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THE PERFECTION OF MULTIPLICATIVE FUNCTIONALS

J. B. Walsh

Suppose M is a multiplicative functional of a Markov process. Then for each positive s and t

$$(1) \qquad M_{s+t}(\omega) = M_s(\omega)\, M_t(\theta_s\omega)$$

for all ω not in some null set N_{st}. If one can choose the exceptional null-set to be independent of s and t, M is called perfect. Now if M is right continuous, one can always eliminate the dependence on t (take the union of N_{st} over rational t) but the dependence on s remains. For additive functionals, it is known (see [1],[2],[3]) that one can often replace an imperfect functional by an equivalent perfect functional – in the case of Hunt processes satisfying hypothesis (L), C. Doléans has shown that one can always do so.

We would like to show that one can do this for exact multiplicative functionals, and that moreover one can entirely eliminate the exceptional set.

This final elimination doesn't appear too significant in itself, but let us note one consequence. If we know only that M satisfies (1) with an exceptional set independent of s and t – that is, if M is perfect in the usual sense – we can't deduce the identity

$$(2) \qquad M_{s+t}(\theta_u\omega) = M_s(\theta_u\omega)\, M_t(\theta_{u+s}\omega)$$

with an exceptional set independent of s,t, and u unless we know, for instance, that M never vanishes. On the other hand, if M satisfies (1) identically, (2) follows trivially. The class of functionals satisfying (2) with an exceptional null-set independent of s,t, and u seems somewhat more natural than that of perfect ones, but bearing in mind the logician's dictum that one cannot improve on perfection, we don't try to give them a new name.

Only the simplest properties of multiplicative functionals intervene in the following, and the underlying Markov process is used only implicitly, but we will need a number of facts about essential limits. The reader will find these discussed in [4] .

Let E be a Borel subset of a compact metric space and let $(\Omega, \underline{\underline{F}}, \underline{\underline{F}}_t, X_t, P^x, \theta_t)$ be a realization of a Markov semigroup (P_t) on E . We suppose that X has right continuous paths and that the fields $\underline{\underline{F}}$ and $\underline{\underline{F}}_t$ are the usual completions of the natural Borel fields $\underline{\underline{F}}^o$ and $\underline{\underline{F}}^o_t$.

We will consider only positive multiplicative functionals bounded by 1. We adopt the usual definition (e.g. [1, p. 97]) except we <u>do not</u> assume they are adapted to the fields $(\underline{\underline{F}}_t)$. Let M_t be a multiplicative functional. We can assume that $t \to M_t(\omega)$ is right continuous and decreasing for all ω .

Let μ be a probability measure on E . If we fix t and apply Fubini's theorem to (1), we see that for P^μ-a.e. ω

$$(3) \qquad M_t(\omega) = M_s(\omega)\, M_{t-s}(\theta_s\omega) \quad \text{for } m\text{ - a.e. } s \leq t ,$$

where m is Lebesgue measure. Thus the multiplicativity of M holds except for sets of zero Lebesgue measure. This leads us to consider using essential limits, which ignore sets of measure zero. Define a new functional $\overline{M}_t$ by

$$(4) \qquad \overline{M}_t = \operatorname{ess}\limsup_{s\downarrow 0} M_{t-s} \circ \theta_s \quad \text{if } t>0$$

$$M_0 = \operatorname{ess}\limsup_{s\downarrow 0} \overline{M}_s$$

$\overline{M}_t$ is decreasing in t since the same is true of M_t .

We needed to know something about the joint measurability of $M_t \circ \theta_s$ in s and t in order to have used Fubini's theorem above. The following lemma provides justification for this, and, combined with Prop. 2.1 of [4], shows that $\overline{M}$ is measurable.

<u>Lemma 1</u>

The functions $(s,t) \to M_t \circ \theta_s$ and $(s,t) \to M_{t-s} \circ \theta_s$ are $dm \times P^\mu$-measurable for each probability measure μ on $\underline{\underline{E}}$. (In the terminology of [4], they are nearly Lebesgue measurable.)

<u>Proof</u> : Since $s \to [f_1(X_{t_1}) \ldots f_n(X_{t_n})] \circ \theta_s$ is $dm \times \underline{\underline{F}}^\circ$-measurable, the monotone class theorem shows that for any $\underline{\underline{F}}^\circ$-measurable Z, $s \to Z \circ \theta_s$ is also $dm \times \underline{\underline{F}}^\circ$-measurable. If Z is only $\underline{\underline{F}}$-measurable, let μ be a probability measure on μ and let $\nu = \mu U_1$, where (U_p) is the resolvant of P_t. There exist $\underline{\underline{F}}^\circ$-measurable Z' and Z'' such that $Z' \leq Z \leq Z''$ and $P^\nu(Z'' > Z') = 0$. But $0 = P^\nu(Z'' > Z') = \int e^{-s} P^\nu\{Z'' \circ \theta_s > Z' \circ \theta_s\} ds$. By Fubini, the set $\{(s,\omega) : Z'' \circ \theta_s > Z'' \circ \theta_s\}$ has $dm \times P^\mu$-measure zero. If we take $Z = M_t$, this gives us the first assertion above, and the second follows because $t \to M_t$ is right continuous. q.e.d.

<u>Remark</u> :

This is the last time we will use the right continuity of X or even the fact that E is a topological space until we talk about strong Markov functionals. We say M is <u>exact</u> if for each $t > 0$, $x \in E$ and sequence $\varepsilon_n \downarrow 0$

$$(5) \qquad M_{t-\varepsilon_n} \circ \theta_{\varepsilon_n} \to M_t \qquad P^x \text{ - a.s.}$$

In general $\overline{M}_t \neq M_t$, but if M is exact, then (5) tells us that $M_t = \operatorname{ess\,lim}_{s \downarrow 0} M_{t-s} \circ \theta_s = \overline{M}_t$ P^x - a.s.

<u>Theorem</u>

Let M be a multiplicative functional. Then $\overline{M}$ (defined by (4)) is an exact perfect multiplicative functional (i.e. (2) holds). If M is exact, then M and $\overline{M}$ are equivalent.

Proof : Let $\Lambda=\{\omega:\text{for m-a.e. } s, M_t(\omega)=M_s(\omega)M_{t-s}(\omega)\ \forall t\geq s\}$. As we remarked, $P^\mu(\Lambda)=1$ for all initial measures μ . Let $\mathcal{S}(\omega)$ be the set of s such that $M_t(\omega)=M_s(\omega)M_{t-s}(\theta_s\omega)$ $\forall\ t\geq s$. If $\omega\in\Lambda$, $\mathcal{S}(\omega)$ will have full Lebesgue measure. Fix an initial measure μ . Using the Markov property of X , $P^\mu\{\theta_s\omega\in\Lambda\}=P^\nu\{\Lambda\}=1$, where $\nu=\mu P_s$. Apply Fubini's theorem: for P^μ-a.e ω , $\theta_s\omega\in\Lambda$ for m - a.e. s . Thus if we define a set Γ by

(6) $$\Gamma = \{\omega:\theta_s\omega\in\Lambda \text{ for m-a.e. } s\geq 0\} ,$$

we have $P^\mu\{\Gamma\}=1$. It is immediate that

(7) $$\theta_t\Gamma\subset\Gamma \quad \text{all } t\geq 0 ,$$

and

(8) if $\theta_t\omega\in\Gamma$ for a sequence of $t\downarrow 0$, then $\omega\in\Gamma$.

Γ will turn out to be the promised "good" set for $\overline{M}$. The key to the proof is the following lemma.

Lemma 2

If $\omega\in\Gamma$ then

(i) $s\to\overline{M}_{t-s}(\theta_s\omega)$ is right continuous and increasing on $[0,t)$;

(ii) $\overline{M}_{t-s}(\theta_s\omega) = M_{t-s}(\theta_s\omega)$ for a.e. $s<t$;

(iii) $t\to\overline{M}_t(\omega)$ is right continuous.

Let us admit this lemma for the minute and see how the proof of Theorem 1 follows.

Let $\omega\in\Gamma$ and $t>0$. Then for a.e. $s\leq t$ we have

(a) $$M_{t-s}(\theta_s\omega) = \overline{M}_{t-s}(\theta_s\omega)$$

(b) For a.e. $\varepsilon<s$ both $M_{s-\varepsilon}(\theta_\varepsilon\omega) = \overline{M}_{s-\varepsilon}(\theta_\varepsilon\omega)$ and $s\in\mathcal{S}(\theta_\varepsilon\omega)$ (so multiplicativity holds).

(a) follows from (ii) and (b) follows from Fubini's theorem and the fact that for a.e. ε the set $\mathcal{S}(\theta_\varepsilon\omega)$ has full Lebesgue measure. If s satisfies (a) and (b) we can write for a.e. $\varepsilon<s$

$$M_{t-\varepsilon}(\theta_\varepsilon\omega) = M_{s-\varepsilon}(\theta_\varepsilon\omega)\, M_{t-s}(\theta_s\omega)$$

$$= \overline{M}_{s-\varepsilon}(\theta_\varepsilon\omega)\, \overline{M}_{t-s}(\theta_s\omega)\,.$$

Take the essential limit as $\varepsilon\downarrow 0$, and use the lemma. We get

$$\overline{M}_t(\omega) = \overline{M}_s(\omega)\, \overline{M}_{t-s}(\theta_s\omega) \qquad \text{for a.e. } s\,. \tag{9}$$

But both sides of (9) are right continuous in s so it is in fact true for all s . Now use the fact that $\theta_u\Gamma\subset\Gamma$ to get (2) . The exactness of $\overline{M}$ follows from Lemma 2 (i).

Finally, if M is exact, $M_t=\overline{M}_t$ P^μ-a.e. for each fixed t , hence for all rational t . By right continuity, they must be equivalent.-q.e.d.

It remains to prove the lemma.

<u>Proof (of Lemma 2)</u>. For any $t>0$

$$\overline{M}_{t-s}\circ\theta_s = \operatorname{ess}\limsup_{u\downarrow s} M_{t-u}\circ\theta_u = \operatorname{ess}\limsup_{u\downarrow s} \overline{M}_{t-u}\circ\theta_u \tag{10}$$

where the last equality is just a property of the lim sup. Fix an $\omega\in\Gamma$. If $\theta_s\omega\in\Lambda$, then $u-s\in\mathcal{S}(\theta_s\omega)$ for a.e. $u>s$ so that

$$M_{t-s}(\theta_s\omega) \leqslant M_{t-u}(\theta_u\omega)$$

(for $M_{u-s}(\theta_s\omega)\leqslant 1$). It is not hard to see from this that $M_{t-s}(\theta_s\omega)$ is essentially increasing in s , hence that it has essential right limits at all points. Combined with (10), this implies (i) . (ii) is a consequence of (i) and Lebesgue's density theorem, for if $f(t)$ is any measurable function having essential right limits everywhere, then f equals its right continuous regularization a.e.

Now consider $\overline{M}_t(\omega)$. Since $\overline{M}_t$ decreases, if $\overline{M}_t=0$ for some t , it is identically zero - hence right continuous - on $[t,\infty)$. Thus suppose $\overline{M}_t>0$. By (i) and (ii) , for a.e. small enough s both $\theta_s\omega\in\Lambda$ and $M_{t-s}(\omega)=\overline{M}_{t-s}(\omega)>0$. For such an s we can write

$$
\text{(11)} \qquad \overline{M}_{t+\varepsilon-s} = \operatorname*{ess\,lim}_{u\downarrow s} M_{t+\varepsilon-u}(\theta_u\omega) = \operatorname*{ess\,lim}_{u\downarrow s} M_{t-u}(\theta_u\omega)\frac{M_{t+\varepsilon-u}(\theta_u\omega)}{M_{t-u}(\theta_u\omega)}
$$

where the denominator is strictly positive for a.e. small enough $u>s$. But the ratio on the right is independent of u for a.e. u : if $\theta_u\omega\in\Lambda$ and $u_0-u\in\mathcal{S}(\theta_u\omega)$, then

$$
M_{t-u}(\theta_u\omega) = M_{u_0-u}(\theta_u\omega)M_{t-u_0}(\theta_{u_0}\omega) \quad ;
$$

The same equation holds with t replaced by $t+\varepsilon$. Taking the essential limit, (11) becomes

$$
= \overline{M}_{t-s}(\theta_s\omega) \ \frac{M_{t+\varepsilon-u_0}(\theta_{u_0}\omega)}{M_{t-u_0}(\theta_{u_0}\omega)}
$$

The ratio tends to 1 as $\varepsilon\downarrow 0$ by right continuity, and we are done.

<u>Corollary</u>.

Let M be an exact multiplicative functional. Then M is equivalent to a perfect multiplicative functional whose exceptional set is empty.

Proof : Let $\overline{M}$ be the regularization of M given by Theorem 1 , and let Γ be the set defined in (6) . Define

$$T(\omega) = \inf \{t \geq 0 : \theta_t \omega \in \Gamma\}$$

and define $\hat{M}$ by

$$\hat{M}_t(\omega) = 1 \quad \text{if} \quad 0 \leq t < T(\omega)$$

and

$$\hat{M}_{T(\omega)+t}(\omega) = \overline{M}_t(\theta_{T(\omega)}\omega) \ .$$

Note that $\hat{M}.=\overline{M}.$ on Γ ; since $\Gamma-\Omega$ is a null set, $\hat{M}$ and $\overline{M}$ are equivalent, hence $\hat{M}$ and M are equivalent. If $\omega \in \Gamma$

(12) $$\hat{M}_{s+t}(\omega) = \hat{M}_s(\omega)\ \hat{M}_t(\theta_s \omega)$$

since $\overline{M}.(\omega)=\hat{M}.(\omega)$ and $\hat{M}.(\theta_s\omega)=\overline{M}.(\theta_s\omega)$ (for $\theta_s\omega\in\Gamma$ by (7)). It is easily verified that (12) holds on $\Omega-\Gamma$ as well once we notice that by (8), $\theta_{T(\omega)}\omega\in\Gamma$.

STRONG MARKOV FUNCTIONALS

So far only the ordinary Markov property has come into play and the multiplicative functionals have not necessarily been adapted. Now we will suppose that X is a right continuous strong Markov process on E and that M is a multiplicative functional adapted to the fields $\underline{\underline{F}}_t$.

An exact multiplicative functional is known to be strongly Markov. In fact, as P.A. Meyer remarked to us, any multiplicative functional M of a strong Markov process which is equivalent to a perfect multiplicative functional $\hat{M}$ must be strongly Markov itself. For, if T is a stopping time, then P^x-a.s.

$$M_{T(\omega)+s}(\omega) = \hat{M}_{T(\omega)+s}(\omega)$$
$$= \hat{M}_{T(\omega)}(\omega)\ \hat{M}_s(\theta_{T(\omega)}\omega)$$
$$= M_{T(\omega)}(\omega)\ M_s(\theta_{T(\omega)}\omega) \quad ,$$

where the first equality comes from equivalence, the second from perfection of $\hat{M}$, and the third from equivalence and the strong Markov property of X . (This is true whether or not M is adapted.)

Let E_M denote the set of permanent points of M , i.e. those x for which $P^x\{M=1\}=1$. If the set E_M is nearly Borel measurable, then the converse is also true : if M is strongly Markov, then M is equivalent to a perfect multiplicative functional with empty exceptional set. This is an immediate consequence of the following proposition.

Proposition

Let M be an adapted strong Markov multiplicative functional such that E_M is nearly Borel. Then M is equivalent to a multiplicative functional of the form $\overline{M}N$, where $\overline{M}$ is defined by (4) and $N_t=I_{\{t<D_K\}}$, D_K being the debut of a thin nearly Borel set.

Proof : Let D_{K_1} be the debut of $K_1=E-E_M$. Note that $M_t \leq I_{\{t<D_{K_1}\}}$ a.s., for by Meyer's section theorem there is a sequence of stopping times $T_n \downarrow D_K$ such that $P\{X_{T_n} \in K_1\} \to P\{D_{K_1}<\infty\}$. The strong Markov property of M implies that $M_{T_n}=0$ a.e. on $\{X_{T_n} \in K_1\}$, hence $M.$ vanishes on $[D_K,\infty)$. $\overline{M}$ is exact, hence $E_{\overline{M}}$ is nearly Borel [1 p. 126] . The same argument shows that $\overline{M} \leq I_{\{t<D_{K_2}\}}$, where $K_2=E-E_{\overline{M}}$. Now set $K=K_1-K_2$. Then we have $M_t \leq \overline{M}I_{\{t<D_K\}}$. Equality is clear if

the process starts from an $x\in K_1$, for then both functionals vanish. If it starts from $x\in E-K_1$, then $M_o=0$ and for any $t>0$ and $\varepsilon<t$

$$M_t = M_\varepsilon M_{t-\varepsilon}\circ\theta_\varepsilon \qquad P^x\text{-a.s.}$$

Let $\varepsilon\downarrow 0$ thru any sequence; $M_\varepsilon \to 1$ and so $M_{t-\varepsilon}\circ\theta_\varepsilon \to M_t$. But this implies that

$$\overline{M}_t = \operatorname*{ess\,lim}_{s\downarrow 0} M_{t-s} = M_t \qquad P^x\text{-a.s} \quad ,$$

and we deduce that $M_t=\overline{M}_t I_{\{t<D_K\}}$ P^x-a.s. for all t .

To see that K is thin, it suffices to remark that if x is regular for $K_1\supset K$, then $M_{t-\varepsilon}\circ\theta_\varepsilon \to 0$ as $\varepsilon\to 0$ thru any sequence, hence $\overline{M}_t=0$ P^x-a.s., hence $x\in K_2$.- q.e.d.

Our results have two immediate applications : the perfection of terminal times and the perfection of cooptional times. The two applications are similar, but as the second has the novelty of involving non-adapted multiplicative functionals, we will give it and leave the question of terminal times to the reader.

A positive random variable L is <u>cooptional</u> if for each $t\geq 0$ one has

$$(13) \qquad L\circ\theta_t = (L-t)^+ \quad P^x \text{ - a.s.} \quad \forall x\in E \ .$$

Set $M_t = I_{\{L>t\}}$. Then M is an exact, tho non-adapted, multiplicative functional and is hence equivalent to a perfect multiplicative functional $\hat{M}$. Define $\hat{L}(\omega)=\inf\{t\geq 0:\hat{M}_t=0\}$. $\hat{L}$ is a perfect cooptional time — i.e. (13) holds for all t simultaneously — and is indistinguishable from L . If we choose $\hat{M}$ as in the proof of the corollary, $\hat{L}$ even satisfies (13) identically in t and ω .

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