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NOTE ON A STOCHASTIC INTEGRAL EQUATION

by N.KAZAMAKI

In the present paper we shall consider the stochastic integral equation

$$(1) \qquad Z_t = x + \int_o^t f(Z_u)dM_u + \int_o^t g(Z_u)dU_u \ , \quad x \in R^1$$

where $M=(M_t)$, $M_o=0$, is a locally square integrable martingale and $U=(U_t)$, $U_o=0$, is a continuous increasing process.

Let $(\Omega, \underline{\underline{F}}, P)$ be a complete probability space, given an increasing right continuous family $(\underline{\underline{F}}_t)$ of sub σ - fields of $\underline{\underline{F}}$. We suppose as usual that $\underline{\underline{F}}_o$ contains all the negligible sets. By a normal change of time $A=(\underline{\underline{F}}_t, a_t)$ we means a family of stopping times of the family $(\underline{\underline{F}}_t)$, finite valued, such that for $\omega \in \Omega$ the sample function $a_.(\omega)$ is strictly increasing, $a_o(\omega)=0$, $a_\infty(\omega)=\lim_{t\to\infty} a_t(\omega)=\infty$ and continuous. We don't distinguish two processes X and Y such that for a.e $\omega \in \Omega$ $X_.(\omega)=Y_.(\omega)$. We assume that the reader knows the usual definitions.

THEOREM.- Assume that the family $(\underline{\underline{F}}_t)$ is quasi-left continuous. Then for coefficients f and g belonging to $C^1(R^1)$ and of bounded slope the equation (1) has one and only one solution.

PROOF.- From the quasi-left continuity of $(\underline{\underline{F}}_t)$, it follows that there exists a unique continuous increasing process $\langle M \rangle$ such that $M^2-\langle M \rangle$ is a local martingale. Define

$$(2) \qquad b_t = t + \langle M \rangle_t + U_t \ , \quad a_t = \inf(u; b_u > t) \ .$$

Then an easy computation shows that $A=(\underline{\underline{F}}_t, a_t)$ and $B=(\underline{\underline{F}}_{a_t}, b_t)$ are normal changes of time.

For every t, put

(3) $\qquad Y_t = Z_{a_t}$, $N_t = M_{a_t}$, $V_t = U_{a_t}$.

The process N is a square integrable martingale and V is the natural increasing process associated to N ; clearly V is, in fact, continuous. It is shown in [1] that we have

(4) $$\int_0^{a_t} f(Z_u)dM_u = \int_0^t f(Y_u)dN_u .$$

Thus , in order to show the existence of the unique solution of (1) , it suffices to consider the following stochastic integral equation

(1*) $$Y_t = x + \int_0^t f(Y_u)dN_u + \int_0^t g(Y_u)dV_u .$$

For simplicity, the proof is spelled out for $0 \leqq t \leqq 1$ only. Without loss of generality, we may assume that $\max(\|f'\|_\infty , \|g'\|_\infty) \leqq 1/2$.

Define in succession

(5)
$$Y_t^o = x$$
$$Y_t^n = x + \int_0^t f(Y_u^{n-1})dN_u + \int_0^t g(Y_u^{n-1})dV_u \quad , n=1,2,\dots$$

Put now

$$c_t^n = f(Y_t^n)-f(Y_t^{n-1}) , \quad d_t^n = g(Y_t^n)-g(Y_t^{n-1}) .$$

As $t = b_{a_t} = a_t + \langle N\rangle_t + V_t$ by the definition of a_t , we have

$$\begin{aligned} D_n(t) &= E[(Y_t^{n+1}- Y_t^n)^2] \\ &\leqq 2E[(\int_0^t c_u^n \, dN_u)^2 + (\int_0^t d_u^n \, dV_u)^2] \\ &\leqq 2 \left\{ E[\int_0^t (c_u^n)^2 d\langle N\rangle_u] + E[V_t \int_0^t (d_u^n)^2 dV_u] \right\} \\ &\leqq 2 \left\{ E[\int_0^t (c_u^n)^2 du] + E[\int_0^t (d_u^n)^2 du] \right\} \\ &\leqq 2(\|f'\|_\infty^2 + \|g'\|_\infty^2) \int_0^t E[(Y_u^n - Y_u^{n-1})^2] \, du \\ &\leqq \int_0^t D_{n-1}(u) \, du \leqq \text{Const.} \times t^n/n! \ . \end{aligned}$$

Since the process $(\int_0^t c_u^n \, dN_u)$ is a martingale, the extension of Kolmogorov's inequality to martingales shows that for any $\varepsilon>0$

$$\varepsilon^2 P(\sup_{0\leq t\leq 1} |\int_0^t c_u^n \, dN_u| \geq \varepsilon) \leq E[(\int_0^1 c_u^n \, dN_u)^2]$$
$$\leq E[\int_0^1 (c_u^n)^2 du]$$
$$\leq \|f'\|_\infty^2 \int_0^1 D_{n-1}(u) \, du$$
$$\leq \text{Const.} \times 1/n! \quad .$$

Similarly, we get by using the Schwarz inequality

$$P(\sup_{0\leq t\leq 1} |\int_0^t d_u^n \, dV_u| \geq \varepsilon) = P(\sup_{0\leq t\leq 1} [\int_0^t d_u^n dV_u]^2 \geq \varepsilon^2)$$
$$\leq P(\sup_{0\leq t\leq 1} V_t \cdot \int_0^t (d_u^n)^2 dV_u \geq \varepsilon^2)$$
$$\leq P(\int_0^1 (d_u^n)^2 du \geq \varepsilon^2)$$
$$\leq \varepsilon^{-2} E[\int_0^1 (d_u^n)^2 du]$$
$$\leq \text{Const.} \times \varepsilon^{-2}/n! \quad .$$

Thus $P(\sup_{0\leq t\leq 1} |Y_t^{n+1} - Y_t^n| \geq 2\varepsilon) \leq \text{Const.} \times \varepsilon^{-2}/n!$. Pick $\varepsilon^{-2} = (n-2)!$. Then $\varepsilon^{-2}/n!$ is the general term of a convergent sum, and so the Borel-Cantelli lemma shows that Y_t^n converges uniformly a.s for $0\leq t\leq 1$ to some random variable Y_t^* ; clearly Y_t^* is $\underline{\underline{F}}_{a_t}$ - measurable and for a.s ω the sample function $Y^*_\cdot(\omega)$ is right continuous.
Because of this, $f(Y_t^n)$ (resp. $g(Y_t^n)$) converges uniformly a.s to $f(Y_t^*)$ (resp. $g(Y_t^*)$).
According to THEOREM 10 of [1] , $\int_0^t f(Y_u^n) dN_u$ converges uniformly in probability to $\int_0^t f(Y_u^*) dN_u$, i.e for each $\varepsilon>0$, $\lim_n P(\sup_{0\leq t\leq 1} |\int_0^t f(Y_u^n) dN_u - \int_0^t f(Y_u^*) dN_u| > \varepsilon) = 0$.

Thus, for some subsequence (n_k), we get

$$(6) \qquad \lim_k \sup_{0\leq t\leq 1} |\int_0^t f(Y_u^{n_k}) \, dN_u - \int_0^t f(Y_u^*) \, dN_u| = 0 \quad \text{a.s}$$

As $\int_0^t g(Y_u^n)dV_u$ converges uniformly a.s to $\int_0^t g(Y_u^*)\, dV_u$, we have

$$(7) \qquad Y_t^* = x + \int_0^t f(Y_u^*)\, dN_u + \int_0^t g(Y_u^*)dV_u \quad .$$

This completes the proof of existence. We are now going to show its uniqueness. Let (Y_t^1) and (Y_t^2) be solutions of (1*). Then the random variable r defined by

$$r = \inf\ (t; \max_i |Y_t^i| \geqq n) \ .$$

is a stopping time of the family $(\underline{\underline{F}}_{a_t})$. We denote $Y_t^i\, I_{[t<r]}$ by $\hat{Y}_t^i$. Then for $t \leqslant r$ we have

$$\hat{Y}_t^2 - \hat{Y}_t^1 = \int_0^t [f(\hat{Y}_u^2) - f(\hat{Y}_u^1)]dN_u + \int_0^t [g(\hat{Y}_u^2) - g(\hat{Y}_u^1)]dV_u \ .$$

From the definition of r , $\hat{D}(t)=E[(\hat{Y}_t^2-\hat{Y}_t^1)^2] \leqq 4n^2 < \infty$. On the other hand , $\hat{D}(t) \leqq \int_0^t \hat{D}(u)\, du$ as in the proof of existence. Thus $\hat{D}(t) \equiv 0$, and making $n \to \infty$ we obtain the uniqueness statement. Consequently $BY^* = (Y_{b_t}^*, \underline{\underline{F}}_t)$ is the unique solution of the equation (1). Hence the theorem is established.

REFERENCE

[1] N.KAZAMAKI ; Some properties of martingale integrals , Ann.Inst.Henri Poincaré, vol.Vll, n°1, 1971.

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