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SOME UNIVERSAL FIELD EQUATIONS

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The results below concern a general stochastic process, a general time T, and the splitting of fields occasioned by T. These things are discussed under this generality in [1] , and may become more relevant now that other times (such as last exit time) are coming into their own. When the general results are applied to the case of a homogeneous Markov process and optional times, they can simplify certain standard arguments to a considerable extent. An example is the treatment of so-called "times of discontinuity of the fields $\mathcal{F}_t$" (see e.g. [2;p. 171 ff]). The main idea here is a consistent use of the left field $\mathcal{F}_{T-}$. Incidentally, formula (5) below may be regarded as a random solution to Zeno's paradox on the flow of time.

$X = \{X_t, t \geq 0\}$ is a Borel measurable stochastic process defined on (Ω, F, P) and taking values in $(E, \mathcal{E})$, a topological space with its Borel field. The topology may be considerably more general than the usual assumption of "locally compact with countable base," but we will leave this point moot. Let o (...) denote the Borel field generated by the random variables within the brackets; $\mathcal{F}_\infty = \mathcal{F}_{\infty-} = o\ (X_s, 0 \leq s < \infty)$, $\mathcal{F}_t = o(X_s, 0 \leq s \leq t)$; we assume that $(\mathcal{F}_\infty, P)$ is a complete probality space and $\mathcal{F}_t$ is augmented with all P-null sets in $\mathcal{F}_\infty$. We are not preoccupied with optional or co-optional times so will review some not-so-well known general facts about an arbitrary positive random variable $T \in \mathcal{F}_\infty$. We define $\mathcal{F}_{T-}$ to be the Borel field generated by sets of the form

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$$\{T > t\} \cap \Lambda_t \quad \text{where} \quad t \geq 0 \quad \text{and} \quad \Lambda_t \in \mathfrak{F}_t \ ;$$

and

$$\mathfrak{F}_{T+} = \bigwedge_{n=1}^{\infty} \mathfrak{F}_{(T+\frac{1}{n})-} \quad .$$

It is known [1] that when T is (loosely) optional, $\mathfrak{F}_{T+}$ coincides with the usual field with this notation. We put

$$\mathfrak{F}'_T = \sigma(X_{T+t}, \ t \geq 0) \ ,$$

$$\mathfrak{F}_{[T,T+u)} = \sigma(X_{T+t}, \ 0 \leq t < u) \ .$$

All these fields are augmented by all P-null sets in $\mathfrak{F}_\infty$ The process X is said to be right (left) continuous or to have right (left) limits iff all its paths have the said property.

<u>Lemma</u>. If X has right (left) limits everywhere, then

(1) $$X_{T+} \in \mathfrak{F}_{T+} \quad [X_{T-} \in \mathfrak{F}_{T-}] \ .$$

<u>Proof</u>. We prove the right version since the other is quite similar. First suppose T is countably valued taking values in the countable set Q . Then for each $q \in Q$, and $A \in \mathcal{E}$:

$$\{T = q; \ X(q) \in A\} = \bigcap_n \{T + \frac{1}{n} > q; \ X(q) \in A\} \cap \{T = q\} \ .$$

Since

$$\{T + \frac{1}{n} > q; \ X(q) \in A\} \in \mathfrak{F}_{T+\frac{1}{n}-}$$

and this set decreases when n increases, the intersection over all n belongs to $\mathfrak{F}_{T+}$. Since $T \in \mathfrak{F}_{T+}$ this establishes (1) for a countably valued T . In the general case we approximate T by $T_n = 2^{-n}[2^n T + 1]$ to get

$$X_{T+} = \lim_n X_{T_n} \in \bigwedge_n \mathfrak{F}_{T_n+} = \mathfrak{F}_{T+} . \quad \|$$

Proposition 1. Let X be either right or left continuous, and T be any positive random variable. Then we have

$$\mathfrak{F}_\infty = \mathfrak{F}_{T-} \bigvee \mathfrak{F}'_T . \tag{2}$$

Remark. The assumption of right continuity can be relaxed. For example, the right stochastic continuity of the post-T process $\{X(T+t), t \geq 0\}$ is sufficient. But we do not know reasonable conditions to insure the latter.

Proof. If T is countably valued, then clearly $X_{T+t} \in \mathfrak{F}_\infty$ for every $t \geq 0$. For a general T , approximate T from right or left according as X is right or left continuous. It follows that the left member of (2) includes the right. To prove the opposite inclusion it is sufficient to show that each X_t belongs to the right member of (2). Let $A \in \mathcal{E}$, then

$$\{X_t \in A\} = \{X_t \in A; T < t\} \cup \{X_t \in A; T = t\} \cup \{X_t \in A; T > t\} ; \tag{3}$$

call the three sets on the right Λ_1, Λ_2 and Λ_3 . Since $T \in \mathfrak{F}_{T-}$,

$$\Lambda_2 = \{X_T \in A; T = t\} \in \sigma(X_T, T) \subset \sigma(X_t) \bigvee \mathfrak{F}_{T-} .$$

By definition of $\mathfrak{F}_{T-}$, $\Lambda_3 \in \mathfrak{F}_{T-}$. Finally, we write

$$\Lambda_1 = \{X(T+(t-T)) \in A;\ T < t\} \ .$$

According as X is right or left continuous, we have

$$I_{\{T<t\}}\, X(T+(t-T)) = \lim_n 1_{\{T<t\}}\, X(T + 2^{-n}[2^n(t-T)\pm 1])$$

with "+" or "-" . For each n , the approximating random variable belongs to $\mathfrak{F}_{T-} \bigvee \mathfrak{F}'_T$ because $\{T < t\} \in \mathfrak{F}_{T-}$. It follows that Λ_1 belongs to the right member of (2) as well as Λ_2 and Λ_3 . Hence so does X_t by (3), since A is arbitrary. ||

<u>Proposition 2</u>. Let X be right continuous and suppose T is such that: for any $M \in \mathfrak{F}'_T$:

$$(4) \qquad P\{M \mid \mathfrak{F}_{T+}\} = P\{M \mid X_T\} \quad \text{on} \quad \{T < \infty\} \ ,$$

then we have

$$(5) \qquad \mathfrak{F}_{T+} = \mathfrak{F}_{T-} \bigvee \sigma(X_T)$$

where strictly speaking X_T should be replaced by $X_T I_{\{T<\infty\}}$ in (5) since X_∞ is not defined.

<u>Proof</u>. Let $\Lambda \in \mathfrak{F}_{T-}$, then since $\mathfrak{F}_{T-} \subset \mathfrak{F}_{T+}$ we have by (4):

$$P\{\Lambda \cap M \mid \mathfrak{F}_{T+}\} = I_\Lambda\, P\{M \mid \mathfrak{F}_{T+}\} = I_\Lambda\, \varphi(X_T)$$

where φ is a function in $\mathcal{E}$. Since X is right continuous, it follows from Proposition 1 that sets of the form $\Lambda \cap M$ above generate $\mathfrak{F}_\infty$. Thus for any $H \in \mathfrak{F}_\infty$:

$$P\{H \mid \mathfrak{F}_{T+}\} \in \mathfrak{F}_{T-} \bigvee \sigma(X_T)$$

since all fields are augmented. In particular, if we take $H \in \mathfrak{F}_{T+} \subset \mathfrak{F}_\infty$, we conclude thus

(6) $$\mathfrak{F}_{T+} \subset \mathfrak{F}_{T-} \bigvee \sigma(X_T) .$$

Conversely since X is right continuous, $X_T \in \mathfrak{F}_{T+}$ by the Lemma. Hence the opposite inclusion to (6) is also true. ||

Corollary 1 below is stated here only for comparison with an older formulation (see [2]), in which X is a homogeneous Markov process and the T_n 's are optional.

<u>Corollary 1</u>. Under the hypothesis of Proposition 2, if $\{T_n\}$ is optional, $T_n \uparrow T$, and

(7) $$X_T I_{\{T<\infty\}} \in \bigvee_n \mathfrak{F}_{T_n+}$$

then

(8) $$\mathfrak{F}_{T+} \subset \bigvee_n \mathfrak{F}_{T_n+} .$$

<u>Proof</u>. We have (see [1])

$$\mathfrak{F}_{T-} = \bigvee_n \mathfrak{F}_{T_n-} \subset \bigvee_n \mathfrak{F}_{T_n+} .$$

Hence (8) follows from (5) and (7). ||

<u>Corollary 2</u>. For a Hunt process, "accessible" = "previsible."

Proof. Let T be previsible and $\{T_n\}$ announce T, namely $T_n < T$ for all n and $T_n \uparrow T$. Then by quasi left continuity,

$$X_T = \lim_n X_{T_n} \quad \text{on} \quad \{T < \infty\} .$$

Since $X_{T_n} \in \mathfrak{F}_{T_n+} \subset \mathfrak{F}_{T-}$ it follows that $X_T I_{\{T<\infty\}} \in \mathfrak{F}_{T-}$ and consequently by (5), $\mathfrak{F}_{T+} = \mathfrak{F}_{T-}$. The conclusion then follows from Dellacherie's criterion: "if $\mathfrak{F}_{T+} = \mathfrak{F}_{T-}$ for each previsible T, then accessible = previsible." ||

For an optional T, the condition (4) is the usual strong Markov property. Can we weaken this condition and still get (5)? Intuitively, a "0-1 law at T" should be sufficient. The following result shows that only the "future germ field" at T is involved, but it seems difficult to disentangle it from the past.

Proposition 3. Suppose X is right continuous. Then for any $t \geq 0$:

$$\mathfrak{F}_{T+t-} \subset \mathfrak{F}_{T-} \vee \mathfrak{F}_{[T,T+t)} ; \tag{9}$$

$$\mathfrak{F}_{T+t+} = \bigwedge_n [\mathfrak{F}_{T-} \vee \mathfrak{F}_{[T,T+t+n^{-1})}] . \tag{10}$$

Proof. A generating set of $\mathfrak{F}_{T+t-}$ is of the following form:

$$\{T+t > r; X_r \in A\} = \{T > r; X_r \in A\} \cup \{T = r; X_r \in A\} \cup \{r-t < T < r; X_r \in A\}$$

where $A \in \mathcal{E}$. Call the three sets on the right Λ_1, Λ_2 and Λ_3. By definition, $\Lambda_1 \in \mathfrak{F}_{T-}$, $\Lambda_2 \in \sigma(T; X_T)$. Using the same approximation as in the proof of Proposition 1, we have

$$\bigwedge_3 = \{0 < r-T < t;\ X(T+(r-T)) \in A\} \in \mathcal{F}_{[T,T+t)} \bigvee \sigma(T)\ .$$

Hence (9) is true. It follows that

$$(11) \qquad \mathcal{F}_{T+t+n^{-1}-} \subset \mathcal{F}_{T-} \bigvee \mathcal{F}_{[T,T+t+n^{-1})}\ .$$

Intersecting over n, we see that (10) is true provided "=" is replaced by "$\subset$". But for any $s \in [0, t+n^{-1})$, we have

$$X_{T+s} \in \mathcal{F}_{T+s+} \subset \mathcal{F}_{T+t+n^{-1}-}\ .$$

Hence the opposite inclusion to (11) is also true, and consequently the right member of (10) is included in

$$\bigwedge_n \mathcal{F}_{T+t+n^{-1}-} = \mathcal{F}_{T+t+}\ .$$

This establishes (10). ||

We do not know if the right member of (10) can be replaced by the smaller Borel field

$$\mathcal{F}_{T-} \bigvee [\bigwedge_n \mathcal{F}_{[T,T+t+n^{-1})}]\ .$$

Under the conditions of Proposition 2, this is true for $t = 0$, and in fact the above is then equal to the even smaller Borel field on the right side of (5). Of course this does not mean that

$$\bigwedge_n \mathcal{F}_{[T,T+n^{-1})} = \sigma(X_T)$$

as a random version of the 0-1 law.

In conclusion, let us remark that the results above may be extended to a process on $(-\infty,+\infty)$ in which case the random variable T will take values in $[-\infty,+\infty]$; see [1].

REFERENCES

[1] K. L. Chung and J. L. Doob. "Fields, Optionality, and Measureability," Amer. J. Math., 87 (1965), 397-424.

[2] R. M. Blumenthal and R. K. Getoor. Markov Processes and Potential Theory, Academic Press, 1968.