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ON BALAYÉES OF EXCESSIVE MEASURES AND FUNCTIONS WITH RESPECT TO RESOLVENTS

by Takesi WATANABE

The balayage with respect to excessive measures and functions of a single kernel was studied by Deny [1]. Meyer [4] introduced the notion of the réduite of supermedian functions with respect to resolvents, and then he [5] applied it to the problem of characterizing the potential kernel of a single kernel. Here we will define the balayées of excessive measures and functions with respect to resolvents and study their properties. The basic method is the passage to the limit from those results obtained by Deny on balayées for a single kernel (see § 1).

In § 2 we will be concerned with the resolvents over a measurable space. The main result is Theorem 8 which states that the balayage operator L_A for excessive measures and the one $\hat{R}_A$ for excessive functions commute with respect to the potential kernel V. Section 3 is devoted to the study of resolvents with the continuous potential kernel over a locally compact space. It is shown (Theorem 11) that, if A is a relatively compact Borel set, the balayée over A of every excessive measure is a V-potential. In § 4 we will consider the case when the resolvent is the Laplace transform of a standard semi-group. We show (Theorem 16) that the balayage operator $\hat{R}_A$, acting on excessive functions, is represented as the kernel of the "harmonic" measure with respect to the penetration time for the set A. Using this and the preceding results, one can give alternative proofs of those results (Theorem 17) obtained by Hunt [2; § 9]. The method developed here enables us to extend Hunt's theorem to more general cases (see [9]).

§ 1. SUMMARY OF THE RESULTS FOR A SINGLE KERNEL

We mostly follow the notation and terminology of Meyer's book [4].

Let $(E,\underline{\underline{E}})$ be a measurable space, and $A \in \underline{\underline{E}}$. Let J_A be the restriction kernel, and f [resp. μ], a function over E [resp. a measure *) over $\underline{\underline{E}}$]. We will write $f|_A$ or f_A [resp. $\mu|_A$ or μ_A] for $J_A f$ [resp. μJ_A]. The notation $[f \geq g]_A$ means that $f(x) \geq g(x)$ for $x \in A$. Similarly, $[\mu \geq \nu]_A$ stands for $\mu_A \geq \nu_A$. A <u>positive function</u> means a nonnegative function allowing the value $+\infty$, unless it is stated as "strictly positive", "finite" and so on. Let μ, ν be σ-finite measures such that $\mu \geq \nu$. There exists a unique σ-finite measure λ such that $\mu = \nu + \lambda$. This measure is denoted by $\mu - \nu$. A sequence $\{\mu_n\}$ of σ-finite measures is said to converge to a σ-finite measure μ if every μ_n is domi-

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*) We consider only positive measures.

nated by some σ-finite measure ν and

(1.1) $$\mu(A) = \lim_{n \to \infty} \mu_n(A) \text{ for every } A \in \underline{\underline{E}} \text{ such that } \nu(A) < \infty .$$

Let K be a kernel. A σ-finite measure of the form μK is called the K-potential of μ. μ is said to belong to the domain of K. A function of the form Kf, where f is a positive function, is said to be the K-potential of f. Let μ be a measure. We often write (μ, f) or $\mu(f)$ for $\int f(x)\,\mu(dx)$.

Let N be a kernel over $(E, \underline{\underline{E}})$ and G, the potential kernel of N:

(1.2) $$G = \sum_{n \geq 0} N^n .$$

A positive function u is called excessive for N if $u \geq Nu$. A positive measure ν is called excessive for N if it is σ-finite and $\nu \geq \nu N$. Let ν [resp. u] be an excessive measure [resp. a finite excessive function]. The measure [resp. function]

(1.3) $$\mu = \nu(I - N) \quad [\text{resp. } f = (I - N)u]$$

is called the charge measure of ν [resp. charge function of u].

Let $A \in \underline{\underline{E}}$. Given any σ-finite measure ν, which is dominated by some excessive measure, define

(1.4) $$\nu L_A := \inf\{\nu' ;\ [\nu' \geq \nu]_A,\ \nu' \text{ is excessive}\} .$$

νL_A is excessive. νL_A is called the <u>balayée of</u> ν <u>over</u> A. Similarly, one defines the <u>balayée</u> (or <u>réduite</u>) $R_A u$ <u>of a positive measurable function</u> u by

(1.5) $$R_A u = \inf \{u' ; [u' \geq u]_A, \quad u' \text{ is excessive}\} .$$

It is shown (Mokobodzki) that $R_A u$ is excessive.

Define the kernels H_A, K_A by

(1.6) $$H_A = \sum_{n \geq 0} [(I - J_A)N]^n J_A ,$$

(1.7) $$K_A = \sum_{n \geq 0} J_A [N(I - J_A)]^n .$$

They are called the <u>balayage kernels</u>.

The whole results of this paper are based on the following results due to Deny [1].

THEOREM 1.

(a) <u>For every excessive measure</u> ν , <u>one has</u>

(1.8) $$\nu L_A = \nu K_A .$$

(b) <u>The charge measure of</u> νL_A <u>is given by</u>

(1.9) $$(\nu L_A)(I-N) = \mu H_A + \lim_{n \to \infty} (\nu N^{\infty}) [(I-J_A)N]^n J_A ,$$

<u>where</u> μ <u>is the charge measure of</u> ν <u>and</u> $\nu N^{\infty} = \lim_{n \to \infty} \nu N^n$. <u>In particular, the charge measure of</u> νL_A <u>is supported in</u> A.

(c) <u>If either</u> ν <u>is a potential or</u> $\nu|_A$ <u>belongs to</u>

the domain of G, νL_A *is a* G-*potential*. *)

(d) *If* ν *is a* G-*potential of a measure* μ *supported in* A,

$$(\mu G)L_A = \mu G \, . \tag{1.10}$$

Note that the assertion (d) is a version of the principle of domination.

THEOREM 2.

(a) *For every excessive function* u, *one has*

$$R_A u = H_A u \, . \tag{1.11}$$

(b) *If* u *is a finite excessive function, the charge function of* $R_A u$ *is given by*

$$(I-N)(R_A u) = K_A f + \lim_{n \to \infty} J_A [N(I-J_A)]^n (N^{\infty} u) \, , \tag{1.12}$$

where f *is the charge function of* u *and* $N^{\infty} u = \lim_{n \to \infty} N^n u$.

(c) *If either* u *is a finite potential or* $G(u|_A)$ *is finite*, then $R_A u$ *is a* G-*potential*.

(d) *If* u *is a* G-*potential of a positive function* f *supported in* A,

$$R_A(Gf) = Gf \, . \tag{1.13}$$

Hereafter we will write νK_A [resp. $H_A u$] for the balayée of an excessive measure ν [resp. an excessive function u] for a single kernel N.

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*) $\nu L_A = \nu K_A \leq \nu J_A \cdot G$. Hence, if $\nu|_A$ belongs to the domain of G, νL_A is a G-potential.

§ 2. BALAYAGE WITH RESPECT TO RESOLVENTS OVER MEASURABLE SPACES

General properties

Let $(V_\alpha)_{\alpha>0}$ be a resolvent over a measurable space and V, the potential kernel of (V_α):

$$V = \sup_{\alpha>0} V_\alpha = \lim_{\alpha\to 0} V_\alpha .$$

To lighten the exposition, we assume that $(V_\alpha)_{\alpha>0}$ is a submarkov resolvent and V is proper throughout the paper.

A supermedian [resp. excessive] measure ν for (V_α) is defined as a σ-finite measure such that, for every $\alpha > 0$,

$$(2.1) \qquad \nu \geq \nu\cdot\alpha V_\alpha \quad [\text{resp. moreover}, \nu = \lim_{\alpha\to\infty}\uparrow \nu\cdot\alpha V_\alpha].$$

We omit the well-known definition of supermedian [resp. excessive] functions.

Let $A \in \underline{\underline{E}}$. Given any σ-finite measure ν dominated by a supermedian measure, define

$$(2.2) \quad \nu L_A := \inf\{\nu';\ [\nu' \geq \nu]_A,\ \ \nu' \text{ is supermedian}\}.$$

νL_A is supermedian. νL_A is called the balayée of ν over A for (V_α). Let u be a positive measurable function. Define

$$(2.3) \quad R_A u := \inf\{u';[u' \geq u]_A,\ \ u' \text{ is supermedian}\}.$$

It is shown (Mokobodzki and Lemma 3) that $R_A u$ is supermedian. $R_A u$ is called the <u>réduite of</u> u <u>over</u> A <u>for</u> (V_α). Define

(2.4) $$\hat{R}_A u := \lim_{\alpha \to \infty} (\alpha V_\alpha)(R_A u) .$$

$\hat{R}_A u$ is excessive. $\hat{R}_A u$ is called the <u>balayée of</u> u <u>over</u> A <u>for</u> (V_α).

L_A, R_A, $\hat{R}_A$ are called the balayage operators.

NOTATION.

We write N_α for αV_α . H_A^α, K_A^α are the balayage kernels for the single kernel N_α. Let G_α be the potential kernel of N_α. By a well-known resolvent identity, one has

(2.5) $$G_\alpha = \sum_{n \geq 0} N_\alpha^n = \sum_{n \geq 0} (\alpha V_\alpha)^n = I + \alpha V .$$

Note that N_α is a submarkov kernel.

LEMMA 3.

Let $\alpha < \beta$.

(a) *If a σ-finite measure ν is excessive for N_β,* then it is excessive for N_α.

(b) *If a positive function u is excessive for N_β, then it is excessive for N_α.*

The proposition (b) was proved by Meyer [5; p.231]. The proof of (a) is the same.

It is obvious that a measure ν [resp. function u] is supermedian for (V_α) if and only if ν [resp. u] is excessive for N_α for every $\alpha > 0$.

THEOREM 4.

L_A <u>has the following properties for supermedian measures</u>.

(a) <u>If a sequence</u> $\{\nu_n\}$ <u>of supermedian measures increases to a supermedian measure</u> ν, <u>then</u> $\nu_n L_A$ <u>increases to</u> νL_A.

(b) <u>Let</u> ν <u>be supermedian</u>. <u>If</u> $A_j \uparrow A$ <u>in</u> $\underline{\underline{E}}$, <u>then</u> $\nu L_{A_j} \uparrow \nu L_A$.

(c) L_A <u>is positively linear for the class of all supermedian measures</u> : if ν_i (i=1,2) <u>is supermedian and</u> $\alpha, \beta \geq 0$,

(2.6) $$(\alpha \nu_1 + \beta \nu_2) L_A = \alpha (\nu_1 L_A) + \beta (\nu_2 L_A) .$$

(d) <u>Let</u> μ <u>be a measure, supported in</u> A <u>and belonging to the domain of</u> V. <u>Moreover, assume that there is an increasing sequence</u> $\{\lambda_n\}$ <u>of supermedian measures such that</u> $\lambda_n \wedge \mu$ <u>increases to</u> μ. <u>Then</u>

(2.7) $$(\mu V) L_A = \mu V .$$

THEOREM 5.

R_A <u>and</u> $\hat{R}_A$ <u>have the following properties for supermedian functions</u>.

(a) <u>If</u> $\{u_n\}$ <u>is an increasing sequence of supermedian functions, then</u> $R_A u_n$ [<u>resp.</u> $\widehat{R}_A u_n$] <u>increases to</u> R_A $(\lim u_n)$ [<u>resp.</u> $\widehat{R}_A$ $(\lim u_n)$].

(b) <u>Let</u> u <u>be supermedian. If</u> $A_j \uparrow A$ <u>in</u> $\underline{\underline{E}}$, <u>then</u> $R_{A_j} u$ [<u>resp.</u> $\widehat{R}_{A_j} u$] <u>increases to</u> $R_A u$ [<u>resp.</u> $\widehat{R}_A u$] .

(c) R_A <u>and</u> $\widehat{R}_A$ <u>are positively linear for the class of supermedian functions</u>.

(d) <u>If</u> $f \in p\underline{\underline{E}}$ *) <u>is supported in</u> A,

(2.8) $$Vf = R_A(Vf) = \widehat{R}_A(Vf) .$$

Theorem 5 was proved by Meyer [5; Théorème 5]. Theorem 4 is proved in the same way. We repeat the proof of Meyer for the convenience of latter reference.

Let ν be supermedian for (V_α). Since ν is excessive for N_α, the balayée of ν for N_α is νK_A^α. By Lemma 3(a) and the following remark, νK_A^α increases to a measure ν', supermedian for (V_α), as $\alpha \to \infty$. Let $\widetilde{\nu}$ be any supermedian measure such that $[\widetilde{\nu} \geq \nu]_A$. Obviously, $\widetilde{\nu} \geq \nu K_A^\alpha$, so that $\nu \geq \nu'$. One has proved that

(2.9) $$\nu L_A = \lim_{\alpha \to \infty} \uparrow \nu K_A^\alpha .$$

Since K_A^α is a kernel, (a) and (c) of Theorem 4 are immediate. The proof of (b) is also easy.

We proceed to the proof of (d). Write μ_n for

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*) positive $\underline{\underline{E}}$-measurable function.

$\lambda_n \wedge \mu$. By Theorem 1 (d) and formula (2.5), one has

$$\mu_n(I + \alpha V) = \mu_n(I + \alpha V)K_A^{\alpha} ,$$

so that

$$\mu_n V - \mu_n VK_A^{\alpha} \leq \alpha^{-1} \mu_n (I+\alpha V)K_A^{\alpha} - \mu_n VK_A^{\alpha} = \alpha^{-1} \mu_n K_A^{\alpha}$$
$$\leq \alpha^{-1} \lambda_n K_A \leq \alpha^{-1} \lambda_n ,$$

since λ_n is supermedian. Letting $\alpha \to \infty$, one gets

(2.10) $$\mu_n V = \lim_{\alpha \to \infty} \mu_n VK_A^{\alpha} = (\mu_n V)L_A .$$

Letting $n \to \infty$, one gets (2.7).

Similarly to (2.9), for every supermedian function u for (V_α),

(2.11) $$R_A u = \lim_{\alpha \to \infty} \uparrow H_A^{\alpha} u .$$

REMARK.

Note that Theorem 5(d) is nothing but the principle of domination of supermedian functions for (V_α). On the other hand, Theorem 4(d) shows that the principle of domination of supermedian measures is valid only under certain additional requirement on μ . Here is a simple example for which the principle of domination is false. Let (V_α) be the resolvent of $N(\geq 3)$-dimensional Brownian motion semi-group. V is the Newtonian potential kernel. Take $A = \{x\}$, $\mu = \varepsilon_x$. It is easy to see that $(\varepsilon_x V)L_{\{x\}} = 0 \neq \varepsilon_x V$ (see also Theorem 12).

<u>The kernels</u> VL_A <u>and</u> $\widehat{R}_A V$.

It is easy to see that the balayage kernels K_A and H_A for a single kernel commute with respect to the potential kernel G :

$$GK_A = H_A G . \tag{2.12}$$

In fact, let $\nu = \varepsilon_x G$ in Theorem 1. By (b) and (c), $(\varepsilon_x G)K_A$ is the G-potential of $\varepsilon_x H_A$, which proves (2.12). The balayage operators for a resolvent (V_α) do not admit the representation as kernels, in general, contrary to those for a single kernel. However, we can prove a relation similar to (2.12).

Define VL_A, $R_A V$, $\widehat{R}_A V$ by

$$VL_A(x,B) := [(\varepsilon_x V)L_A](B) ,$$

$$R_A V(x,B) := [R_A(VI_B)](x) ,$$

$$R_A V(x,B) := [\widehat{R}_A(VI_B)](x) .$$

It is immediate from Theorem 4 and 5 that each of VL_A, $R_A V$, $\widehat{R}_A V$ defines a proper kernel dominated by V. Moreover, using formula (2.9) and (2.11), one sees that

$$\mu \cdot VL_A = (\mu V)L_A, \quad R_A V \cdot f = R_A(Vf), \quad \widehat{R}_A V \cdot f = \widehat{R}_A(Vf), \tag{2.13}$$

$$\widehat{R}_A V = \lim_{\alpha \to \infty} \uparrow \alpha V_\alpha \cdot R_A V . \tag{2.14}$$

Obviously, $\widehat{R}_A V \cdot f$ is excessive for every $f \in p\underline{\underline{E}}$. Later we will show that

$$VL_A = \widehat{R}_A V .$$

LEMMA 6.

(a) If μ is a measure belonging to the domain of V, $\mu\cdot R_A V$ and $\mu\cdot\hat{R}_A V$ are excessive for (V_α).

(b) For every $f \in p\underline{\underline{E}}$, $VL_A\cdot f$ is excessive.

PROOF.

(a) For every $f \in p\underline{\underline{E}}$,

$$(\mu V,f) \geq (\mu\cdot R_A V,f) = (\mu, R_A V\cdot f) = (\mu, R_A(Vf))$$

$$= (\mu, \lim_{\alpha\to\infty}\uparrow H_A^\alpha (Vf))$$

$$= \lim_{\alpha\to\infty} (\mu H_A^\alpha \cdot V, f) .$$

Therefore, $\mu\cdot R_A V$ is a σ-finite measure and it is the increasing limit of V-potentials $\mu H_A^\alpha\cdot V$. One has proved that $\mu\cdot R_A V$ is excessive. By (2.14),

(2.15) $$\mu\cdot\hat{R}_A V = \lim_{\alpha\to\infty}\uparrow(\mu\cdot\alpha V_\alpha)\cdot R_A V ,$$

which proves that $\cdot\hat{R}_A V$ is excessive.

(b) The proof is similar to (a).

LEMMA 7.

(a)

(2.16) $$VL_A \leq R_A V .$$

(b) Suppose that μ is a measure dominated by a supermedian measure ν. Then

(2.17) $$\mu\cdot VL_A = \mu\cdot R_A V .$$

(c)

(2.18) $$VL_A = \hat{R}_A V .$$

PROOF.

(a) It is enough to show that, for every bounded $f \in p\underline{\underline{E}}$,

(2.19) $$(\varepsilon_x \cdot VL_A, f) \leq (\varepsilon_x, R_A V \cdot f) .$$

By (2.5), the following inequality is obvious:

$$(\varepsilon_x V \cdot K_A^\alpha, f) \leq (\varepsilon_x, G_\alpha K_A^\alpha \cdot \alpha^{-1} f) .$$

The left side increases to $((\varepsilon_x V)L_A, f) = (\varepsilon_x \cdot VL_A, f)$. On the other hand, by (2.12),

$$G_\alpha K_A^\alpha \cdot \alpha^{-1} f = H_A^\alpha G_\alpha \cdot \alpha^{-1} f = H_A^\alpha (\alpha^{-1} f + Vf) ,$$

$$H_A^\alpha \cdot \alpha^{-1} f \leq \alpha^{-1} \| f \|_\infty H_A^\alpha \cdot 1 \leq \alpha^{-1} \| f \|_\infty \to 0 \quad (\alpha \to \infty) ,$$

where $\| f \|_\infty = \sup | f(x) |$. Therefore,

$$\lim_{\alpha \to \infty} (\varepsilon_x, G_\alpha K_A^\alpha \cdot \alpha^{-1} f) = \lim_{\alpha \to \infty} (\varepsilon_x, H_A^\alpha (Vf))$$

$$= (\varepsilon_x, R_A(Vf)) = (\varepsilon_x, R_A V \cdot f) ,$$

so that (2.19) was proved.

(b) Let $\{\mu_n\}$ be a sequence of bounded measures increasing to μ. Let f be a bounded function in $p\underline{\underline{E}}$ which is supported in a set $B \in \underline{\underline{E}}$ such that $\nu(B) < \infty$. It is enough to show that, for every such f,

(2.20) $$(\mu_n \cdot VL_A, f) \geq (\mu_n, R_A V \cdot f) .$$

The proof is similar to that of (2.19). By (2.5),

$$(\alpha^{-1}\mu_n \cdot H_A^\alpha G_\alpha, f) \geq (\mu_n, H_A^\alpha Vf) .$$

The right-hand side increases to $(\mu_n, R_A(Vf)) =$ $(\mu_n, R_A V\cdot f)$. On the other hand,

$$\alpha^{-1}\mu_n \cdot H_A^\alpha G_\alpha = \alpha^{-1}\mu_n \cdot G_\alpha K_A^\alpha = (\alpha^{-1}\mu_n + \mu_n V)K_A^\alpha ,$$

$$\alpha^{-1}\mu_n \cdot K_A^\alpha \leq \alpha^{-1}\nu \cdot K_A^\alpha \leq \alpha^{-1}\nu ,$$

so that $(\alpha^{-1}\mu_n \cdot H_A^\alpha G_\alpha, f) \rightarrow ((\mu_n V)L_A, f) = (\mu_n \cdot VL_A, f)$, which proves (2.20).

(c) For each $\alpha > 0$, the measure $\varepsilon_x \cdot \alpha V_\alpha$ is dominated by the excessive measure $\alpha \cdot \varepsilon_x V$. Hence, by (b),

$$((\varepsilon_x \cdot \alpha V_\alpha) \cdot VL_A, f) = ((\varepsilon_x \cdot \alpha V_\alpha) \cdot R_A V, f), \quad f \in p\underline{\underline{E}} .$$

By Lemma 6 (b),

the left-hand side $= \alpha V_\alpha (VL_A \cdot f)(x) \rightarrow (VL_A \cdot f)(x)$.

But, by (2.14),

the right-hand side $= \alpha V_\alpha (R_A V\cdot f)(x) \rightarrow (\hat{R}_A V\cdot f)(x)$.

This proves (2.18).

One now summarize the preceding results.

THEOREM 8.

VL_A <u>and</u> $\hat{R}_A V$ <u>represent the same proper kernel</u>. <u>This kernel has the following properties</u>.

(a) <u>For every</u> μ <u>belonging to the domain of</u> V, $\mu \cdot VL_A$ <u>is the smallest supermedian measure among those</u>

which dominate μV over A. Moreover, $\mu \cdot VL_A$ is excessive for (V_α).

(b) For every $f \in p\underline{\underline{E}}$, $VL_A \cdot f$ is excessive. It is the smallest one among those supermedian functions which dominate Vf over A except on sets of potential zero. *)

It remains to prove the latter half of (b). But since $VL_A \cdot f = \hat{R}_A V \cdot f = \hat{R}_A(Vf)$, this is a special case of the next theorem (b).

THEOREM 9.

(a) If ν is excessive, νL_A is also excessive.

(b) Let u be a supermedian function. $\hat{R}_A u$ is excessive and it is the smallest one among those supermedian functions which dominate u over A except on sets of potential zero.

PROOF.

(a) Choose a sequence $\{\mu_n V\}$ of V-potentials increasing to ν. By Theorem 8(a), $(\mu_n V)L_A = \mu_n \cdot VL_A$ is excessive. Since $\nu L_A = \lim_{n \to \infty} \uparrow (\mu_n V)L_A$ by Theorem 3 (b), νL_A is excessive.

(b) Note that $[R_A u = \hat{R}_A u]_{E \setminus N}$, where N is a set of potential zero (Meyer [4; p. 195]). Since $[R_A u = u]_A$,

*) A set $B \in \underline{\underline{E}}$ is said to be of potential zero if $VI_B = 0$.

it follows that $[\hat{R}_A u = u]_{A \setminus N}$. Hence $\hat{R}_A u$ belongs to the class of functions described in (b).

Suppose that u' is a supermedian function such that $[u' \geq u]_{A \setminus N}$, where N is a set of potential zero. Since $N_\alpha = \alpha V_\alpha$, one has

$$[(I - J_A)N_\alpha]^n J_A u' \geq [(I - J_A)N_\alpha]^n J_A u \qquad (n \geq 1),$$

so that, by (1.6),

$$H_A^\alpha u' \geq H_A^\alpha u \text{ on } E \setminus N .$$

Therefore, letting $\alpha \to \infty$, one has $[R_A u' \geq R_A u]_{E \setminus N}$, which implies that $u' \geq \hat{R}_A u' \geq \hat{R}_A u$.

§ 3. RESOLVENTS OVER LOCALLY COMPACT SPACES

Let E be a locally compact space. We assume that E is σ-compact. (Many results are true without this restriction.) Excessive measures are restricted to Radon excessive measures. *) We omit the name "Radon". (V_α) is a submarkov resolvent of dispersion-kernels. We also assume that V is a dispersion-kernel such that VI_K is bounded for every compact set K.

NOTATION.

$\underline{\underline{B}}(E)$ [resp. $\underline{\underline{B}}_u(E)$] stands for the σ-algebra of all Borel [resp. universally measurable] subsets of E.

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*) Although such restriction is sometimes inconvenient, the present case is not the case.

$\underline{\underline{C}}_c(E)$ [resp. $\underline{\underline{C}}_b(E)$] is the collection of all continuous functions with compact supports, [resp. bounded continuous functions].

A dispersion-kernel K is said to be <u>continuous</u> [resp. <u>strictly positive</u>] if K maps $\underline{\underline{C}}_c(E)$ into $\underline{\underline{C}}_b(E)$ [resp. if all of the measures $K(x,\cdot)$ are not zero].

A resolvent (V_α) is said to be <u>weakly continuous</u> if $\alpha V_\alpha(x,\cdot)$ converges weakly* to ε_x for every x. It is obvious that, if (V_α) is weakly continuous, every V_α (and hence V) is strictly positive.

The following theorem is well-known.

THEOREM 10.

<u>Let</u> (V_α) <u>be weakly continuous</u>.

(a) <u>Every supermedian measure is excessive</u>.

(b) <u>If</u> ν <u>is a</u> V-<u>potential of a finite Radon measure</u> μ, <u>the charge measure</u> μ <u>is uniquely determined by the following formula</u>:

(3.1) $$(\mu,f) = \lim_{\alpha\to\infty} \alpha\left[(\nu,f) - (\nu,\alpha V_\alpha f)\right], \quad f\in\underline{\underline{C}}_c(E).$$

(c) <u>If</u> V <u>is a continuous kernel, the result</u> (b) <u>is valid without the restriction that</u> μ <u>is finite</u>.

PROOF.

(a) The proof is omitted.

(b) Since μ is finite, it follows that, for every $f \in \underline{\underline{C}}_c^+(E)$,

(3.2) $$(\mu, f) = \lim_{\alpha \to \infty} (\mu, \alpha V_\alpha f) = \lim_{\alpha \to \infty} \alpha (\mu, [V - \alpha VV_\alpha] f)$$
$$= \lim_{\alpha \to \infty} \alpha [(\nu, f) - (\nu, \alpha V_\alpha f)] .$$

(c) Let A be a relativelt compact set in $\underline{\underline{B}}_u(E)$. Since V is strictly positive and continuous, one can choose $g \in \underline{\underline{C}}_c^+(E)$ such that

(3.3) $$c = \inf_{x \in A} Vg(x) > 0 .$$

Let $f \in \underline{\underline{C}}_c^+(E)$. By the above remark, one can choose $g \in \underline{\underline{C}}_c^+(E)$ such that $f \leq Vg$. Hence $\alpha V_\alpha f \leq \alpha V_\alpha Vg \leq Vg$. But since $(\mu, Vg) = (\nu, g) < \infty$, the dominated convergence theorem is applied to obtain the first equality in (3.2).

THEOREM 11.

<u>Suppose that</u> V <u>is a strictly positive, continuous kernel. Let</u> ν <u>be an excessive measure for</u> (V_α) <u>and let</u> $A \in \underline{\underline{B}}_u(E)$ <u>be relatively compact</u>. <u>Then,</u> νL_A <u>is a</u> V-<u>potential of a measure</u> μ <u>supported in</u> $\overline{A}$.

PROOF.

We keep the notation N_α, G_α etc. in § 2. Since A is relatively compact, $\nu|_A$ belongs to the domain of G_α. By Theorem 1, νK_A^α is a G_α-potential of a measure μ_α supported in A;

(3.4) $$\nu K_A^{\alpha} = \mu_\alpha G_\alpha \ .$$

Choose $g \in \underline{\underline{C}}_c^+(E)$ satisfying (3.3). One has

$$(\nu, g) \geq (\nu K_A^{\alpha}, g) = (\mu_\alpha, G_\alpha g)$$
$$\geq \alpha(\mu_\alpha, Vg)$$
$$\geq c[\alpha \mu_\alpha(E)] ,$$

so that the measures $\{\alpha \mu_\alpha\}_{\alpha > 0}$ are uniformly bounded. Hence there is a subnet $\{\beta \mu_\beta\}$ which converges weakly* to a measure μ supported in $\overline{A}$. Since Vf is continuous for every $f \in \underline{\underline{C}}_c^+(E)$, one has

$$(\nu K_A^{\beta}, f) = (\mu_\beta G_\beta, f) = (\mu (I + \beta V), f)$$
$$= (\mu_\beta, f) + (\beta \mu_\beta, Vf)$$
$$\longrightarrow (\mu, Vf) = (\mu V, f) \ .$$

On the other hand, νK_A^{α} increases to νL_A as $\alpha \to \infty$, the subnet $\{\nu K_A^{\beta}\}$ converges weakly* to νL_A. Hence one has proved that $\nu L_A = \mu V$.

Here is a sufficient condition under which the domination principle for excessive measures is valid.

THEOREM 12.

<u>Suppose that</u> (V_α) <u>is weakly continuous. Moreover, suppose that either of the following conditions is fulfilled</u>.

(i) V <u>is a continuous kernel</u>.

(ii) <u>For every</u> $\alpha > 0$, V_α <u>is a continuous kernel</u>.

<u>Then the principle of domination for excessive measures is valid for every open set</u> A.

PROOF.

Let ν be a V-potential of a measure μ supported in A. Let ν' be an excessive measure for (V_α) such that $[\nu' \geq \nu]_A$. One has to show that

(3.5) $$\nu' \geq \nu = \mu V .$$

<u>Case</u> (i).

Take any compact subset K of A. It is enough to show that $\nu' \geq \mu J_K \cdot V$. Let B be a relatively compact open neighborhood of K, contained in A. Define $\tilde{\nu} = \nu' \wedge (\mu J_K \cdot V)$. By Theorem 11, $\tilde{\nu} L_B = \tilde{\mu} V$ with $\tilde{\mu}$ a measure supported in $\overline{B}$. Noting that $[\tilde{\nu} = \mu J_K \cdot V]_B$, one has

$$[\tilde{\mu} V = \mu J_K \cdot V]_B, \qquad \tilde{\mu} V \leq \mu J_K \cdot V .$$

Therefore, for every $f \in \underline{\underline{C}}_c(E)$ which is supported in B,

$$\begin{aligned}(\mu J_K, f) &= \lim_{\alpha \to \infty} \alpha [(\mu J_K \cdot V, f) - (\mu J_K \cdot V, \alpha V_\alpha f)] \\ &\leq \lim_{\alpha \to \infty} \alpha [(\tilde{\mu} V, f) - (\tilde{\mu} V, \alpha V_\alpha f)] \\ &= (\tilde{\mu}, f) ,\end{aligned}$$

which proves that $\mu J_K \leq \tilde{\mu}$. Hence,

$$\mu J_K \cdot V \leq \tilde{\mu} V = \tilde{\nu} L_B \leq \tilde{\nu} \leq \nu' .$$

Case (ii).

Apply the result of case (i) to the resolvent $\{ V_{\alpha+\beta}\}_{\beta>0}$ and the V_α -potential μV_α, and then take the limit $(\alpha \to 0)$.

Let $\{A_n\}$ be a sequence of relatively compact open sets, increasing to E. An excessive measure is said to be a potential if

$$\inf_n \nu L_{E\setminus A_n} = 0 . \tag{3.6}$$

Let E_p be the set of points $x \in E$ such that $V(x,\cdot)$ is a potential. By (2.9) and (3.6), it is not difficult to see that $E_p \in \underline{\underline{B}}_u(E)$.

Theorem 11 and the following results generalize those results in Kunita and the author [3; § 7] (see also Meyer [6; Chap. III, § 1]).

THEOREM 13.

Let (V_α) be weakly continuous and V, a continuous kernel. Let $\{\mu_n V\}$ be a sequence of V-potentials which are dominated by a Radon measure λ .

(a) There exists at least one cluster point of $\{\mu_n\}$ in weak* topology.

(b) If μ is a cluster point of $\{\mu_n\}$ and if $\mu_n V$

converges weakly* to a Radon measure ν, *) then

(3.7) $$\nu \geq \mu V .$$

(c) Equality holds in (3.7) for every cluster measures μ of $\{\mu_n\}$ if and only if, for each $f \in \underline{\underline{C}}_c^+(E)$ and every $\varepsilon > 0$, there exists a relatively compact set $A \in \underline{\underline{B}}_u(E)$ such that

(3.8) $$\int_{E \setminus A} Vf(x)\, \mu_n(dx) < \varepsilon \quad \text{for every } n .$$

In this case, $\{\mu_n\}$ is weakly* convergent.

THEOREM 14.

Suppose that (V_α) and V satisfy the conditions of the preceding theorem. Then, every potential is a V-potential of a measure supported in E_p.

PROOF OF THEOREM 13.

(a) Let A be relatively compact. Take $g \in \underline{\underline{C}}_c^+(E)$ as in (3.3). One has

$$\infty > (\lambda, g) \geq (\mu_n V, g) \geq a \cdot \mu_n(A) ,$$

which proves (a).

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*) We don't know if ν is excessive in general. ν is excessive if either of the following conditions is satisfied. (i) Every V_α is a continuous kernel. (ii) $\mu_n V$ converges to ν in the sense of (1.1).

(b) Take a subnet $\{\mu_{n'}\}$ converging to μ. For each $f \in \underline{\underline{C}}_c^+(E)$,

$$(\nu, f) = \lim(\mu_{n'}V, f) = \lim(\mu_{n'}, Vf)$$
$$\geq (\mu, Vf) = (\mu V, f) .$$

Inequality "$\geq$" follows from the fact that Vf is positive and continuous.

(c) Suppose that (3.8) is valid. Take $g \in \underline{\underline{C}}_c^+(E)$ such that $[g=1]_A$ and $0 \leq g \leq 1$. One has

$$(3.9) \qquad (\mu_{n'}V, f) = (\mu_{n'}, g \cdot Vf) + (\mu_{n'}, (1-g) \cdot Vf)$$
$$\leq (\mu_{n'}, Vf) + \varepsilon .$$

Taking the limit for n', one has

$$(\nu, f) \leq (\mu, Vf) + \varepsilon .$$

Suppose that (3.8) is false. Let $\{A_k\}$ be a sequence of relatively compact open sets, increasing to E, such that $A_n \subset \overline{A}_n \subset A_{n+1}$. Choose $f \in \underline{\underline{C}}_c^+(E)$, $\varepsilon > 0$, $n(k)$ such that

$$\int_{E \setminus A_k} Vf(x)\, \mu_{n(k)}(dx) > \varepsilon , \qquad k=1,2,\ldots .$$

By an evaluation similar to (3.9), it is not difficult to see that a cluster measure of $\{\mu_{n(k)}\}$ does not satisfy the equality in (3.7). The final statement is obvious from Theorem 10(c).

PROOF OF THEOREM 14.

Let $\{A_n\}$ be a sequence of relatively compact sets, increasing to E. By Theorem 11, each νL_{A_n} is a V-potential of a finite measure μ_n. Since $\mu_n V = \nu L_{A_n} \uparrow \nu$, it is enough to verify condition (3.8). Take any $f \in \underline{\underline{C}}_c^+(E)$ and $\varepsilon > 0$. Since ν is a potential, there exists a compact set A such that $(\nu L_{E\setminus A}, f) < \varepsilon$. Using Theorem 12, one gets

$$\begin{aligned}\varepsilon > (\nu L_{E\setminus A}, f) &\geq ((\mu_n V) L_{E\setminus A}, f)\\ &\geq ((\mu_n J_{E\setminus A} \cdot V) L_{E\setminus A}, f)\\ &= (\mu_n J_{E\setminus A} \cdot V, f)\\ &= \int_{E\setminus A} Vf(x)\, \mu_n(dx)\ ,\end{aligned}$$

so that ν is the V-potential of the weak* limit μ of $\{\mu_n\}$. It is obvious that μ is supported in E_p.

§ 4. RESOLVENTS OF STANDARD SEMI-GROUPS

Let E be a locally compact Hausdorff space with a denumerable base. Let $(P_t)_{t\geq o}$ be a <u>standard semi-group</u> of submarkov kernels in the sense of Meyer [7]. Let $(U_\alpha)_{\alpha > o}$ be the resolvent of (P_t) and U, the potential kernel of (U_α);

$$U_\alpha = \int_0^\infty e^{-\alpha t} P_t dt, \quad U = \int_0^\infty P_t dt .$$

As in § 3, we assume that UI_K is bounded for every compact set K. Note that (U_α) is weakly continuous.

Let $X = (\Omega, \underline{\underline{F}}, (\underline{\underline{F}}_t), (X(t)), (\underset{\sim}{P}^\mu))$ be a standard realization of (P_t). Let $A \in \underline{\underline{B}}(E)$ and W_A be the penetration time *) of $(X(t))$ for the set A:

(4.1) $$W_A = \inf \left\{ t > 0;\ \int_0^t I_A \circ X(s) ds > 0 \right\}$$

$$= + \infty \quad \text{if the set } \{ \quad \} \text{ is empty .}$$

Define

(4.2) $$P_{W_A}(x,B) = \underset{\sim}{P}^x \{ X(W_A) \in B \} .$$

LEMMA 15 (H. Rost [8]).

<u>Let</u> $f \in p\underline{\underline{B}}(E)$. <u>Then</u>, $P_{W_A} Uf$ <u>is the increasing limit of a sequence</u> $\{Uf_n\}$ <u>of</u> U-<u>potentials such that each charge function</u> f_n <u>is supported in</u> A.

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*) The author wishes to thank Professor Meyer for the suggestion of the use of penetration times. In the beginning the author used the usual hitting time T_A in place of W_A and proved Theorem 16 only when A is open.

THEOREM 16.

(a) <u>For every</u> $A \in \underline{\underline{B}}(E)$,

(4.3) $$\hat{R}_A U = P_{W_A} \cdot U .$$

(b) <u>For every excessive function</u> u,

(4.4) $$\hat{R}_A u = P_{W_A} u .$$

PROOF.

(a) Since $\hat{R}_A U = UL_A$, it is enough to show that

(4.5) $$(\varepsilon_x U)L_A = \varepsilon_x P_{W_A} \cdot U .$$

Let $f \in p\underline{\underline{B}}(E)$ be supported in A. By the strong Markov property,

$$(\varepsilon_x U, f) = Uf(x) = P_{W_A} Uf(x) = (\varepsilon_x P_{W_A} \cdot U, f),$$

so that $[\varepsilon_x U = \varepsilon_x P_{W_A} \cdot U]_A$. One has proved the inequality "$\leq$" in (4.5).

To show the converse, write ν' for $(\varepsilon_x U)L_A$. Let $\{\mu_n U\}$ be a sequence of U-potentials increasing to ν'. Take any $f \in p\underline{\underline{B}}(E)$. Due to Lemma 15, choose $\{Uf_k\}$, increasing to $P_{W_A} Uf$, such that each f_k is supported in A. One has

$$\begin{aligned}(\varepsilon_x P_{W_A} U, f) &= \lim_{k \to \infty} (\varepsilon_x, Uf_k) = \lim_{k \to \infty} (\nu', f_k) \\ &= \lim_{k \to \infty} \lim_{n \to \infty} (\mu_n U, f_k) = \lim_{n \to \infty} \lim_{k \to \infty} (\mu_n, Uf_k)\end{aligned}$$

$$= \lim_{n \to \infty} (\mu_n, P_{W_A} Uf)$$

$$\leq \lim_{n \to \infty} (\mu_n, Uf) = (\nu', f) ,$$

which proves that $(\varepsilon_x U) L_A = \nu' \geq \varepsilon_x P_{W_A} \cdot U$.

(b) This is an immediate consequence of (a).

The following theorem is essentially due to Hunt [2; § 9].

THEOREM 17.

Suppose that U is a continuous kernel. Let ν be an excessive measure. Let $\{A_n\}$ be a sequence of relatively compact sets in $\underline{\underline{B}}(E)$, increasing to E. Then, there exists a unique sequence $\{\mu_n\}$ of finite measures such that

(4.6) $$\mu_n U \uparrow \nu ,$$

(4.7) $$\mu_n = \mu_{n+1} P_{W_{A_n}} .$$

In this case, one has

(4.8) $$\mu_n U = \nu L_{A_n} ,$$

so that μ_n is supported in $\overline{A}_n$.

PROOF.

By Theorem 11 and Theorem 10(c), νL_{A_n} is a U-potential of a unique measure μ_n. Noting that

$(\nu L_{A_{n+1}})L_{A_n} = \nu L_{A_n}$ and making use of Theorem 8 and 17, one gets

$$\mu_{n+1}P_{W_{A_n}} \cdot U = (\mu_{n+1}U)L_{A_n} = (\nu L_{A_{n+1}})L_{A_n}$$

$$= \nu L_{A_n} = \mu_n U .$$

By Theorem 10(c), one has (4.7). (4.6) is obvious.

For the uniqueness, suppose that $\{\mu_n\}$ satisfies (4.6) and (4.7). By a calculation similar to the above, one sees that

$$\mu_n U = \mu_{n+1}P_{W_{A_n}} \cdot U = (\mu_{n+1}U)L_{A_n} = \cdots$$

$$= (\mu_{n+m}U)L_{A_n}$$

$$\rightarrow \nu L_{A_n} \qquad (m \rightarrow \infty) ,$$

which proves (4.8) and the uniqueness of $\{\mu_n\}$.

REMARKS

(a) In the context of this section, G. Hunt [2; p. 86~88] defined a balayage operator of excessive measures (denoted by L_A in his paper, but $\overline{L}_A$ here) as follows.

Let $A \in \underline{\underline{B}}(E)$ (or more generally, nearly analytic). Let $\alpha > 0$ and H_A^α, the α-harmonic measure kernel for the set A. Let ν be an α-excessive measure $(\alpha > 0)$

and $\{\mu_n U_\alpha\}$, a sequence of U_α-potentials increasing to ν. He proved that $\mu_n H_A^\alpha \cdot U_\alpha$ increases to an α-excessive measure (denoted by $\nu \overline{L}_A^\alpha$), independent of the choice of $\{\mu_n\}$:

$$(4.9) \qquad \nu \overline{L}_A^\alpha := \lim_{n \to \infty} \uparrow \mu_n H_A^\alpha \cdot U_\alpha \ .$$

Then he proved that, if ν is (0-) excessive, $\nu \overline{L}_A^\alpha$ increases to an excessive measure (denoted by $\nu \overline{L}_A$) as $\alpha \to 0$:

$$(4.10) \qquad \nu \overline{L}_A := \lim_{\alpha \to 0} \uparrow \nu \overline{L}_A^\alpha \ .$$

This balayage operator $\overline{L}_A$ coincides with ours L_A, if A is open (or more generally, nearly open). In fact, all of the preceding results are valid for the balayage operator L_A^α with respect to the resolvent $(U_{\alpha+\beta})_{\beta>0}$. If A is open, $P_{W_A}^\alpha = H_A^\alpha$. *) Therefore,

$$\lim_{n \to \infty} \mu_n H_A^\alpha \cdot U_\alpha = \lim_{n \to \infty} (\mu_n U_\alpha) L_A^\alpha = \nu L_A^\alpha ,$$

as far as $\{\mu_n U\}$ increases to ν as $n - \infty$. It is easy to see that $\nu L_A^\alpha \uparrow \nu L_A$ as $\alpha \to 0$ for every excessive measure ν. Hence $\nu L_A = \nu \overline{L}_A$.

(b) M. Weil [10] discovered a nice method of constructing approximate P_t-processes, based on Theorem 17. In

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*) $P_{W_A}^\alpha(x,B) = \underset{\sim}{E}^x\{ e^{-\alpha W_A} I_B \circ X(W_A)\}$.

Weil's construction in the present context, the use of the penetration times is not indispensable, since it is enough to take a sequence of relatively compact <u>open sets</u> as $\{A_n\}$ in Theorem 17, for which the penetration times are identical with the hitting times. However, in a more general context in [9], the use of penetration times becomes essential.

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