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A SEMI-MARKOVIAN MODEL FOR THE BROWNIAN MOTION

Séminaire de Probabilités VII (Lecture Notes in Mathematics, Vol.321)

4. An Example: Repelling potentials

By mistake two pages of the article "A Semi-Markovian Model for the Brownian Motion" were not printed in the last volume of the Lecture Notes of the Séminaire de Probabilités de Strasbourg. Taking the opportunity of their being supplied in this volume we want to show that for an interaction by a repelling potential in a gas of sufficiently low density in $\ell > 2$ dimensions our model holds, i.e. the condition of Lemma 1 is fulfilled. We maintain the notations of the previous article with the exception of a change of the scale of length, which is more convenient for the physical interpretation of the result. We set the radius of the interaction region $R = 1$ and then have to introduce the density n of the gas. The distributions of the interaction parameters are easily adopted. If we denote again $c(V_0) = C_{\ell-1} \int |V_0 - v| dv(v)$, then the interarrival times are exponentially distributed with parameter $n c(V_0)$. The distribution of the phase coordinates remain the same with $R = 1$. Furthermore, let ϕ be the potential as in 1.1 with the additional properties: ϕ is spherically symmetric, so that there exists a function φ on \mathbb{R}^+ with $\phi(x) = \varphi(|x|)$, and ϕ is repelling, i.e. φ is monotonously decreasing.

For the verification of the condition of Lemma 1 we have to regard the distribution of the interaction times. The interaction time of a particle is gained by the motion of the particle relative to the heavy particle which can be described by the motion of a particle of mass μ in a central field of the potential ϕ . Let θ be the time this particle stays in the interaction region. θ only depends on the absolute value u of the velocity outside the interaction region and of the

impact parameter ρ . $\theta = \theta(u, \rho)$ is given by the formula:

$$\theta(u, \rho) = 2 \int_{h(u, \rho)}^1 \frac{dr}{\sqrt{f(u, \rho, r)}}$$

with $f(u, \rho, r) = u^2 \left(1 - \frac{\rho^2}{r^2}\right) - \frac{2}{u} \cdot \varphi(r)$

and $h(u, \rho) = \max\{r: r \leq 1, f(u, \rho, r) = 0\}$.

Since $f(u, \rho, 1) \geq 0$ and $-\infty < f(u, \rho, 0+) < 0$ hold, $h(u, \rho)$ is well defined. The following lemma gives an upper bound for θ

Lemma 3:
$$\theta(u, \rho) \leq \frac{2h(u, \rho) \cdot \sqrt{1-h^2(u, \rho)}}{u \cdot \rho} \leq \frac{1}{u \rho}$$

Proof: Fix u, ρ with $u\rho \neq 0$. For $h = h(u, \rho)$ we have $f(u, \rho, h) = 0$ which implies $\frac{2}{u} \cdot \varphi(h) = u^2 \left(1 - \frac{\rho^2}{h^2}\right)$.

For $h \leq r \leq 1$ there follows:

$$\begin{aligned} f(u, \rho, r) &= u^2 \left(1 - \frac{\rho^2}{r^2}\right) - \frac{2}{u} \cdot \varphi(r) \geq u^2 \left(1 - \frac{\rho^2}{r^2}\right) - \frac{2}{u} \cdot \varphi(h) \\ &= u^2 \rho^2 \left(\frac{1}{h^2} - \frac{1}{r^2}\right) \end{aligned}$$

$$\Rightarrow \theta(u, \rho) = 2 \int_h^1 \frac{dr}{\sqrt{f(u, \rho, r)}} \leq \frac{2}{u\rho} \int_h^1 \frac{dr}{\sqrt{\frac{1}{h^2} - \frac{1}{r^2}}} = \frac{2h \cdot \sqrt{1-h^2}}{u\rho}$$

For $0 \leq h \leq 1$ $2h\sqrt{1-h^2} \leq 1$ holds, which completes the proof.

With the estimation of Lemma 3 we are now able to gain the mentioned result. Fix V_0 and let $F = F_{V_0}$ be the distribution function of the interaction times of the single particles.

We have to consider $\int_0^\infty \exp\{-nc(V_0) \int_0^\xi (1 - F(\xi)) d\xi\} d\xi$.

Let $u = |v - V_0|$ where v is the initial velocity of the interacting particle, and ρ be the impact parameter of the interacting particle. The distribution of the interaction parameters given in 1.1 yield that u and ρ are independently distributed. Call $\bar{v} = v - V_0$ the distribution of u . We have

$$\frac{\int d\bar{v}(u)}{u} = \frac{\int dv(v)}{\int |v - V_0| dv(v)} = \frac{C_{k-1}}{c(V_0)}$$

ρ is distributed with respect to the Lebesgue-measure on $[0,1]$.

From Lemma 3 we get:

$$1 - F(\xi) = \Pr\{\theta > \xi\} \leq \Pr\left\{\frac{1}{u\rho} > \xi\right\}.$$

For $u < \frac{1}{\xi}$: $\frac{1}{u\rho} > \xi$ holds for all $\rho \in [0,1]$,

for $u \geq \frac{1}{\xi}$: $\frac{1}{u\rho} > \xi$ holds for $\rho < \frac{1}{u\xi}$.

$$\text{So } 1 - F(\xi) \leq \int_{[0, \frac{1}{\xi})} d\bar{v}(u) + \int_{[\frac{1}{\xi}, \infty)} \frac{1}{u\xi} d\bar{v}(u) \leq \frac{1}{\xi} \int \frac{d\bar{v}(u)}{u} = \frac{C_{\ell-1}}{c(V_0)} \cdot \frac{1}{\xi}.$$

This inequality is important for ξ great. For $\xi < \frac{C_{\ell-1}}{c(V_0)}$ we replace it by $1 - F(\xi) \leq 1$.

For $\zeta \geq \frac{C_{\ell-1}}{c(V_0)}$ we have:

$$\int_0^{\zeta} (1 - F(\xi)) d\xi \leq \frac{C_{\ell-1}}{c(V_0)} + \int \frac{C_{\ell-1}}{c(V_0)} \cdot \frac{1}{\xi} d\xi$$

$$= \frac{C_{\ell-1}}{c(V_0)} \left(1 + \log \zeta - \log \frac{C_{\ell-1}}{c(V_0)}\right)$$

and $\exp\{-nc(V_0) \int_0^{\zeta} (1 - F(\xi)) d\xi\} \geq k \cdot \zeta^{-nC_{\ell-1}}$ with a constant k .

So we finally get the result:

Proposition 6: For $nC_{\ell-1} \leq 1$ the condition

$$\int_0^{\infty} \exp\{-nc(V_0) \int_0^{\zeta} (1 - F(\xi)) d\xi\} d\zeta = +\infty$$

of Lemma 1 is fulfilled.

We close with the remark that the estimation

$$1 - F(\xi) \leq \int_{[0, \frac{1}{\xi})} d\bar{v}(u) + \int_{[\frac{1}{\xi}, \infty)} \frac{1}{u\xi} d\bar{v}(u)$$

alone even with a further exact calculation is not sufficient enough to get better results for $nC_{\ell-1} > 1$.