeta)$$

<u>Theorem 4:</u> i) The random variables $Y_n = X_{T_n} = \omega_n(0)$ $(n \geq 1)$ form a Markov chain with state space E, transition kernel $K = SR$ and initial distribution $\nu = S\mu_o$ where μ_o is the marginal distribution of ω_o.

ii) The random variables $Z_n = (X_{T_n}, T_n)$ $(n \geq 1)$ form a Markov chain with state space $E \times \mathbb{R}^+$, transition probability uniquely defined by:

$$P((X_{T_{n+1}}, T_{n+1}) \in A \times B \mid (X_{T_n}, T_n)) = \tilde{K}(X_{T_n}, A \times (B - T_n))$$

and initial distribution ν':

$$\nu'(A \times B) = \int \mu(d(\omega_o, u)) 1_B(-u + \zeta(\omega)) S(\omega_o, A)$$

($A \subset E$ and $B \subset \mathbb{R}^+$ Borel sets).

Proof: For $n \geq 1$, $Y_1, \ldots, Y_n$ are $\mathcal{F}_n$-measurable. Let $\mathcal{F}_n^o = \sigma(\mathcal{F}_{n-1}, Y_n)$. For $n \geq 1$, we have $\sigma(Y_1, \ldots, Y_n) \subset \mathcal{F}_n^o \subset \mathcal{F}_n$. This implies for each Borel set $A \subset E$:

$P(Y_{n+1} \in A | Y_1, \ldots, Y_n) = E(E(P(Y_{n+1} \in A | \mathcal{F}_n) | \mathcal{F}_n^o) | Y_1, \ldots, Y_n)$.

Since $(\omega_n)_{n \geq 1}$ is a Markov chain with transition operator Q, we have:

$P(Y_{n+1} \in A | \mathcal{F}_n) = Q(\omega_n, \{\omega_{n+1}(0) \in A\}) = S(\omega_n, A)$

$E(S(\omega_n, A) | \mathcal{F}_{n-1}) = (QS)(\omega_{n-1}, A) = \int S(\omega_{n-1}, dy_n)(RS)(y_n, A)$

which implies by the definition of conditional distributions:

$E(S(\omega_n, A) | \mathcal{F}_n^o) = E(E(S(\omega_n, A) | \mathcal{F}_{n-1}) | \omega_n(0)) = (RS)(\omega_n(0), A)$.

So $(Y_n)_{n \geq 1}$ is a Markov chain with state space E and transition operator K. The assertion about the initial distribution is clear.

The proof of ii) is analogous.

Now we define the process $(\tilde{X}_t)_{t \in \mathbb{R}^+}$ on $(\Omega, \mathcal{F}, P)$ by:

$$\tilde{X}_t = X_{T_n} \quad \text{for} \quad T_n \leq t < T_{n+1}$$
$$\tilde{X}_t = \Delta \quad \text{for} \quad t \geq \lim_{n \to \infty} T_n$$

Then Theorem 4ii) just means that $(\tilde{X}_t)_{t \in \mathbb{R}^+}$ is a semi-Markov process with the kernel $\tilde{K}$. We call it the associated semi-Markov process.

The Markov chain $(Y_n)_{n \geq 1}$ is called the imbedded Markov chain. It is clear that it is also the imbedded Markov chain to the associated semi-Markov process. Analytically this is expressed by the property:

$$\tilde{K}(x, A \times \mathbb{R}^+) = K(x, A) \quad (x \in E,\ A \subset E \text{ Borel set}).$$

3.2 Finite-dimensional distributions

The probability measures $R(x, \cdot)$ define the distributions of the blocks in dependence of their initial values. We may regard their finite dimensional distributions $R_x^{t_1, \ldots, t_r}$ $(t_1, \ldots, t_r \in T)$. For $r \geq 1$ fixed they define a linear positive contraction $R^{(r)}$ on the Banach space of real-valued bounded measurable functions on E^r supplied with the sup-norm into the Banach space $L^{(r)}$.

$$(R^{(r)} f)(x, t_1, \ldots, t_r) = R_x^{t_1, \ldots, t_r} f = \int R(x, d\omega) f(\omega(t_1), \ldots, \omega(t_r))$$

where we set $f(\omega(t_1), \ldots, \omega(t_r)) = 0$, if $t_i \geq \zeta(\omega)$ for at least one i $(i = 1, \ldots, r)$.

From the finite-dimensional distributions of the blocks we derive the finite-dimensional distributions of the process $(X_t)_{t\in\mathbb{R}^+}$ by choosing the "right" blocks. For this we only need the associated semi-Markov process.

We start with the one-dimensional distributions. So let f be a real-valued bounded measurable function on E. For $t \in \mathbb{R}^+$ we have:

$$f(X_t((\omega_0, u), (\omega_j)_{j\geq 1})) = \sum_{n=0}^{\infty} f(\omega_n(t-T_n))1_{[T_n,T_{n+1})}(t)$$

$$Ef(X_t) = \sum_{n=0}^{\infty} Ef(\omega_n(t-T_n))1_{[T_n,T_{n+1})}(t)$$

For shortening the notation we always set $f(\omega(t)) = 0$ for $\omega \in D_f^+(E)$ and $t<0$ or $t \geq \zeta(\omega)$. Then:

$$f(\omega_n(t-T_n))1_{[T_n,T_{n+1})}(t) = f(\omega_n(t-T_n)).$$

For $n=0$ we have

$$Ef(\omega_0(t-T_0)) = \int \mu(d(\omega_0, u))f(\omega_0(t+u))$$

and for $n\geq 1$

$$Ef(\omega_n(t-T_n)) =$$

$$= \int \mu(d(\omega_0,u)) \int Q(\omega_0,d\omega_1)\int\ldots\int Q(\omega_{n-1},d\omega_n)f(\omega_n(t-[-u+\zeta(\omega_0)+\ldots+\zeta(\omega_{n-1})])).$$

We shall transform this representation.

$$\int Q(\omega_{n-1},d\omega_n)f(\omega_n(t-T_n)) = \int S(\omega_{n-1},dy_n)\int R(y_n,d\omega_n)f(\omega_n(t-T_n))$$

$$= \int S(\omega_{n-1},dy_n)(R^{(1)}f)(y_n,t-T_n)$$

$$\int Q(\omega_{n-2},d\omega_{n-1})\int Q(\omega_{n-1},d\omega_n)f(\omega_n(t-T_n))$$

$$= \int S(\omega_{n-2},dy_{n-1})\int R(y_{n-1},d\omega_{n-1})\int S(\omega_{n-1},dy_n)(R^{(1)}f)(y_n,t-T_{n-1}-\zeta(\omega_{n-1}))$$

$$= \int S(\omega_{n-2},dy_{n-1})(K^{(1)}R^{(1)}f)(y_{n-1},t-T_{n-1}).$$

By recursion we get for $n\geq 1$:

$$\int Q(\omega_0,d\omega_1)\int\ldots\int Q(\omega_{n-1},d\omega_n)f(\omega_n(t-T_n)) = \int S(\omega_0,dy_1)(K^{(1)n-1}R^{(1)}f)(y_1,t-T_1)$$

$$Ef(X_t) = \int \mu(d(\omega_0,u))f(\omega_0(t+u)) + \sum_{n=0}^{\infty}\int \mu(d(\omega_0,u))\, S(\omega_0,dy_1)(K^{(1)^n}R^{(1)}f)(y_1,t-(-u+\zeta(\omega_0))$$

Let $N^{(r)}$ be the strong limit $\sum_{n=0}^{\infty} K^{(r)n}$. $N^{(1)}$ is intensively studied in Çinlar [2]. Since $|(R^{(1)}f)(x,\tau)| < \| f \| \cdot R(x,\{\zeta(\omega) > \tau\})$, one easily sees that Proposition (9) of [2] can be applied, which shows that $N^{(1)}R^{(1)}f$ exists. We get the final result:

$$Ef(X_t) = \int \mu(d(\omega_o,u))[f(\omega_o(t+u)) + \int S(\omega_o,dy)(N^{(1)}R^{(1)}f)(y,t+u-\zeta(\omega))].$$

In a similar way we obtain the more-dimensional distributions.

Let f be a real-valued bounded measurable function on E^r. For $t_1 \leq t_2 \leq \ldots \leq t_r$ we have

$$Ef(X_{t_1},X_{t_2},\ldots,X_{t_r}) =$$

$$= E\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\sum_{n_r=0}^{\infty} f(\omega_{n_1}(t-T_{n_1}),\omega_{n_1+n_2}(t-T_{n_1+n_2}),\ldots,\omega_{n_1+\ldots+n_r}(t-T_{n_1+\ldots+n_r}))$$

Using the same methods as before, we can derive analogous formulae. Besides the operators $R^{(r)}$ and $K^{(r)}$, they will contain combinations of them, acting on some coordinates like $R^{(r')}$ and on some like $K^{(r'')}$ and leave the rest unaffected. Since the principle is clear, but the formulae become too confusing, we shall not perform their derivation.

3.3 Ergodic behaviour

A basic tool in the study of semi-Markov processes is the method of the imbedded Markov chain. By means of the associated semi-Markov process we can directly adopt the results for the imbedded Markov chain $(Y_n)_{n\geq 1}$.

Furthermore, we can use the imbedded Markov chain for the derivation of results about the original Markov chain $(\omega_n)_{n\geq 1}$, namely those concerning the ergodic behaviour, as we shall show now. By this we can apply ergodic theorems for Markov chains with a locally compact state space with a denumerable base to the imbedded Markov chain and transfer the results to the Markov chain $(\omega_n)_{n\geq 1}$.

We shall start with invariant measures on E resp. $D_f^+(E)$. The kernel $R[S]$ maps measures on $D_f^+(E)$ into measures on E [vice versa]. We shall show that by means of this mapping Q-invariant measures will be mapped into K-invariant measures [vice versa] and that for invariant measures these correspondences are inverse. The mapping π_o maps measures on $D_f^+(E)$ into measures on E. For invariant measures this mapping is the same as the one induced by the kernel R.

<u>Proposition 4:</u> i) If μ is a Q-invariant measure on $D_f^+(E)$, then $\pi_o(\mu) = \mu S$, and this measure is K-invariant.

ii) If ν is a K-invariant measure on E, then νR is Q-invariant.

iii) The correspondences between Q-invariant and K-invariant measures defined in i) and ii) are inverse.

<u>Proof:</u> i) We have $\mu = \mu Q = \mu SR$.
Let $\nu = \mu S$. Then $\nu = \mu S = \mu SRS = \nu RS = \nu K$. So ν is K-invariant.
Now let $B \subset E$ be a Borel set.

$$\pi_o(\mu)(B) = \mu(\pi_o^{-1}(B)) = (\mu SR)(\pi_o^{-1}(B)) = \int (\mu S)(dx) R(x, \pi_o^{-1}(B))$$
$$= \int (\mu S)(dx)\, 1_B(x) = (\mu S)(B) = \nu(B).$$

The proof of ii) is analogous to the first part of the proof of i).
iii) Let μ be Q-invariant. Then μ is assigned the measure $\nu = \mu S$, and ν is assigned the measure $\nu R = \mu SR = \mu Q = \mu$. The inverse relation will be gained by starting with a K-invariant measure ν in the same way.

Proposition 4 can especially be utilized for the questions concerning the existence and uniqueness of invariant measures.

If μ_o is the distribution of ω_o, then $\mu_n = \mu_o Q^n$ is the distribution of ω_n, and $\nu_{n+1} = \mu_n S$ is the distribution of Y_{n+1}. If we proceed otherwise from ν_1, the distribution of Y_1, then $\nu_{n+1} = \nu_1 K^n$ is the distribution of Y_{n+1}, and $\mu_n = \nu_n R$ is the distribution of ω_n. We shall show that the weak convergence of μ_n to a limit μ implies the weak convergence ν_n to a limit ν. For the inversion, which is the interesting one, we need an additional supposition. The measures μ and ν are Q- resp. K-invariant and are in relation to each other according to the preceding proposition.

<u>Proposition 5:</u> i) Let μ_o be a measure on $D_f^+(E)$. If the measures $\mu_n = \mu_o Q^n$ converge weakly to a measure μ, then the measures $\nu_{n+1} = \mu_n S$ converge weakly to the measure $\nu = \mu S$.

ii) Suppose that for real-valued bounded continuous functions f on $D_f^+(E)$ the functions Rf are continuous on E. Let ν_1 be a measure on E. If the measures $\nu_{n+1} = \nu_1 K^n$ converge weakly to a measure ν, then the measures $\mu_n = \nu_n R$ converge weakly to the measure $\mu = \nu R$.