Séminaire de philosophie et mathématiques

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Descartes, Euler, Poincaré, Pólya and Polyhedra

DESCARTES, EULER, POINCARÉ, PÓLYA—AND POLYHEDRA

by Peter HILTON and Jean PEDERSEN

1. INTRODUCTION

When geometers talk of *polyhedra*, they restrict themselves to configurations, made up of *vertices*, *edges* and *faces*, embedded in three-dimensional Euclidean space. Indeed, their polyhedra are always homeomorphic to the two-dimensional sphere S^2. Here we adopt the topologists' terminology, wherein dimension is a topological invariant, intrinsic to the configuration, and *not* a property of the ambient space in which the configuration is located. Thus S^2 is the surface of the 3-dimensional ball; and so we find, among the geometers' polyhedra, the five Platonic "solids", together with many other examples. However, we should emphasize that we do not here think of a Platonic "solid" as a solid: we have in mind the bounding surface of the solid, not the interior. It seems to us that geometers are sometimes able to be cavalier about this distinction (so that, for them, a *polygon* may be the closed polygonal path or the homeomorph of a disk), but we will need, in what follows, to be precise about meanings.

In this article we retrace an interesting historical path in the study of polyhedra and even carry the story further ourselves—though with modest expectations! We begin with a result due to Descartes (1596-1650). Let us consider a convex polyhedron P, homeomorphic to S^2. Euclid proved that the sum of the face angles at any vertex P is less than 2π; the difference between this sum and 2π is called the *angular defect* at that vertex. If we sum the angular defects over all the vertices of P we obtain the total angular defect Δ: Descartes proved, using methods of spherical trigonometry, that $\Delta = 4\pi$ for every convex polyhedron P. Thus in Figure 1 (b) there are 8 identical vertices on the cube and the angular defect at every vertex is $\frac{\pi}{2}$, so that the total angular defect Δ is 4π.

Notice that the polyhedra shown in Figure 2 are not homeomorphic to S^2 and they fail to satisfy the formula.
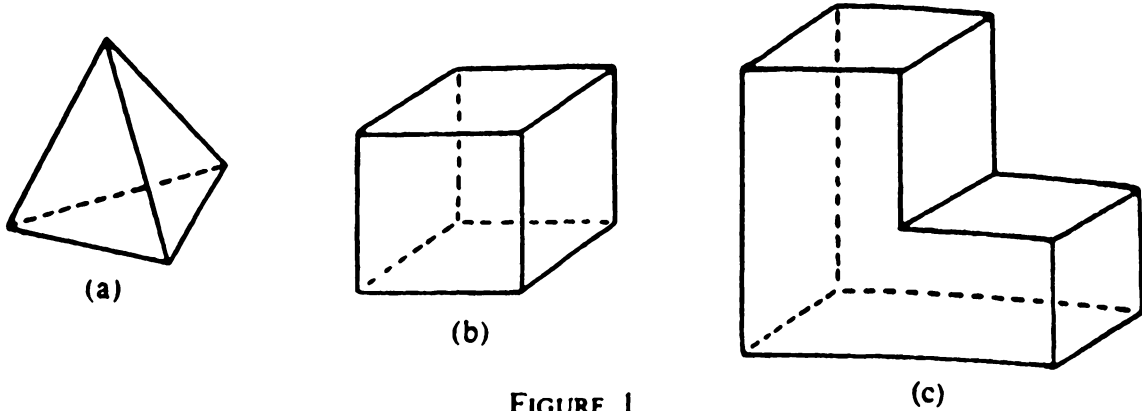
FIGURE 1
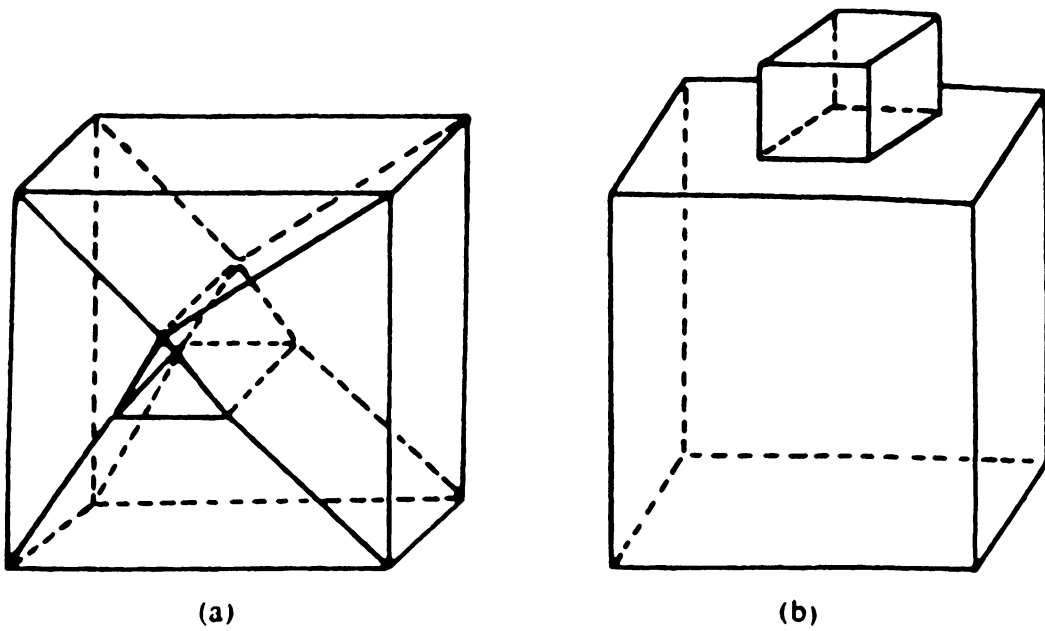
FIGURE 2

Pólya gave an argument in a lecture at Stanford University on March 6. 1974 (see [1]) to deduce Descartes' theorem, using the fact that the Euler characteristic of any polyhedron homeomorphic to S^2 is 2. Here the Euler characteristic χ (P) is given by the formula

$$\chi(P) = V - E + F,$$ (1.1)

where V is the number of vertices of P, E is the number of edges of P, and F is the number of faces of P. Thus Pólya's proof (which appears in slightly modified form in [2]) shows that $\Delta = 2\pi\chi$ and hence $\Delta = 4\pi$ since χ (P) = 2 when P is homeomorphic to S^2.

However Pólya's proof really demonstrates a much more general fact; namely that

$$\Delta = 2\pi\chi \tag{1.2}$$

for *any* 2-dimensional polyhedral manifold [1]. Thus if S is any rectilinear surface, subdivided into vertices, edges and faces in such a way that every edge is incident with exactly two faces, then formula (1.2) holds for S. Of course, we have to interpret Δ somewhat more generally in the sense that, since we no longer require convexity, we must allow the angular defect at any vertex to be negative.

Let us now take S to be any closed surface, orientable or not. Then we may find a homeomorphic rectilinear model T of S, and we may compute Δ (T), χ (T). Since we know that χ (T) is a *topological* invariant of S—a result due to Poincaré—it follows that Δ (T), too, is a topological invariant of S, a result which is surely rather surprising.

In the next section we give, in its more general setting, Pólya's proof of the relation (1.2), and point to the topological significance of the result. In Section 3 we consider analogous formulae for Δ (P), where P is a polyhedron of dimension greater than 2. Now Schläfli [9] generalized Euler's formula to spheres of higher dimension. He succeeded in demonstrating that if P is a polyhedral subdivision of the n-dimensional sphere S^n and if N_i is the number of i-dimensional cells in the subdivision, then

$$\chi (P) = 2 \quad \text{if } n \text{ is even,}$$
$$\chi (P) = 0 \quad \text{if } n \text{ is odd,} \tag{1.3}$$

where

$$\chi (P) = \sum_{i=0}^{n} (-1)^i N_i, \tag{1.4}$$

We call this alternating sum (1.4) the *Euler-Poincaré characteristic* of P and note that it may be defined for any polyhedron P, of any dimension. Poincaré [10] proved that χ (P) is a topological invariant. This means that if X is any geometric configuration embedded in some Euclidean space (of arbitrary dimension) and if P, Q are any two polyhedra, subdivided into cells of dimension 0, 1, 2, ..., n (vertices, edges, faces, ...), such that P and Q are each homeomorphic to X, then χ (P) = χ (Q). This result is one of the great triumphs of homology

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[1] Here, of course, we use the term "polyhedron" in the more general sense favored by topologists. Thus a polyhedron, in this broader sense, certainly need not be 2-dimensional; and an n-dimensional polyhedron need not be homeomorphic to an n-dimensional sphere.

theory [12, p. 167]. For there are natural numbers $p_0, p_1, ..., p_n$ measuring the number of "holes" in X of dimensions 0, 1, ..., n, and one may show that, for any polyhedron P homeomorphic to X,

$$\Sigma (-1)^i N_i = \Sigma (-1)^i p_i. \qquad (1.5)$$

The numbers $p_0, p_1, ..., p_n$ are called the *Betti numbers* of X; they are the dimensions of the homology groups of X in dimensions 0, 1, ..., n. For an n-dimensional sphere S^n, we have

$$p_0(S^n) = p_n(S^n) = 1, p_i(S^n) = 0, i \neq 0, n; \qquad (1.6)$$

thus (1.5) and (1.6) explain Schläfli's result (1.3).

For any polyhedron P, we may continue to define the total angular defect $\Delta(P)$ exactly as in the two-dimensional case. However, $\Delta(P)$ obviously depends only on the two-dimensional structure of P—its vertices, edges and faces—so that we cannot expect, for higher-dimensional polyhedra, either that $\Delta(P)$ will be an invariant or that it will be related to the Euler-Poincaré characteristic. However, we may still attempt to generalize Pólya's argument and thus to express $\Delta(P)$ as a function of V, E and F (or, in our present notation, N_0, N_1 and N_2).

We prove in Section 3 that, indeed, $\Delta(P)$ may be expressed in terms of N_0, N_1, N_2, provided only that the cellular structure on P has the property that there exists an integer q such that every edge of P is incident with exactly q faces. We give three examples of standard cellular subdivisions of S^n with this property. Reverting to the language of geometers (as exemplified by Coxeter [3]), such structures on S^n are called *polytopes*, and the three polytopes considered are called, in [3], respectively *simplexes*, *cross polytopes*, and *parallelotopes*—and will be so referred to by us. The numbers q in these cases are, respectively $n, 2n - 2, n$. We compute Δ in these three cases. We remark that the fact that Δ is, in these cases, a function of N_0, N_1, and N_2 shows that it is a combinatorial, rather than a geometric, invariant; that is, we may pull and push the n-sphere around, squeeze it, squash it, elongate it, stretch it, without altering Δ. Once again our intuition may be at fault!

We close this article with a brief resumé of the history of the question. In this resumé, as in the article itself, we do not take account of another direction in which it may be said that formula (1.2) has been generalized—in the direction of differential geometry. For formula (1.2) contains the seeds of the celebrated

Gauss Bonnet formula for smooth manifolds; an excellent account of the development in this direction is to be found in the article by Chern ([13]; see especially formula (4) on p. 343).

2. Pólya's proof of Descartes' theorem

We start from the position that Euler's formula for a polyhedral 2-sphere S^2 is known; that is to say, if P is a polyhedron homeomorphic to S^2 with V vertices, E edges and F faces, then

$$V - E + F = 2 . \tag{2.1}$$

In Figure 1 (a), for example, $V = 4$, $E = 6$, $F = 4$. Thus $4 - 6 + 4 = 2$, verifying (2.1). Euler's formula is discussed in many elementary books on polyhedra and many proofs have been given. The book by Courant and Robbins, *What is Mathematics?* [4] contains a proof using networks. Pólya's book, *Mathematics and Plausible Reasoning*, Vol. I, [1], has a sequence of problems that leads the reader to a proof. Lakatos' *Proofs and Refutations* [8] is cleverly written in the format of a dialogue between a mathematics teacher and his extremely bright students (who continually find counterexamples to the proposed theorems). The "general" proof must be attributed to Poincaré [10] who, as explained in the Introduction, proved that the generalized Euler-Poincaré characteristic is a topological invariant which takes the value 2 on any even-dimensional sphere.

We now show how Pólya deduced Descartes theorem from (2.1); this argument is essentially that given in [2].

Let P be a polyhedron homeomorphic to S^2, subdivided into vertices, edges and faces in such a way that every edge is incident with exactly two faces. Number the vertices 1, 2, ..., V and let the sum of the plane face angles at the i-th vertex be σ_i. Then the angular defect at the i-th vertex is

$$\delta_i = 2\pi - \sigma_i .$$

Note that δ_i will be positive if P is convex, but that, in general, δ_i may be negative or zero. Let

$$\Delta = \sum_{i=1}^{V} \delta_i .$$

We want to show that $\Delta = 4\pi$.

Proceed by numbering the faces $1, 2, ..., F$ and let S_j be the number of sides [1]) of the j-th face. Then

$$(S_1 - 2)\,\pi + (S_2 - 2)\,\pi + (S_3 - 2)\,\pi + ... + (S_F - 2)\,\pi$$

$$= V\,(2\pi) - \sum_{i=1}^{V} \delta_i = V\,(2\pi) - \Delta.$$

Rearranging the terms on the left yields

$$\pi\left(\sum_{j=1}^{F} S_j\right) - 2\pi F = 2\pi V - \Delta. \tag{2.2}$$

Now, since the total number of sides of the faces which make up the polyhedron P is twice the number of edges, E, on P, we have $\sum_{j=1}^{F} S_j = 2E$, so that

$$\pi\,(2E) - 2\pi F = 2\pi V - \Delta$$

or

$$\Delta = 2\pi V - 2\pi E + 2\pi F$$

$$= 2\pi\,(V - E + F)$$

But, by Euler's formula (2.1), $V - E + F = 2$. Thus

$$\Delta = 2\pi\,(2) = 4\pi.$$

Our first observation is that Pólya's argument immediately generalizes to arbitrary 2-dimensional polyhedra (in the topologists' sense!). Thus let P be any 2-dimensional polyhedron, subdivided into vertices, edges and faces in such a way that every edge is incident with exactly two faces. Define the *Euler characteristic*, $\chi\,(P)$, by

$$\chi\,(P) = V - E + F. \tag{2.3}$$

where P has V vertices, E edges and F faces. Define the total angular defect Δ as above; that is

$$\Delta = \sum_{i=1}^{V} \delta_i.$$

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[1]) It is very important to the understanding of this proof to distinguish between the meaning of a *side* and an *edge*. If a line segment joining two vertices is considered in relation to a face, to whose boundary it belongs, it is called a *side* of that face; if it is considered in relation to the whole polyhedron (forming the common boundary of two neighboring faces) it is called an *edge* of that polyhedron. Thus we see that we may think of the polyhedron as being formed by taking the individual faces and joining the sides of the faces to each other in pairs so that each pair then becomes a single edge of the polyhedron.

where δ_i is the sum of the plane face angles at the i-th vertex and $\delta_i = 2\pi - \sigma_i$. Then Pólya's argument immediately yields the theorem

THEOREM 1. $\Delta(P) = 2\pi\chi(P)$.

A polyhedron P of the type discussed in this theorem is described in the literature of topology as a two-dimensional *pseudomanifold*. Included in this category is the family of *closed surfaces*. If S is such a surface we may take a *rectilinear model* of S, that is, a polyhedron P, homeomorphic to S, and subdivided into vertices, edges and faces as above. Closed surfaces are either *orientable* or non-orientable. An orientable closed surface of *genus* g $(g \geqslant 0)$ may be thought of as formed by attaching g handles to a sphere S^2. Thus if $g = 0$ we have the sphere; if $g = 1$ we have the torus; if $g = 2$ we have the double torus... In general, for an orientable surface S of genus g,

$$\chi(S) = 2 - 2g. \tag{2.4}$$

Observe that $\chi = 2$ for all of the models displayed in Figure 1. When the manifold is homeomorphic with a torus $\chi = 0$. Figure 2 (a) serves to illustrate this example of Theorem 1. Notice that the figure has 14 vertices, 29 edges and 15 faces (2 triangles and 13 quadrilaterals). The computation for the sum of the angular deficiencies produced at all of the 14 vertices may be verified to be $2\pi\chi$. This computation may be displayed, instructively, as follows:

$$\Delta = 14(2\pi) - \{2(3-2)\pi + 13(4-2)\pi\}$$
$$= 14(2\pi) - 29(2\pi) + 15(2\pi)$$
$$\overset{V \quad E \quad F}{= 2\pi(14-29+15)}$$
$$= 0.$$

sphere
(split along arc A B)   +   Möbius band   =   projective plane

FIGURE 3

P

R

Q

Q
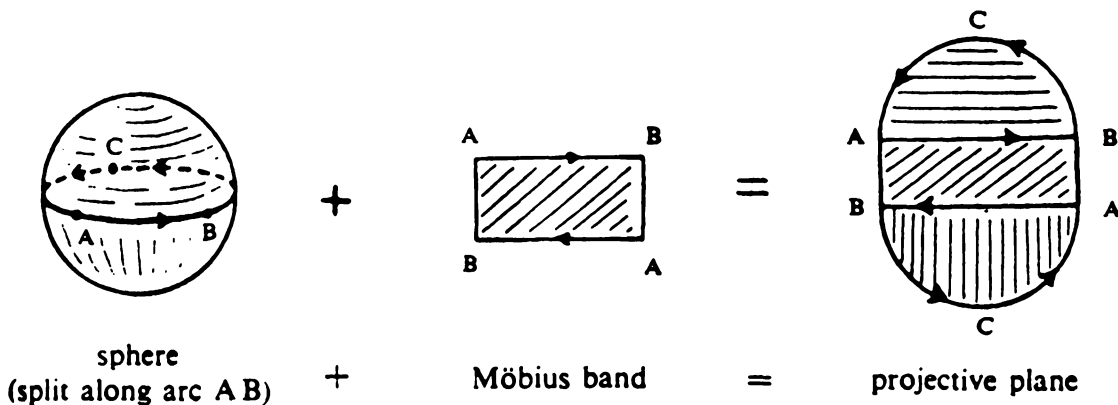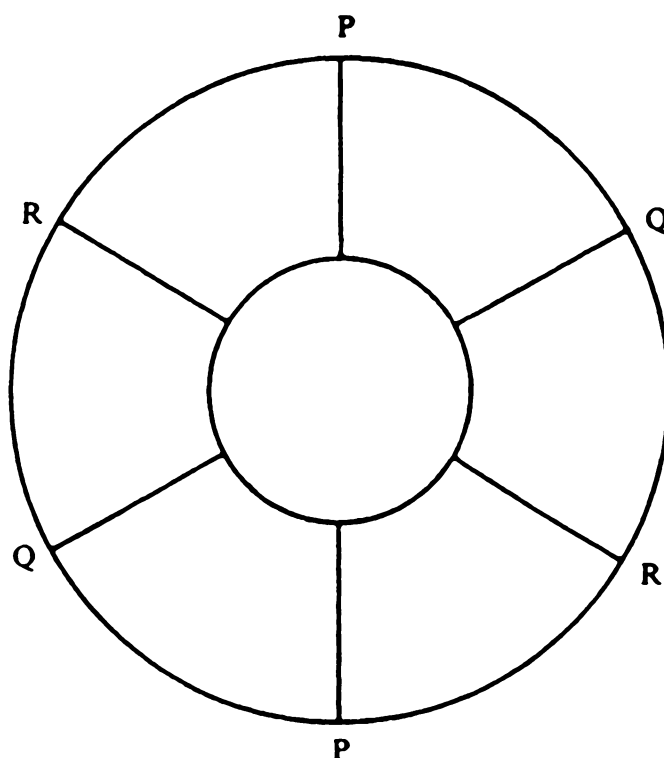
R

P

FIGURE 4

A non-orientable surface may be formed from a sphere S^2 by adding cross-caps (i.e., Möbius bands). If one cross-cap is added, we have the projective plane (see Figure 3). In general, for a non-orientable surface S with k cross-caps

$$\chi(S) = 2 - k. \qquad (2.5)$$

We now exemplify Theorem 1 for the projective plane ($k = 1$). A cellular subdivision of the projective plane is shown in Figure 4 (where, for aesthetic reasons, we have maintained the rounded edges rather than draw, artificially, a strictly polyhedral figure). The cells consist of 6 quadrilaterals and one hexagon, so that the sum of all the face angles may be expressed by $6(4-2)\pi + 1(6-2)\pi$. There are 9 vertices, 15 edges and 7 faces. We display the computation for Δ in the same manner as the last example so that it may suggest the general approach.

$$\Delta = 9(2\pi) - \{6(4-2)\pi + 1(6-2)\pi\}$$
$$= 9\underset{V}{(2\pi)} - 15\underset{E}{(2\pi)} + 7\underset{F}{(2\pi)}$$
$$= 2\pi(9-15+7)$$
$$= 2\pi(1)$$
$$= 2\pi.$$

Theorem 1 exhibits a remarkable fact about the total angular defect of P. For, quite apart from the precise relationship between Δ and χ which it expresses, it shows that $\Delta(P)$ depends only on the topological type of P. It would be remarkable enough that $\Delta(P)$ is independent of the cellular subdivision of P; but, in fact, it does not vary if P is replaced by some other polyhedron homeomorphic to P. Thus $\Delta(P)$ may be said, paradoxically, to be defined by the geometry of P—and to be independent of that geometry! In fact the situation is even more remarkable, since the Euler characteristic is not only a topological invariant but even a *homotopy* invariant; this means that we may deform P continuously without changing $\chi(P)$—and thus without changing $\Delta(P)$.

3. THE ANGULAR DEFECT IN HIGHER DIMENSIONS

We look now at the possibility of obtaining a formula for the total angular defect for a polyhedron of arbitrary dimension. We will largely confine attention to *polytopes* (see [3]), that is, homeomorphs of [1]) S^{n-1}, for some $n \geq 3$. As explained in the Introduction, we will no longer expect to find any significant relationship with the Euler characteristic, so we will concentrate on the question of whether, for such a polytope P, we may obtain a formula for $\Delta(P)$ in terms of V, E and F. Our first result is very general, but will prove to be applicable for certain standard polytopes.

THEOREM 2. *Let* P *be an arbitrary polyhedron in which every edge is incident with precisely* q *faces, then*

$$\Delta(P) = \pi(2V - qE + 2F).$$ (3.1)

Proof. We have only to make a small modification of Pólya's argument. We proceed as in the proof of Theorem 1 as far as the relation (2.2). But now

$$\sum_{j=1}^{F} S_j = qE,$$

so that (2.2) implies that

$$q\pi E - 2\pi F = 2\pi V - \Delta,$$

from which (3.1) immediately follows.

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[1]) We explain later in the section why it is more convenient to talk of S^{n-1} than of S^n.

The restriction in the hypothesis of Theorem 2, that every edge be incident with precisely q faces, is very severe, except in the case that P is 2-dimensional. What is remarkable is that it is satisfied in the case of three standard polytopes. These we now describe. In doing so it will be convenient sometimes to adopt the notation of the Introduction, replacing V, E, F by N_0, N_1, N_2, and, generally, using N_i to designate the number of cells of dimension i in the polytope P.
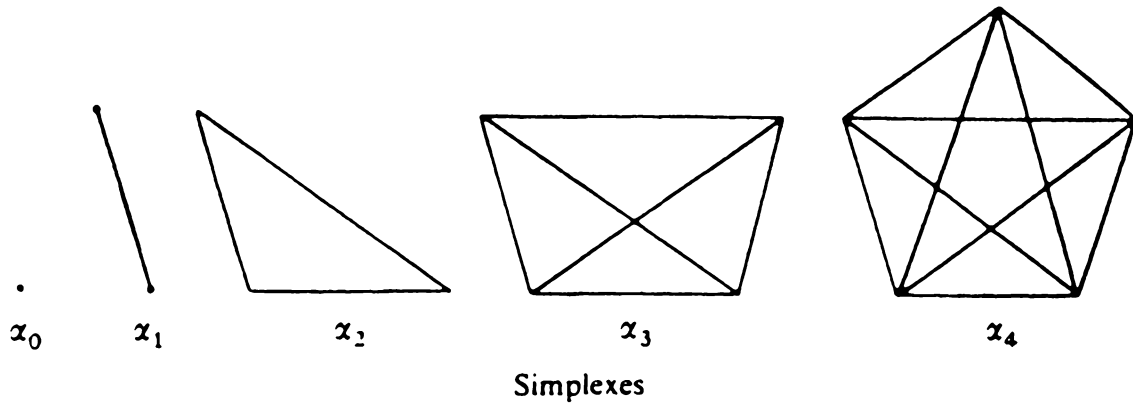

Simplexes

FIGURE 5

SIMPLEXES are produced, as illustrated in Figure 5, by beginning with a single point x_0; we then take this existing structure, introduce another point and join it to the previous one, thus producing x_1 (an edge or line segment); again, we begin with this existing structure, introduce a single point, not in the linear space spanned by x_1, and join it to each of the existing points to produce x_2 (a triangle or 2-simplex); we continue by taking the structure of x_2, introducing a single point, not in the linear space spanned by x_2, and joining that point to each of the existing points to obtain x_3 (a tetrahedron or 3-simplex); etc. In the general case the $(n + 1)$ points we have introduced are the vertices of an n-dimensional simplex, or n-simplex, x_n, whose cells are themselves simplexes formed by subsets of the $(n + 1)$ points, so that there are $\binom{n+1}{1}$ vertices, $\binom{n+1}{2}$ edges, $\binom{n+1}{3}$ faces, $\binom{n+1}{4}$ tetrahedra, etc. Hence we see that, if P_1 is the boundary of x_n

$$N(P_1)_k = \binom{n+1}{k+1}, \quad 0 \leqslant k \leqslant n - 1.$$

When all the edges are equal these structures are called regular simplexes, in [3] denoted x_i. The x_i of Figure 5 should be viewed as though they are in perspective since they were intentionally drawn to show a symmetric placement of the vertices in x_4.

If we remove the interior of α_n we obtain a cellular subdivision of S^{n-1}. It is for this reason that we prefer to speak in this section of S^{n-1} rather than S^n. Since every proper subset of the $(n+1)$ vertices of α_n span a cell of S^{n-1}, we see that, for this polytope, every edge is incident with precisely $(n-1)$ faces, so that we may apply Theorem 2 with $q = n - 1$. Since for this polytope, with $n \geqslant 3$,

$$V = \binom{n+1}{1}, \quad E = \binom{n+1}{2}, \quad F = \binom{n+1}{3}, \tag{3.2}$$

we have, from Theorem 2,

COROLLARY 1. Let P_1 be the polytope obtained by subdividing S^{n-1} as the boundary of an n-simplex, $n \geqslant 3$. Then

$$\Delta (P_1) = -\frac{\pi}{6} (n-4)(n+1)(n+3).$$

Proof. We have, from (3.1) and (3.2)

$$\Delta (P_1) = \pi \left(2(n+1) - (n-1) \frac{n(n+1)}{2} + 2 \frac{(n+1)n(n-1)}{6} \right)$$

$$= \frac{\pi}{6} (n+1)(12 - 3n(n-1) + 2n(n-1))$$

$$= -\frac{\pi}{6} (n+1)(n^2 - n - 12).$$

It is interesting to note that, while a simplex is convex, $\Delta (P_1)$ is negative for $n \geqslant 5$ (and zero for $n = 4$).

We now turn to our second example of a polytope.

$$\beta_1 \qquad\qquad \beta_2 \qquad\qquad \beta_3 \qquad\qquad \beta_4$$
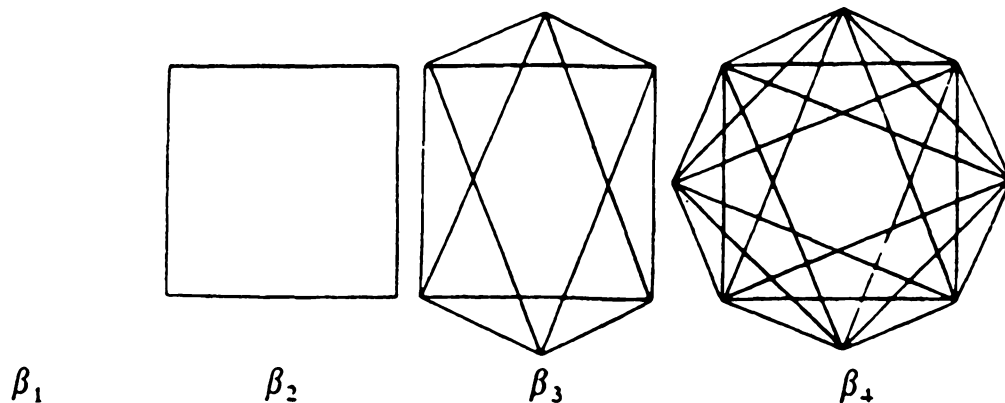
Cross Polytopes

FIGURE 6

CROSS POLYTOPES may be introduced by recognizing that an important aspect of n-dimensional space is the possibility of having n mutually perpendicular lines through any point 0. For example, each regular simplex α_{n-1} (of Figure 5) involves n points equidistant from 0. Now if we choose to take points equidistant from 0 in both directions we obtain the cellular subdivision of the $(n-1)$-sphere called a *cross polytope*. These have $2n$ vertices and their $(n-1)$-cells consist of 2^n of the α_{n-1}'s. Figure 6 illustrates the cases where n is equal to 1, 2, 3, and 4 respectively. Thus β_1 is a pair of points (vertices) and we can think of progressing from β_i to β_{i+1} by beginning with β_i, introducing a pair of diametrically opposed points (vertices), not in the linear space spanned by β_i, and then joining each of these points to the existing points of β_i (but not to each other). The polytope β_n, which we will call P_2, is, in fact, homeomorphic to S^{n-1}. It can easily be shown by induction that

$$N(P_2)_k = 2^{k+1}\binom{n}{k+1}, \quad 0 \leqslant k \leqslant n-1.$$

Thus, in particular, for this polyhedron P_2,

$$V = 2n, \quad E = 2n(n-1), \quad F = \frac{4}{3}n(n-1)(n-2) \tag{3.3}$$

We now prove

PROPOSITION 1. *In the polytope* P_2 *every edge is incident with precisely* $(2n-4)$ *faces,* $n \geqslant 2$.

Proof. We first assert that it is plain that in β_n every vertex is incident with precisely $(2n-2)$ edges. This follows by an easy induction on n. For β_{n-1} has $(2n-2)$ vertices and every vertex is, by induction, incident with $(2n-4)$ edges. Thus a vertex of β_{n-1} is incident with $((2n-4)+2)$ edges of β_n, while a new vertex of β_n is incident with $(2n-2)$ edges of β_n.

Now suppose that, in β_{n-1}, every edge is incident with $(2n-6)$ faces—this is certainly true if $n = 3$. Then an edge of β_{n-1} is incident with $((2n-6)+2)$ faces of β_n, while a new edge of β_n is incident with $(2n-4)$ faces of β_n (since a vertex of β_{n-1} is incident with $(2n-4)$ edges of β_{n-1}).

This proof illustrates how we pass from β_{n-1} to β_n by "stepping up dimensions by 1". This is the point of view of topologists, who introduced such an idea into combinatorial topology without, perhaps, realizing that it had already been introduced by the geometers. Topologists call the passage from β_{n-1} to β_n

suspension, and apply this idea to arbitrary topological spaces. Thus the suspension of X is obtained by joining X to two independent points or, equivalently, by taking two cones with base X and joining them together along their bases.

Returning to P_2, we are now ready to prove

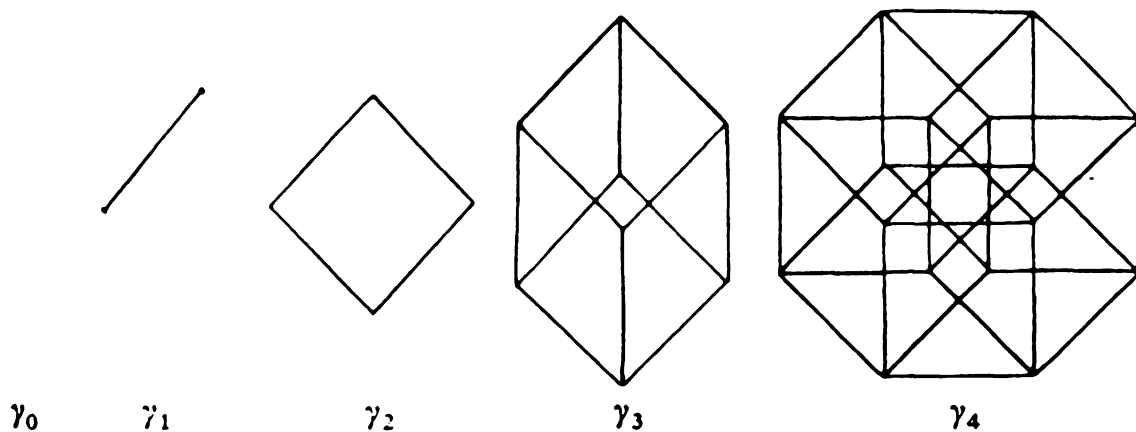COROLLARY 2. *Let* P_2 *be the polytope obtained by subdividing* S^{n-1} *as a cross polytope. Then*

$$\Delta (P_2) = -\frac{4\pi}{3} n (n^2 - 3n - 1).$$

Proof. We assemble the facts from (3.1), (3.3) and Proposition 1 to infer that

$$\Delta (P_2) = \pi \left(4n - 4n (n-1)(n-2) + \frac{8}{3} n (n-1)(n-2)\right)$$

$$= \frac{4\pi n}{3} (3 - (n-1)(n-2))$$

$$= -\frac{4\pi}{3} n (n^2 - 3n - 1).$$

Here we note that $\Delta (P_2)$ is negative for $n \geqslant 4$.

Finally we turn to our third example of a polytope.

γ_0 γ_1 γ_2 γ_3 γ_4

Parallelotopes

FIGURE 7

PARALLELOTOPES are illustrated in Figure 7. The passage from γ_i to γ_{i+1} is achieved by translating γ_i (not along any of its own lines) from its initial to a final position and then joining in pairs each of the original points with the

corresponding point of the translated figure. If all edges have the same length the polytope is called a *measure polytope*. The quantities N_k can be computed by considering the passage from γ_i to γ_{i+1}. Thus we readily obtain the inductive relation

$$N\,(\gamma_{i+1})_k \;=\; 2N\,(\gamma_i)_k \;+\; N\,(\gamma_i)_{k-1}\,, \qquad k \leqslant i\,. \tag{3.4}$$

Now γ_n is, combinatorially, a hypercube—we specialize the construction by taking γ_1 to be the unit interval and always translating orthogonally by unit distance. Thus the boundary of γ_n is topologically S^{n-1}. We call the boundary P_3 and infer from (3.4) that

$$N\,(P_3)_k \;=\; 2^{n-k}\binom{n}{k}, \qquad 0 \leqslant k \leqslant n-1\,.$$

Note that, for $n = 3$, we get, combinatorially, the unit cube, with 8 vertices, 12 edges, and 6 faces. In general the polytope P_3, with $n \geqslant 3$, yields the values

$$V = 2^n, \quad E = 2^{n-1}\,n, \quad F = 2^{n-3}\,n\,(n-1)\,. \tag{3.5}$$

By an argument very similar to, but simpler than, that of Proposition 1, we may show

PROPOSITION 3. *In the polytope* P_3, *with* $n \geqslant 3$, *every edge is incident with* $(n-1)$ *faces.*

We are now ready to prove

COROLLARY 4. *Let* P_3 *be the polytope obtained by subdividing* S^{n-1} *as the boundary of an* n-*dimensional parallelotope,* $n \geqslant 3$. *Then*

$$\Delta\,(P_3) \;=\; -2^{n-2}\,\pi\,(n^2-n-8)\,.$$

Proof. From (3.1), (3.5) and Proposition 3 we have

$$\begin{aligned}
\Delta\,(P_3) &= \pi\,(2^{n+1} - 2^{n-1}\,n\,(n-1) + 2^{n-2}\,n\,(n-1)) \\
&= 2^{n-2}\,\pi\,(8 - n\,(n-1)) \\
&= -2^{n-2}\,\pi\,(n^2-n-8)\,.
\end{aligned}$$

Here we note that $\Delta\,(P_3)$ is negative for $n \geqslant 4$.

The fact that $\Delta\,(P_1)$, $\Delta\,(P_2)$, and $\Delta\,(P_3)$ are all different (except for $n = 3$) shows that the total angular deficiency has no chance of being a topological

invariant for polyhedra of dimension $\geqslant 3$. On the other hand it is still striking that Δ depends only on the cellular structure and is independent of the underlying geometric structure.

Remarks. (a) The polytopes P_1, P_2, P_3 not only enjoy the property that each edge of P_i is incident with the same number of faces of P_i, $i = 1, 2, 3$—the property we used to calculate $\Delta(P_i)$ from Theorem 1—they also enjoy the property that each face has the same number of sides. This latter property could also have been used to calculate $\Delta(P)$. Thus if P is a polyhedron subdivided so that each face has the same number s of sides, then one may show that

$$\Delta(P) = 2\pi V - \pi F(s-2). \tag{3.6}$$

It is easy to deduce either of the formulae (3.1), (3.6) from the other if the polyhedron P enjoys *both* the relevant properties. For if every edge of P is incident with q faces and every face of P has s sides, then

$$qE = sF. \tag{3.7}$$

Of course there is an equality corresponding to (3.7) in higher dimensions.

(b) The polytopes P_2 and P_3 may be regarded as *dual*, in the sense that there is a one-one correspondence between the cells of P_2 of dimension k and the cells

FIGURE 8

Data for P_1, P_2, P_3 when $n = 4$

Name of polytope (P_i)	N_0	N_1	N_2	N_3	Number of sides on each N_2	Number of faces incident with each edge	Δ
Simplex (P_1)	5	10	10	5	3	3	0
Cross polytope (P_2)	8	24	32	16	3	4	-16π
Parallelotope (P_3)	16	32	24	8	4	3	-16π

of P_3 of dimension [1] $(n-1) - k$. Moreover, the incidence relations are carried over by this duality; thus if, in P_2, every $(i-1)$-face is incident with [2] s_i i-faces, then, in P_3, every $(n-i)$-face is incident with s_i $(n-i-1)$-faces (and there is a symmetrical statement interchanging P_2 and P_3). In this sense P_1 is selfdual. Figure 8 displays these dualities for $n = 4$, as well as the value of Δ.

3. HISTORICAL COMMENT AND SUMMARY

René Descartes (1596–1650) and Leonhard Euler (1707–1783) worked on these subjects independently—yet, as we have seen, Pólya (1887–   ) has shown that their seemingly different formulae for convex polyhedra homeomorphic to S^2 are entirely equivalent to each other. One might believe from the evidence that Euler may have known about Descartes' work on this subject. That would be an erroneous assumption since Descartes' work on this matter [5] was not printed until a century after Euler's death (see [1], p. 56).

Euler [6] offered a variety of verifications but no formal proof of his formula. We have observed that each of the formulae is somewhat surprising by itself and that their connection rather defies intuition since at first glance they seem to be dealing with different qualitative aspects of polyhedra. As a matter of fact neither Euler's nor Descartes' formula is easy to *prove* independently; yet, as we have seen, it is not at all difficult to follow Pólya's proof that the two formulae are equivalent.

The formulae diverge in higher dimensions so that their relationship is a special phenomenon of dimension 2. Euler's formula was generalized by Ludwig Schläfli [9], a Swiss mathematician of the 19th century (1814–1895), who described, in effect, the Euler-Poincaré characteristic of an n-dimensional sphere S^n, subdivided as a *polytope*, a combinatorial structure attributed by Coxeter to Reinhold Hoppe [11]. Poincaré (1854-1912) gave a definition of the Euler-Poincaré characteristic for arbitrary polyhedra, and one proves now, by invoking the topological invariance of the homology groups (see [12]) that the Euler-Poincaré characteristic is a topological invariant.

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[1] The precise form of this duality shows how "correct" it is to regard S^{n-1} as $(n-1)$-dimensional, rather than n-dimensional.

[2] In fact, $s_i = 2(n-i-1)$.

On the other hand, there will be no straightforward generalization to higher dimensions of Descartes' formula for the total angular defect of a polyhedron homeomorphic to S^2, since this defect ceases in higher dimensions to be a topological invariant. However it remains, under suitable restrictions on the cellular structure, a combinatorial invariant in a certain strict sense and thus independent of the underlying geometry of the polyhedron.

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