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BRANCHING PROPERTY OF MARKOV PROCESSES

Masao Nagasawa

Introduction

In a probabilistic treatment of semi-linear parabolic equations key steps were to construct a certain class of Markov processes on a "big" state space and to prove the branching property of semi-groups of the processes (cf. Ikeda-Nagasawa-Watanabe [1]). In this paper branching property will be characterized in terms of multiplicativity (or decomposability) of semigroup of a Markov process on a multiplicative state space (cf. § 1 for definitions). This characterization will then be applied to branching Markov processes with age and sign in § 2, and to a branching Markov process treated recently by Sirao [7] in § 3.

1. Multiplicativity of Markov processes

Definition 1.1. A measure space $\mathcal{S}$ is called multiplicative if a multiplication with a unit is defined in it, i.e. if

(1) for $\underline{\underline{a}}, \underline{\underline{b}} \in \mathcal{S}$ a multiplication $\underline{\underline{a}} \cdot \underline{\underline{b}} \in \mathcal{S}$ is defined and measurable,

(2) there exists a unit $\partial \in \mathcal{S}$ such that

$$\partial \cdot \underline{\underline{a}} = \underline{\underline{a}} \cdot \partial = \underline{\underline{a}} \qquad \text{for every } \underline{\underline{a}} \in \mathcal{S} .$$

Given a multiplicative space $\mathcal{S}$, taking $N = \{0,1,2,\ldots\}$ and $J = \{0,1\}$, the state space E of Markov processes treated in this section is one of $\mathcal{S}$, $\mathcal{S} \times N$, $\mathcal{S} \times J$, and $\mathcal{S} \times N \times J$. Theorems will be stated for $E = \mathcal{S} \times N \times J$, but all arguments are valid for other cases. Only we need to do is to get rid of unnecessary variables.

We denoted the space of measurable functions on E by $\mathcal{L}(E)$ (and $\mathcal{L}(\mathcal{S})$ etc.) and $\mathcal{L}^+$ denotes the space of non-negative elements of $\mathcal{L}$.

<u>Definition 12</u>. $F \in \mathcal{L}(E)$ is said to be <u>multiplicative</u> if

(3) $F(\partial, 0, 0) = 1$,

(4) $F(\underline{a} \cdot \underline{b}, k, j) = g(k, j) F(\underline{a}) F(\underline{b})$,

where $F(\underline{a}) = F(\underline{a}, 0, 0)$ and

(5) $g(k, j) = (-1)^j \lambda^k$

for a fixed positive constant λ. *)

Let $(X_t, B_t, P_\cdot)$ be a strong Markov process **) on the state space E where $\{ (\partial, k, j);\ k \in N,\ j \in J \}$ are traps and let τ be a fixed quasi-hitting time ***) satisfying

(6) $$P_\cdot[\tau = t] = 0 \qquad \text{for all } t \geq 0 .$$

Introducing a sequence of Markov times

(7) $$\tau_0 \equiv 0$$
$$\tau_1 = \tau$$
$$\tau_n = \tau_{n-1} + \tau \circ \theta_{\tau_{n-1}}, \qquad \text{for } n = 2, 3, \ldots,$$

we define kernels; for a measurable subset B of E,

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*) We will take $\lambda = 2$ in applications.

**) X_t is supposed to be measurable.

***) A B_t Markov time τ is said to be <u>quasi-hitting time</u> if $t \leq \tau$ implies $\tau = t + \tau \circ \theta_t$, where θ_t is the shift operator.

$$U_t^{(r)}(\cdot,B) = P_\cdot[X_t \in B,\ \tau_r \le t < \tau_{r+1}], \qquad r=0,1,2,\dots ; \tag{8}$$

for a measurable subset $\Gamma \times B$ in $[0,\infty)\times E$,

$$\psi(\cdot,\ \Gamma\times B) = P_\cdot[\tau\in\Gamma,\ X_\tau \in B] . \tag{9}$$

Let μ and ν be finite measure on E. We assume in addition the support of ν is $\mathcal{S}\times N \times\{0\}$. Then a product measure $\mu*\nu$ on E is defined by

$$\int_E \mu*\nu\,(d(\underline{c},k,j))F(\underline{c},k,j) = \int_E \mu(d(\underline{c}_1,k_1,j))\int_E \nu(d(\underline{c}_2,k_2,0))\cdot F(\underline{c}_1\cdot\underline{c}_2,k_1+k_2,j) , \tag{10}$$

for $F\in\mathcal{L}(E)^+$. $\nu*\mu$ can be defined in the same way interchanging the order in (10).

When μ is a finite measure on $[0,\infty)\times E$, and ν depends on t and measurable in t, then a product measure $\mu*\nu\,(\Gamma\times B)$ is defined by

$$\begin{aligned}\int_{[0,\infty)\times E} & \mu*\nu\,(ds,d(\underline{c},k,j))F(s,\underline{c},k,j) \\ &= \int_{[0,\infty)\times E} \mu(ds,d(\underline{c}_1,k_1,j))\int_E \nu_s(d(\underline{c}_2,k_2,0))F(s,\underline{c}_1\cdot\underline{c}_2,k_1+k_2,j)\end{aligned} \tag{11}$$

for $F\in\mathcal{L}([0,\infty)\times E)^+$.

In this section we assume that the Markov process $(X_t,B_t,P_\cdot)$ on E satisfies the following two conditions:

<u>Condition 1</u>. The support of $U_t^{(0)}((\underline{a},0,0),B)$ is $\mathcal{S}\times N\times\{0\}$, and for $F\in\mathcal{L}(E)$ of the form $F(\underline{a},k,j) = g(k,j)F(\underline{a})$, where $F(\underline{a}) = F(\underline{a},0,0)$,

$$(12)\quad \int_E U_t^{(o)}((\underline{\underline{a}}\cdot\underline{\underline{b}},k,j),d(\cdot))F(\cdot)$$

$$= g(k,j)\int_E U_t^{(o)}((\underline{\underline{a}},0,0),d(\cdot))*U_t^{(o)}((\underline{\underline{b}},0,0),d(\cdot))F(\cdot)\ :$$

Condition 2. *) For $F\in\mathcal{L}([0,\infty)\times E)$ of the form $F(s,\underline{\underline{a}},k,j)=g(k,j)F(s,\underline{\underline{a}})$ where $F(s,\underline{\underline{a}})=F(s,\underline{\underline{a}},0,0)$,

$$(13)\quad \int_0^t\int_E \psi((\underline{\underline{a}}\cdot\underline{\underline{b}},k,j),ds,d(\cdot))F(s,\cdot)$$

$$= g(k,j)\Big\{\int_0^t\int_E \psi((\underline{\underline{a}},0,0),ds,d(\cdot))*U_s^{(o)}((\underline{\underline{b}},0,0),d(\cdot))F(s,\cdot)$$

$$+\int_0^t\int_E U_s^{(o)}((\underline{\underline{a}},0,0),d(\cdot))*\psi((\underline{\underline{b}},0,0),ds,d(\cdot))F(s,\cdot)\Big\}\ ,$$

and $\psi((\partial,k,j),\cdot)$ has no mass on $[0,\infty)\times E$.

Remark. As a special case of (13),

$$\int_0^t\int_E \psi((\underline{\underline{a}},k,j),ds,d(\cdot))F(s,\cdot)=g(k,j)\int_0^t\int_E \psi((\underline{\underline{a}},0,0),ds,d(\cdot))F(s,\cdot).$$

All equalities in the paper should be so understood, unless otherwise stated, that if the left hand side has definite value then so does the right hand side and both sides coincide.

Lemma 1.1. Let $F\in\mathcal{L}(E)$ be multiplicative, then

$$(14)\quad U_t^{(r)}F(\underline{\underline{a}}\cdot\underline{\underline{b}},k,j)=g(k,j)\sum_{i=o}^{r}U_t^{(i)}F(\underline{\underline{a}})U_t^{(r-i)}F(\underline{\underline{b}}),\qquad r=0,1,2,\ldots,$$

where $U_t^{(i)}F(\underline{\underline{a}})$ stands for $U_t^{(i)}F(\underline{\underline{a}},0,0)$.

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*) The condition 1 and 2 are rephrasing of Property B.III. of [1], P. 262.

Proof. When $r=0$, (14) is implied by (12). Assume that (14) holds for $0,1,2,\ldots,r$, then by the strong Markov property and the induction hypothesis

$$U_t^{(r+1)}F(\underline{\underline{a}}\cdot\underline{\underline{b}},k,j) = \int_0^t\int_E \psi((\underline{\underline{a}}\cdot\underline{\underline{b}},k,j),ds,d(\underline{\underline{c}},k',j'))U_{t-s}^{(r)}F(\underline{\underline{c}},k',j')$$

$$= g(k,j)\Big\{\int_0^t\int_E \psi((\underline{\underline{a}},0,0),ds,d(\underline{\underline{c}}_1,k_1,j'))\int_E U_s^{(0)}((\underline{\underline{b}},0,0),d(\underline{\underline{c}}_2,k_2,0))$$

$$\cdot\sum_{i=o}^{r} g(k_1,j')U_{t-s}^{(i)}F(\underline{\underline{c}}_1)g(k_2,0)U_{t-s}^{(r-i)}F(\underline{\underline{c}}_2)$$

$$+\int_0^t\int_E \psi((\underline{\underline{b}},0,0),ds,d(\underline{\underline{c}}_2,k_2,j'))\int_E U_s^{(0)}((\underline{\underline{a}},0,0),d(\underline{\underline{c}}_1,k_1,0))$$

$$\cdot\sum_{i=o}^{r} g(k_1,0)U_{t-s}^{(i)}F(\underline{\underline{c}}_1)g(k_2,j')U_{t-s}^{(r-i)}F(\underline{\underline{c}}_2)\Big\} .$$

On the other hand, because

$$U_s^{(0)}U_{t-s}^{(r)}F(\cdot) = \int_s^t\int_E \psi(\cdot,du,d(\underline{\underline{b}},k,j))U_{t-u}^{(r-1)}F(\underline{\underline{b}},k,j) ,$$

by the strong Markov property and because it is continuous in s by the assumption (6), we have

$$U_t^{(r+1)}F(\underline{\underline{a}}\cdot\underline{\underline{b}},k,j)$$

$$= g(k,j)\sum_{i=o}^{r}\Big[\int_0^t\Big\{d(-U_s^{(0)}U_{t-s}^{(i+1)}F(\underline{\underline{a}}))U_s^{(0)}U_{t-s}^{(r-i)}F(\underline{\underline{b}})$$

$$+ U_s^{(0)}U_{t-s}^{(i)}F(\underline{\underline{a}})d(-U_s^{(0)}U_{t-s}^{(r+1-i)}F(\underline{\underline{b}}))\Big\}\Big] ,$$

where d applies to s, but since $d(-U_s^{(0)}U_{t-s}^{(0)}F) \equiv 0$ because

$U_s^{(0)}U_{t-s}^{(0)}F = U_t^{(0)}F$ does not depend on s, it takes the following form

$$g(k,j) \sum_{i=o}^{r+1} \int_0^t d(-U_s^{(0)}U_{t-s}^{(i)}F(\underline{\underline{a}})U_s^{(0)}U_{t-s}^{(r+1-i)}F(\underline{\underline{b}}))$$

$$= g(k,j) \sum_{i=o}^{r+1} U_t^{(i)}F(\underline{\underline{a}})U_t^{(r+1-i)}F(\underline{\underline{b}}) ,$$

which completes the proof.

Theorem 1. Let $F \in \mathcal{L}(E)$ be multiplicative. If

(15) $$U_tF(\cdot) = \sum_{r=o}^{\infty} U_t^{(r)}F(\cdot)$$

has definite value, then it is multiplicative, i.e.

(16) $$U_tF(\underline{\underline{a}}\cdot\underline{\underline{b}},k,j) = g(k,j)U_tF(\underline{\underline{a}})U_tF(\underline{\underline{b}}) ,$$

where $U_tF(\underline{\underline{a}}) = U_tF(\underline{\underline{a}},0,0)$.

Proof is straightforward by summing up both sides in (14) over $r=0,1,2,\ldots$;

$$U_tF(\underline{\underline{a}}\cdot\underline{\underline{b}},k,j) = g(k,j) \sum_{r=0}^{\infty} \sum_{i=o}^{r} U_t^{(i)}F(\underline{\underline{a}})U_t^{(r-i)}F(\underline{\underline{b}})$$

$$= g(k,j)U_tF(\underline{\underline{a}})U_tF(\underline{\underline{b}}) .$$

§ 2. Branching Markov processes with age and sign *)

Given a metric space S, let

$$\mathcal{S} = \bigcup_{n=o}^{\infty} S^n ,$$

where

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*) Lemma 5.1 in Nagasawa [2] is valid only when $q_o \equiv 0$. We can correct the lemma modifying the definition of branching law, but simpler construction is given in this section and Theorem 1 in § 1 will be applied to show the branching property.

$S^{o} = \{\partial\}$, ∂ an extra point, and

$S^{n} = S \times \ldots \times S$, n-fold Cartesian product of S. *)

If we define a multiplication of $\underline{\underline{x}} = (x_1, \ldots, x_n)$ and $\underline{\underline{y}} = (y_1, \ldots, y_m)$ by

$$\underline{\underline{x}} \cdot \underline{\underline{y}} = (x_1, \ldots, x_n, y_1, \ldots, y_m) ,$$

and

$$\partial \cdot \underline{\underline{x}} = (x_1, \ldots, x_n) ,$$

then $\mathcal{S}$ turns out to be multiplicative.

Given a strong Markov process $(W, B_t, x_t, P_x, x \in S)$ with right continuous paths on the state space S, the canonical realization of n-fold direct product of it is defined on the fundamental space

$$W^{n} = W \times \ldots \times W, \text{ n-fold product of } W .$$

When $\underline{\underline{w}} = (w_1, \ldots, w_n) \in W^n$, we put

$$\underline{\underline{x}}_t(\underline{\underline{w}}) = (x_t(w_1), \ldots, x_t(w_n)), \text{ and}$$

$$P_{(x_1, \ldots, x_n)} = P_{x_1} \times P_{x_2} \times \ldots \times P_{x_n} .$$

Then $(W^n, N_t, \underline{\underline{x}}_t, P_{\underline{\underline{x}}}, \underline{\underline{x}} \in S^n)$ **) is a strong Markov process with right continuous paths. (cf. Theorem 3.1 of [1].)

Let A_t be a right continuous non-negative additive functional of x_t, then

$$A_t(\underline{\underline{w}}) = A_t(w_1) + \ldots + A_t(w_n)$$

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*) In [1] and [2], symmetric product was employed instead of Cartesian product. However arguments becomes simpler if we take usual Cartesian product as pointed out by Sawyer [4].

**) $N_t = \sigma(x_s, s \leq t)$.

is that of $\underline{\underline{x}}_t(\underline{\underline{w}})$. Taking up a Poisson process (W', p_t, P'_k) with rate 1 which is independent of $\underline{\underline{x}}_t$, define

$$\Omega_n = W^n \times W' ,$$

$$P^o_{(\underline{\underline{x}},k)} = P_{\underline{\underline{x}}} \times P'_k ,$$

and for $(\underline{\underline{w}}, w') \in \Omega_n$

$$\begin{aligned} Z^o_t(\underline{\underline{w}}, w') &= (\underline{\underline{x}}_t(\underline{\underline{w}}), p_{A_t(\underline{\underline{w}})}(w')), \quad \text{if } A_t(\underline{\underline{w}}) < \infty \\ &= \Delta \quad \text{(an extra point), if } A_t(\underline{\underline{w}}) = \infty . \end{aligned}$$

Then $(\Omega_n, Z^o_t, P^o_{(\underline{\underline{x}},k)})$ is a strong Markov process on $S^n \times N$ (cf. [2], where a different construction is given and it is called a Markov process with age since p_{A_t} can be understood as "age" of particles.)

A strong Markov process with right continuous paths on $E = \mathcal{S} \times N \times J \cup \{\Delta\}$ is defined on

$$\Omega = \{\omega_{\partial k_j}\} \cup (\bigcup_{n=1}^{\infty} \Omega_n \times J) \cup \{\omega_\Delta\}$$

by

i) $\{(\partial, k, j);\ k \in N,\ j \in J\}$ are traps,

ii) $P^o_{(\underline{\underline{x}},k,j)} = P^o_{(\underline{\underline{x}},k)} \times \delta_{jj'}$

iii) for $\omega = (\underline{\underline{w}}, w', j) \in \Omega$

$$\begin{aligned} &Z^o_t(\omega) = (\underline{\underline{x}}_t(\underline{\underline{w}}), p_{A_t(\underline{\underline{w}})}(w'), j) \\ (1) \qquad &Z^o_t(\omega_{\partial k_j}) = (\partial, k, j) \\ &Z^o_t(\omega_\Delta) = \Delta . \end{aligned}$$

The next step is to define a branching distribution. Given a sequence of measurable functions $\{q_n(x)\}$ on S with

(2) $$\sum_{n=0}^{\infty} |q_n(x)| = 1, \quad x \in S ,$$

where $|q_n| = q_n^+ + q_n^-$, $q_n^+ = q_n \vee 0$ and $q_n^- = (-q_n) \vee 0$, and given a system of kernels $\pi_n(x,d\underline{y})$ on $S \times S^n$, which is a probability measure in $d\underline{y}$ and measurable in x, we define μ_i^+ and μ_i^- for $F \in \mathcal{L}(E)^+$ by, when $\underline{x} \in S^n$ $(n \geq 1)$,

(3) $$\mu_i^{\pm} F(\underline{x},k,j) = \int \mu_i^{\pm}((\underline{x},k,j),d(\underline{y},k',j'))F(\underline{y},k',j')$$

$$= q_o(x_i)F(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n,k,j^{\pm})$$

$$+ \sum_{m=1}^{\infty} \int_{S^m} q_m^{\pm}(x_i)\,\pi_m(x_i,dy_1 \times \ldots \times dy_m)F(x_1,\ldots,x_{i-1},y_1,\ldots,y_m, x_{i+1},\ldots,x_n,k,j^{\pm}) ,$$

where the first term is $q_o(x)F(\partial,k,j^{\pm})$ when $n = 1$, and j^+ and j^- are functions of j defined by the table 1.

j	j^+	j^-
0	0	1
1	1	0

Table 1 .

Furthermore we put $\mu^{\pm}F(\partial,k,j) = F(\partial,k,j)$.

Let $e_1,\ldots,e_n$ be n-random variables *) independent of each other and Z_t^o with exponential distribution $P_x[t < e_k | N_t] = e^{-t}$, and set

(4) $$\zeta_k(w_k) = \sup\{ t;\ A_t(w_k) \leq e_k\},$$

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*) We can always assume the existence of such random variables enlarging W and B_t if necessary.

and

$$(5) \qquad \tau(\underline{w}) = \min_{1 \le k \le n} \zeta_k(w_k) .$$

ζ_k is the life time of k-th particle killed by a multiplicative functional $M_t = e^{-A_t}$.

Then a branching distribution is defined by

$$(6) \quad \mu(\omega, B) = \sum_i I_{[\tau = \zeta_i]}(\mu_i^+(Z_\tau^0, B) + \mu_i^-(Z_\tau^0, B)) .$$

Because μ is an instantaneous distribution (cf. definition 2.1 of [1]), we can construct a big Markov process on E by piecing path Z_t^0, $t \le \tau$ together by the distribution μ. Then the big process $(Z_t, B_t, P(\underline{x}, k, j))$ is strong Markov with right continuous paths *) (Theorem 2.2 of [1] and cf. Chapter 3 of [4]).

From construction it is easily seen that the big process $(Z_t, P_\cdot)$ on E satisfies condition 1 and 2 in § 1.

If we define for $f \in \mathcal{L}(S)$

$$(7) \qquad \tilde{f}(\underline{x}, k, j) = (-1)^j \lambda^k \hat{f}(\underline{x})$$

where

$$(8) \qquad \hat{f}(\underline{x}) = \prod_{j=1}^{n} f(x_j), \qquad \text{if } \underline{x} = (x_1, \dots, x_n)$$

$$= 1, \qquad \text{if } \underline{x} = \partial$$

$$= 0, \qquad \text{if } \underline{x} = \Delta,$$

then $\tilde{f} \in \mathcal{L}(E)$ is multiplicative, and therefore applying Theorem 1, we have

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*) B_t is not equal to $N_t \equiv \sigma(Z_s; s \le t)$ in general. τ is B_t Markov time but not necessarily N_t Markov time.

Theorem 2. *The big Markov process* $(Z_t, P_\cdot)$ *on* E *has the branching property*

$$(9) \qquad U_t\widetilde{f} = \widetilde{(U_t\widetilde{f})|_S} \quad . \quad *)$$

Corollary. *If* $U_t f(x,0,0)$ *has definite value,* $u(t,x) = U_t\widetilde{f}(x,0,0)$ *is a solution of*

$$(10) \quad u(t,x) = T_t f(x) + \int_0^t \int K(x,ds,dy) \sum_{n=0}^{\infty} q_n(y) \int_{S^n} \pi_n(y,d\underline{z}) u_{t-s}(\underline{z}) \quad ,$$

where

$$T_t f(x) = E_x[f(x_t)] \quad ,$$

$$K(x,ds,dy) = E_x[I_{dy}(x_s) dA_s] \ .$$

Remark. As for applications cf. Sirao [6], Nagasawa-Sirao [3], and Nagasawa [2].

§ 3. Branching Markov processes with differential space **)

Let S be a locally compact Hausdorff space with a countable open base with C^∞- structure. A big state space $\mathcal{S}$ is defined constructively as follows:

$$\underline{\underline{S}}^{(0)} = \bigcup_{n=0}^{\infty} S^n \quad ,$$

(we use the notation $\underline{\underline{S}}^{(0)}$, which was denoted as $\mathcal{S}$ in § 2). Introducing an operation D, the role of which will be specified later on, $\underline{\underline{S}}^{(1)}$ is defined to be the collection of all elements of the form

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*) For a function F on E , $F|_S$ is a function on S defined by $F|_S(x) = F(x,0,0)$, $x \in S$.

**) The construction and the proof of branching property given in this section are simpler but essentially the same as Sirao's [7]. In [7], multiplication is assumed to be commutative.

$$(\underline{\underline{x}}_0, D(\underline{\underline{x}}_1), \ldots, D(\underline{\underline{x}}_m)), \qquad m=1,2,\ldots,$$

where $\underline{\underline{x}}_i \in \underline{\underline{S}}^{(0)}$, at least one of $\underline{\underline{x}}_i$ $(i>0)$ is not equal to ∂, and of all elements obtained by possible permutation of components. When ∂ and $D(\partial)$ appear in the components, they must be erased off, for example,

$$(\partial, D(\underline{\underline{x}}_1), D\partial, D(\underline{\underline{x}}_3)) \longrightarrow (D(\underline{\underline{x}}_1), D(\underline{\underline{x}}_3)) :$$

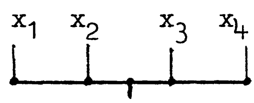
After $\underline{\underline{S}}^{(0)}, \underline{\underline{S}}^{(1)}, \ldots, \underline{\underline{S}}^{(n-1)}$ are defined, $\underline{\underline{S}}^{(n)}$ is the collection of all elements of the form

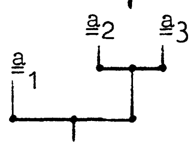
$$(\underline{\underline{a}}_0, D(\underline{\underline{a}}_1), \ldots, D(\underline{\underline{a}}_m)) , \qquad m=1,2,\ldots,$$

where $\underline{\underline{a}}_0 \in \underline{\underline{S}}^{(0)} \cup \underline{\underline{S}}^{(1)} \cup \ldots \cup \underline{\underline{S}}^{(n-1)}$, $\underline{\underline{a}}_j \in \underline{\underline{S}}^{(n-1)}$ $(j \geq 1)$, and of all elements obtained by possible permutation of components: Then finally we put

$$\mathcal{S} = \bigcup_{n=0}^{\infty} \underline{\underline{S}}^{(n)} . \tag{1}$$

Elements of $\mathcal{S}$ can be expressed by diagrams, for example,

$(x_1, x_2, Dx_3, Dx_4) \longleftrightarrow$ 

$(\underline{\underline{a}}_1, D(\underline{\underline{a}}_2 D\underline{\underline{a}}_3)) \longleftrightarrow$ 

<u>Definition 3.1</u>. The <u>stripping operator</u> γ is a mapping from $\mathcal{S}$ to $\underline{\underline{S}}^{(0)}$ which maps $\underline{\underline{a}} \in \mathcal{S}$ to an element in $\underline{\underline{S}}^{(0)}$ obtained by stripping off all D operation in $\underline{\underline{a}}$.

For example, when $\underline{\underline{a}} = (x_0, x_1, Dx_2, D(x_3, D(x_4, x_5)))$,

$$\gamma \underline{\underline{a}} = (x_0, x_1, \ldots, x_5) \in \underline{\underline{S}}^{(0)} .$$

The Cartesian product topology is given to S^n. Collecting all elements of the same form, we consider it as a subspace of

$\mathcal{S}$ and give the induced topology by γ^{-1}. For example, (x_1, Dx_2) belongs to a subspace $S \times DS$ with the induced topology from S^2 by γ^{-1}, ((Dx_1, x_2) belongs to a different subspace $DS \times S$). All different subspaces of $\mathcal{S}$ should be considered as discrete. Then $\mathcal{S}$ is a locally compact Hausdorff space with a countable open base.

If we define a multiplication in $\mathcal{S}$ by

(2) $$\underline{\underline{a}} \cdot \underline{\underline{b}} = (\underline{\underline{a}}, \underline{\underline{b}}) \qquad \text{for } \underline{\underline{a}},\ \underline{\underline{b}} \in \mathcal{S},$$

$$\partial \cdot \underline{\underline{a}} = \underline{\underline{a}} \cdot \partial = \underline{\underline{a}}\ ,$$

$$D \cdot \partial = \partial\ ,$$

then $\mathcal{S}$ is multiplicative.

Given a strong Markov process $(W, x_t, P_x,\ x \in S)$ on S with right continuous paths and let $T_t(x,dy)$ be the transition semi-group of the process.

We assume in this section that

(3) $$T_t D = D T_t$$

where D is a first order differential operator.

Remark. For example, when S is n-dimensional Eucledian space and if $T_t(x,dy)$ is given in terms of a C^{∞}-function $p_t(x)$ which vanishes at ∞ by

$$\int T_t(x,dy) f(y) = \int p_t(x-y) f(y) dy\ ,$$

where dy is the Lebesgue measure, then the assumption is satisfied.

Step 1. For each subspace $\mathcal{S}_1 \subset \mathcal{S}$, first we define a Markov process on $\mathcal{S}_1$ as follows: If $\gamma \underline{\underline{a}} = (x_1, \ldots, x_n), \underline{\underline{a}} \in \mathcal{S}_1$, taking n-fold direct product $(W^n, \underline{\underline{x}}_t, P_{\underline{\underline{x}}})$ of the given Markov process (x_t, P_x) on S, put

(4) $$X_t = \gamma^{-1}\underline{\underline{x}}_t ,$$

$$P_{\underline{\underline{a}}} = P_{\gamma \underline{\underline{a}}} .$$

Then $(W^n, X_t, P_{\underline{\underline{a}}})$ is a right continuous strong Markov process on $\mathcal{S}_1$ and the transition semigroup $T_t(\underline{\underline{a}}, d\underline{\underline{b}})$ is given by

(5) $$T_t(\gamma^{-1}(x_1,\dots,x_n), d(\gamma^{-1}(y_1,\dots,y_n)) = T_t(x_1, dy_1) \times \dots \times T_t(x_n, dy_n) ,$$

where γ is the stripping operator from $\mathcal{S}_1$ to S^n.

Secondly, taking a Poisson process (W', p_t, P'_k) with rate nc (c is a positive constant) which is independent of $(X_t, P_{\underline{\underline{a}}})$, put

$$\Omega_{\mathcal{S}_1} = W^n \times W' \times J, \qquad J = \{0, 1\}$$

$$Z^o_t(\underline{\underline{w}}, w', j) = (X_t(\underline{\underline{w}}), p_t(w'), j) ,$$

$$P^o_{(\underline{\underline{a}}, k, j)} = P_{\underline{\underline{a}}} \times P'_k .$$

Finally, collecting the processes thus defined on all subspaces, we can get a big (but still "minimal") Markov process $(\Omega^o, B^o_t, Z^o_t, P^o_\bullet)$ on $E = \mathcal{S} \times N \times J$, which is strong Markov with right continuous paths, where $\Omega^o = \bigcup \Omega_{\mathcal{S}_1}$ (union over all subspaces $\mathcal{S}_1$ of $\mathcal{S}$).

Step 2. Now, consider Z^o_t on a subspace $\mathcal{S}_1 \subset \mathcal{S}$ and assume $\gamma \underline{\underline{a}} = (x_1, \dots, x_n)$, $\underline{\underline{a}} \in \mathcal{S}_1$. We define the first branching time τ in the same way as in the previous section; taking n-random variables $e_1, \dots, e_n$ independent of each other and Z^o_t with exponential distribution, put

$$(6)\qquad \zeta_k = \sup\{t;\ ct \le e_k\},$$

$$\tau = \min_{1\le k\le n} \zeta_k .$$

Step 3. For $\underline{\underline{a}} \in \mathcal{S}$, $\gamma\underline{\underline{a}} = (x_1,\ldots,x_n)$, we define $\underline{\underline{a}}_m^{pq} \in \mathcal{S}$ as follows:

i) $\underline{\underline{a}}_m^{oo}$ is an element obtained by replacing x_m in $\underline{\underline{a}}$ by ∂,

ii) $\underline{\underline{a}}_m^{pq}$ ($p \neq 0$ or $q \neq 0$) is obtained by replacing x_m in $\underline{\underline{a}}$ by

$$\underbrace{x_m \cdots x_m}_{p}\ \underbrace{D(x_m) \cdots D(x_m)}_{q} .$$

Given constants c_{pq} with

$$\sum_{p,q=0}^{\infty} |c_{pq}| = 1 ,$$

kernels μ_m^+ are defined by

$$(7)\ \mu_m^+((\underline{\underline{a}},k,j),d(\underline{\underline{b}},k',j')) = \sum_{\substack{p,q=0\\ c_{pq}\ge 0}}^{\infty} |c_{pq}|\,\delta(\underline{\underline{a}}_m^{pq},d\underline{\underline{b}})\cdot\delta_{k,k'}\cdot\delta_{j^+,j'} ,$$

and μ_m^- replacing $c_{pq} \ge 0$ by $c_{pq} < 0$ and j^+ by j^- on the right hand side of (7), where j^+ and j^- are defined by table 1 in § 2. Then a branching distribution is given by

$$(8)\quad \mu(\omega,B) = \sum_m I_{[\tau=\zeta_m]}(\mu_m^+(Z^o_\tau,B) + \mu_m^-(Z^o_\tau,B)) .$$

Step 4. Applying Theorem of piecing out to the Markov process $(Z^o_t, P^o_\cdot)$ and the branching distribution μ, we can construct a "big" Markov process $(Z_t, B_t, P_\cdot)$ on E which is strong Markov with right continuous paths. (Theorem 2.2 of [1], cf. Chapter 3 of [4]). It is not difficult to see that by construction the big Markov process satisfies the condition 1 and

2 in § 1, and therefore <u>the big Markov process is multiplicative</u> by Theorem 1.

For $f \in C^{\infty}(S)$, we define a function $\tilde{f}$ on E inductively:

$$(9-1) \qquad \hat{f}(\partial) = 1$$

$$\hat{f}(x_1,\dots,x_n) = \prod_{i=1}^{n} f(x_i) \;;$$

when $\underline{\underline{a}} = (\underline{\underline{x}}_0, D(\underline{\underline{x}}_1),\dots,D(\underline{\underline{x}}_m)) \in \underline{\underline{S}}^{(1)}$

$$(9-2) \qquad \hat{f}(\underline{\underline{a}}) = \hat{f}(\underline{\underline{x}}_0) D\hat{f}(\underline{\underline{x}}_1)\dots D\hat{f}(\underline{\underline{x}}_m) \,,$$

where

$$(9-3) \qquad D\hat{f}(x_1,\dots,x_n) = \sum_{i=1}^{n} D_i f(x_1,\dots,x_n)$$

and D_i is a first order differential operator applied to the i-th variable x_i;

when $\underline{\underline{a}} = (\underline{\underline{a}}_0, D(\underline{\underline{a}}_1),\dots,D(\underline{\underline{a}}_m)) \in \underline{\underline{S}}^{(n)}$

$$(9-4) \qquad \hat{f}(\underline{\underline{a}}) = \hat{f}(\underline{\underline{a}}_0) D\hat{f}(\underline{\underline{a}}_1)\dots D\hat{f}(\underline{\underline{a}}_m)$$

where D is operated to $\hat{f}(\underline{\underline{a}}_i)$ regarded as a function of $\gamma \underline{\underline{a}}_i$; finally we define $\tilde{f}$ by

$$(9-5) \qquad \tilde{f}(\underline{\underline{a}},k,j) = (-1)^j \lambda^k \hat{f}(\underline{\underline{a}}), \quad \lambda > 0, \text{ a fixed constant.}$$

$\tilde{f}$ thus defined is multiplicative.

Let $U_t^{(r)}$ and ψ be defined for the big Markov process $(Z_t, P_\cdot)$ as in § 1. Then we have

<u>Lemma 3.1</u>. <u>For</u> $F \in \mathcal{L}([0,\infty)\times E)$,

$$(10) \qquad \int_0^t \int_E \psi((\underline{a},0,0),ds,d(\underline{b},k,j))(-1)^j 2^k F(s,\underline{b})$$

$$= \int_0^t c\,ds \int T_s(\underline{\underline{a}}, d\underline{\underline{b}}) \sum_m \sum_{p,q=o}^{\infty} c_{pq} F(s, \underline{\underline{b}}_m^{pq}) .$$

Proof. Because

$$E_x[\zeta \in ds,\ x_\zeta \in dy] = c\,ds\ T_s(x,dy)e^{-cs}, \quad \text{and}$$

$$E_x[x_s \in dy,\ s < \zeta] = T_s(x,dy)e^{-cs} ,$$

if $\gamma \underline{\underline{a}} = (x_1, \dots, x_n)$, denoting $\gamma \underline{\underline{b}} = (y_1, \dots, y_n)$, the left hand side of (10) takes the following form

$$\sum_{m=1}^{n} \int_0^t \int E_{x_m}[\zeta_m \in ds,\ x_{\zeta_m} \in dy_m] \prod_{\substack{l=1 \\ l \neq m}}^{n} E_{x_l}[x_s \in dy_l,\ s < \zeta_l]$$

$$\cdot \sum_{k=o}^{\infty} e^{-nct} \frac{(2nct)^n}{n!} \sum_{p,q=o}^{\infty} c_{pq} F(s, \underline{\underline{b}}_m^{pq})$$

$$= \sum_m \int_0^t c\,ds \int \prod_{l=1}^{n} T_s(x_l, dy_l) \sum_{p,q} c_{pq} F(s, \underline{\underline{b}}_m^{pq})$$

$$= \int_0^t c\,ds \int T_s(\underline{\underline{a}}, d\underline{\underline{b}}) \sum_m \sum_{p,q} c_{pq} F(s, \underline{\underline{b}}_m^{pq}) ,$$

which completes the proof.

Lemma 3.2. *For* $f \in C^{\infty}(S)$,

(11) $U_t^{(r)} \tilde{f}(D\underline{\underline{a}}) = D U_t^{(r)} \tilde{f}(\underline{\underline{a}})$ *for* $\underline{\underline{a}} \neq \partial$, $r = 0, 1, 2, \dots,$

where $U_t^{(r)} f(\underline{\underline{a}})$ *stands for* $U_t^{(r)} f(\underline{\underline{a}}, 0, 0)$.

Proof. When $r = 0$, by definition of $\tilde{f}$ and $U_t^{(0)}$ we have

$$U_t^{(0)}\tilde{f}(D\underline{\underline{a}}) = \int U_t^{(0)}((D\underline{\underline{a}},0,0),d(D\underline{\underline{b}},k,0))\tilde{f}(D\underline{\underline{b}},k,0)$$

$$= \int U_t^{(0)}((\underline{\underline{a}},0,0),d(\underline{\underline{b}},k,0))D\tilde{f}(\underline{\underline{b}},k,0)$$

$$= DU_t^{(0)}\tilde{f}(\underline{\underline{a}}) ,$$

where we applied commutativity $DU_t^{(0)} = U_t^{(0)}D$ in the last step.

Assume (11) holds for $0,1,2,\dots,r$, then by the strong Markov property of $(Z_t,P_\cdot)$ and lemma 3.1,

$$U_t^{(r+1)}\tilde{f}(D\underline{\underline{a}})$$

$$= \int_0^t \int \psi((D\underline{\underline{a}},0,0),ds,d(D\underline{\underline{b}},k,j))U_{t-s}^{(r)}\tilde{f}(D\underline{\underline{b}},k,j)$$

$$= \int_0^t cds \int T_s(D\underline{\underline{a}},d(D\underline{\underline{b}})) \sum_m \sum_{p,q} c_{p,q}U_{t-s}^{(r)}\ \tilde{f}(D\underline{\underline{b}}_m^{pq},0,0)$$

$$= \int_0^t cds \int T_s(\underline{\underline{a}},d\underline{\underline{b}}) \sum_m \sum_{p,q} c_{pq}\ U_{t-s}^{(r)}\ D\tilde{f}(\underline{\underline{b}}_m^{pq},0,0) .$$

But here, firstly, D commutes with $U_{t-s}^{(r)}$ by induction hypothesis: secondly because

$$\gamma\underline{\underline{b}}_m^{pq} = (y_1,\dots,y_{m-1},\underbrace{y_m,\dots,y_m}_{p+q},y_{m+1},\dots,y_n) \text{ if } \gamma\underline{\underline{b}} = (y_1,\dots,y_n),$$

we can write as

$$DU_{t-s}^{(r)}\ \tilde{f}(\underline{\underline{b}}_m^{pq},0,0) = \sum_{i=1}^{n} D_i\ U_{t-s}^{(r)}\tilde{f}(\underline{\underline{b}}_m^{pq},0,0) ,$$

where D_i operates to y_i, and then $D = \sum_{i=1}^{n} D_i$ commutes with $T_t(\underline{\underline{a}},d\underline{\underline{b}})$.

Finally we have

$$U_t^{(r+1)}\tilde{f}(D\underline{\underline{a}}) = D\int_0^t cds \int T_s(\underline{\underline{a}},d\underline{\underline{b}}) \sum_m \sum_{p,q} U_t^{(r)} f(\underline{\underline{b}}_m^{pq},0,0)$$

$$= DU_t^{(r+1)}\tilde{f}(\underline{\underline{a}}) \ ,$$

completing the proof.

Summing up (11) over $r = 0,1,2,\ldots,$

Lemma 3.3. When $U_t \tilde{f}(D\underline{\underline{a}}) = \sum_{r=0}^{\infty} U_t^{(r)}\tilde{f}(D\underline{\underline{a}})$ converges,

(12) $$U_t\tilde{f}(D\underline{\underline{a}}) = DU_t\tilde{f}(\underline{\underline{a}}), \qquad \text{for } \underline{\underline{a}} \neq \partial .$$

Theorem 3. When $U_t \tilde{f}(\underline{\underline{a}})$ has definite value, it has the branching property:

(13) $$U_t\tilde{f}(\underline{\underline{a}},k,j) = \widetilde{(U_t\tilde{f})|_S}(\underline{\underline{a}},k,j) \ .$$

Proof. Because $U_t\tilde{f}$ is multiplicative and satisfies (12), we can decompose it into multiplication and differentiation successively until we get the right hand side.

For example, when $\underline{\underline{a}} = (x_1,\ldots,x_n,Dy_1,\ldots,Dy_m)$, $x_1,\ldots,x_n,y_1,\ldots,y_m \in S$,

(14) $$U_t\tilde{f}(\underline{\underline{a}},0,0) = \prod_{i=1}^{n} U_t\tilde{f}(x_i,0,0) \prod_{j=1}^{m} DU_t\tilde{f}(y_j,0,0) \ ,$$

which will be of use in the following

Corollary. Choose $\lambda = 2$ in (9 - 4). If $U_t\tilde{f}(x,0,0)$ has definite value, then

(15) $$u(t,x) = U_t\tilde{f}(x,0,0)$$

is a solution of

(16) $$u(t,x) = T_tf(x) + \int_0^t cds \int_S T_s(x,dy) \sum_{p,q} c_{pq}(u(t-s,y))^p(Du(t-s,y))^q .$$

Proof. By the strong Markov property applied to the first branching time τ, we have

$$u(t,x) = T_t f(x) + \int_0^t c\,ds \int_S T_s(x,dy) \sum c_{pq} U_{t-s} \tilde{f}(y,\underbrace{\ldots}_{p},y,Dy,\underbrace{\ldots}_{q},Dy) .$$

Because of (14),

$$U_{t-s}\tilde{f}(y,\ldots,y,Dy,\ldots,Dy) = (u(t-s,y))^p (Du(t-s,y))^q ,$$

and hence $u(t,x) = U_t\tilde{f}(x,0,0)$ is a solution of (16) .

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