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Nilpotency in Group Theory and Topology

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by

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0. Introduction

Our object in this paper is to illustrate certain key features of the methodology of mathematics by means of the example of the development of the concept of nilpotence. We hope thereby to help to dispel certain impressions created by the pontifical utterances of those on the fringes of mathematics--philosophers, physicists--who see mathematics as a static entity and who have no real conception of the nature of *mathematical activity*, which is a dynamical process involving dialectical change leading to higher syntheses.[1]

The methodological features to which we wish to draw special attention are *generalization*, *relativization*, *application* and *reasoning by analogy*. All these will be illustrated in the course of the mathematical development of the next three sections, but we would wish to say a few words here by way of introduction before embarking on the mathematical discussion.

As to *generalization*, this process is, of course, familiar to all mathematicians. It is an art, in the sense that there is no unique choice of generalization of a given concept--the criteria determining the validity of a given generalization reside in a subtle blend of the scope of the generalization and the availability of significant theorems analysing the generalized concept. Generalization is also an art in the sense that

[1]We borrow here what is valuable from marxist terminology.

there can be no algorithmic rule determining when it is appropriate to generalize and which collection of familiar concepts should be subsumed in a common generalization. The satisfaction of the following principles is clearly necessary (and just as clearly not sufficient) to justify a given generalization: (a) the 'collection of familiar concepts' should be bigger than a singleton set, (b) theorems in the generalized context should cast light on special cases where the assertions contained in those theorems were hitherto unknown.

It is, of course, true that *relativization* is a special case of *generalization*—in other words, 'generalization' is a generalization of 'relativization'! Nevertheless, it does seem to deserve explicit mention since it is a common or, as one might dare to say, a *standard* type of generalization. The classical method of relativizing was to pass from a single object X to a pair of objects (X,Y), where Y is a subobject of X; moreover, it might be necessary to impose some special condition on Y as a subobject. We give an example of this type of relativization in connection with Definition 1.1 where we consider a pair (N,N') consisting of a group N and a normal subgroup N' of N. Other important examples consist of a topological space X and a closed subspace Y of X; and a manifold M and its boundary ∂M. However, the categorical point of view suggests that we should not confine ourselves to pairs of objects (X,Y) in which Y is a subobject of X. Rather we should broaden the concept of relativization as follows. Given any category $\mathcal{C}$, we form the category $\mathcal{C}_r$ of $\mathcal{C}$-*morphisms*; thus an object of $\mathcal{C}_r$ is a morphism $f\colon Y \to X$ of $\mathcal{C}$, and a morphism of $\mathcal{C}_r$, from f to f' is a pair of morphisms (g,h) such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ Y' & \xrightarrow{f'} & X' \end{array}$$

commutes. Composition of morphisms in $\mathcal{C}_r$ is defined in the obvious way. Then a *relativization* of $\mathcal{C}$ is a full subcategory of $\mathcal{C}_r$. This definition is implicitly brought into play in Section 2 when we relativize the notion of *nilpotent space* to obtain the notion of *nilpotent fibre map;* since the notion of nilpotent space is a *homotopy* (rather than a *homology*) notion, the 'principles' of Eckmann-Hilton duality dictate that, in relativizing, we pass from spaces to fibre maps. Another example where relativization involves maps which are not embeddings is in the concept of *relative projective* in homological algebra; thus an object is described as $\mathcal{E}$-projective if it is projective relative to a class $\mathcal{E}$ of epimorphisms.

It is particularly appropriate today, when there is so much emphasis placed on applied mathematics, to highlight, in an article devoted to an exemplification of mathematical method itself, the *role of application inside mathematics*. In the text of this article we show how group-theoretical theorems may be applied to topology and how topological ideas and situations and constructions may be applied in group theory. Of course, we could give many examples of the application of topological theorems to algebra; let us only instance the famous result of Adams on the non-existence of real division algebras of dimension $\neq 1, 2, 4, 8$ over $\mathbb{R}$. Thus it might be claimed that, within mathematics, the method of progress by successful application is even richer than in so-called applied mathematics, since we may make unrestricted *two-way* applications. It is true that mathematics can provide theorems in physics, and physics can provide inspiration for mathematics. But there seems to be no sense in which physics, or any other extramathematical discipline, can provide *theorems* of mathematics. At least it will be claimed that the *methods* of applied mathematics are by no means special to applied mathematics, but instead form part of all good mathematics. There is, of course, an important philosophical distinction in that, when we form a mathematical model of a 'real', non-mathematical situation, there is no meaning to the demand for accurate and precise mathematical description of the problem situation; whereas, when we apply a mathematical theorem from one mathematical domain to study another mathematical domain, the strict relevance of the theorem can be assured. Thus our motives in studying the significance of a theorem for our particular mathematical purposes, do not coincide with the set of motives of the theoretical scientist who must beware that the mathematical model may have been so crude an approximation to reality that the conclusions drawn from it are unacceptable. Nevertheless, it would appear that no case can be made out for separating off pure and applied mathematics as separate disciplines. There should only be mathematicians, and students of mathematics, learning how to do mathematics, to make mathematics and to use mathematics.

The pattern of development in the subsequent sections of this article illustrate one point which, it would seem, it is very necessary to emphasize, in view of the expectations entertained by many advocating more emphasis on applied mathematics, and in view of the claims made for *problem-solving* as a dominant thread in mathematical education. I refer to the fact that *applications* first make their appearance relatively late in our exposition. The reason is clear: in order to make useful applications there must be a substantial body of theory to apply; and, in order to solve significant problems, it must be possible to set those problems in an appropriate theoretical context. In the traditional method of teaching applied mathematics, the necessity for a strong theoretical base in mathematics is usually masked by one of two devices. Either the application is given immediately following the elaboration of the appropriate mathematical theory, or the appropriate theory is simply presented to the student more or less ready-made and he is left to wonder how it was so clear (to the textbook writer or the lecturer) that that was indeed the appropriate theory. It seems fair to say that, by those devices, it is possible to describe solutions of problems already solved but not to describe how problems are solved. Genuine applied mathematics is very difficult; and an essential prerequisite is a strong mathematical preparation.

In talking of *reasoning by analogy* we do not intend to convey the impression that we use analogy to achieve mathematical proof--although we would also not wish to deny the possibility of doing so in a specific mathematical context. Here, in this article, we confine ourselves to the elaboration of a situation in which we employ intuition and experience to suggest that an idea, taken from a certain mathematical situation, might prove fruitful, if intelligently interpreted, in a somewhat different situation. This type of reasoning is, of course, of the very stuff of rational behavior, and we owe to René Thom the observation that a principal defect of an elementary mathematics education based on elementary set theory (Venn diagrams) is precisely that it is bound to ignore reasoning by analogy. We also owe to Thom the exciting possibility of building reasoning by analogy itself on the foundation of mathematical analysis.[1]

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[1] 'All these points show what limitations are set by set theory for the description of the usual thought processes. Our usual thinking depends on deep psychic mechanisms, as for example 'analogy', which cannot be reduced to set-theoretical operations. An important role is played in such cases by the organizing isomorphism between semantic fields which are connected by homology with each other'. *(my translation)* R. Thom, 'Moderne' Mathematik-Ein erzieherischer und philosophischer Irrtum?, Mathematiker über die Mathematik, Springer (1974), p. 388.

This article would become inordinately long were we to include all proofs. We have sketched some, totally omitted others. The results cited are to be found in published articles by the author, Guido Mislin, Joseph Roitberg and David Singer, together and severally; the main part in the monograph [HMR]. Since our principal purpose is to illustrate certain characteristic methods of mathematics in action, and not to announce new results in group theory and topology, we have felt justified in departing from the usual canons of mathematical exposition, and hope that no loss of clarity will result.

I would like to thank M. Maurice Loi for inviting me to speak in his seminar at the École Normale Supérieure, and thus stimulating me to give thought to the problems adumbrated in this article.

1. Concept of nilpotence in group theory

Let N be a group. We define the *lower central series* of N by the rule

$$(1.1) \qquad \Gamma^1 N = N, \ \Gamma^{i+1} N = [N, \Gamma^i N], \ i \geq 1;$$

and we say that N is *nilpotent of class* $\leq c$, and write $\mathrm{nil}\, N \leq c$, if $\Gamma^{c+1} N = \{1\}$. Thus the concept of nilpotent group *generalizes* that of commutative group: N is commutative if and only if $\mathrm{nil}\, N \leq 1$. We write $\mathcal{N}$ for the category of nilpotent groups, $\mathcal{N}_c$ for the full subcategory of groups N with $\mathrm{nil}\, N \leq c$.

Examples (a) Let $N = N(p) = \{a,b \mid a^p = b^p = [a,b]\}$, where p is a fixed prime number. It is easy to see that the centre, ZN, of N coincides with the commutator subgroup $[N,N] = \Gamma^2 N$, and is cyclic of order p. For $[a,b]^p = [a,b^p] = 1$. Moreover N/ZN is generated by the residue classes $\bar{a}$, $\bar{b}$ of a, b and $\bar{a}^p = \bar{b}^p = 1$. Thus we have a *central* extension

$$(1.2) \qquad \mathbb{Z}/p \rightarrowtail N(p) \twoheadrightarrow \mathbb{Z}/p \times \mathbb{Z}/p,$$

showing that $|N| = p^3$ and $\mathrm{nil}\, N = 2$. If $p = 2$, N is the celebrated *quaternionic* group of order 8.

(b) Let $F = F(x_\alpha)$ be the free group on the symbols (x_α) and let $N(x_\alpha) = F/\Gamma^{c+1} F$. Then $N(x_\alpha)$ is the *free nilpotent group of class* c *on the symbols* (x_α). Every group in $\mathcal{N}_c$ is the homomorphic image of $N(x_\alpha)$, for some suitable choice of (x_α). This example also shows that there are groups of arbitrary nilpotency—hardly surprising!

Our first main result generalizes the observation, based on (1.2) in Example (a), that $N(p) \in N_2$.

Theorem 1.1. *Let* $N' \rightarrowtail N \twoheadrightarrow N''$ *be a central extension of groups (that is,* N' *is in the center of* N *and* $N/N' = N''$*). Then*[1] $\operatorname{nil} N'' \leq \operatorname{nil} N \leq \operatorname{nil} N'' + 1$.

Proof. That $\operatorname{nil} N'' \leq \operatorname{nil} N$ is obvious from (1.1) and requires no hypothesis of centrality. On the other hand, if $\Gamma^{c+1}N'' = \{1\}$, then $\Gamma^{c+1}N \subseteq N'$ and $\Gamma^{c+2}N \subseteq [N,N'] = \{1\}$, since N' is central.

We now seek to *generalize* Theorem 1.1. We must bear in mind that, given an extension of groups $N' \rightarrowtail N \twoheadrightarrow N''$, we cannot infer the nilpotence of N from that of N' and N'' (the converse implication, on the other hand, obviously holds). For let $N = S_3$, the symmetric group on 3 symbols, 1, 2, 3. If x is the cyclic permutation (123), then x generates a normal subgroup $N' = \mathbb{Z}/3$ and $N'' = N/N' = \mathbb{Z}/2$. Thus N' and N'' are nilpotent--indeed, commutative--but N is not nilpotent. For an easy calculation shows that $\Gamma^i N = N'$, $i \geq 2$. Thus our generalization cannot consist of simply discarding the centrality condition in Theorem 1.1; we must weaken it judiciously. We are indeed led to the following *relativization* of the concept of nilpotency.

Definition 1.1. Let N' be normal in N, written $N' \lhd N$. Then the *(relative) lower central series of* N *in* N' is given by

$$(1.3) \qquad \Gamma^1_N N' = N', \ \Gamma^{i+1}_N N' = [N, \Gamma^i_N N'], \ i \geq 1.$$

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[1] Our result remains valid if we adopt the convention, as we will henceforth, that N not nilpotent $\Leftrightarrow$ $\operatorname{nil} N = \infty$.

We say that the embedding of N' in N is *nilpotent of class* $\leq c$ and write $\mathrm{nil}_N N' \leq c$ if $\Gamma_N^{c+1} N' = \{1\}$.

Notice that we have simultaneously *relativized* nilpotency ($\Gamma_N^i N = \Gamma^i N$) and *generalized* centrality (N' is central in N if and only if $\mathrm{nil}_N N' \leq 1$). Notice also that each $\Gamma_N^i N'$ is normal in N. We now generalize Theorem 1.1.

<u>Theorem 1.1g</u>. *Let* $N' \rightarrowtail N \twoheadrightarrow N''$ *be an extension of groups. Then*

$$\max(\mathrm{nil}\, N'', \mathrm{nil}_N N') \leq \mathrm{nil}\, N \leq \mathrm{nil}\, N'' + \mathrm{nil}_N N'.$$

<u>Proof</u>. We easily generalize the argument of Theorem 1.1. In particular, if $\Gamma^{c+1} N'' = \{1\}$, $\Gamma_N^{d+1} N' = \{1\}$, then $\Gamma^{c+1} N \subseteq N' = \Gamma_N^1 N'$, $\Gamma_N^{c+2} N \subseteq \Gamma_N^2 N'$, ..., $\Gamma^{c+d+1} N \subseteq \Gamma_N^{d+1} N' = \{1\}$.

Let G be a group. In the theory of G-modules, there is also a notion very much akin to that of the lower central series. Indeed, if A is a G-module, we define the *lower central* G-*series* of A by the rule

$$\Gamma_G^1 A = A,\ \Gamma_G^{i+1} A = \mathrm{gp}(a - xa),\ x \in G,\ a \in \Gamma_G^i A,\ i \geq 1; \tag{1.4}$$

and we say that A is G-*nilpotent of class* $\leq c$, written $\mathrm{nil}_G A \leq c$, if $\Gamma_G^{c+1} A = \{0\}$. Notice that each $\Gamma_G^i A$ is a *submodule* of A, and that

$$\Gamma_G^{i+1} A = \Gamma_G^i(\Gamma_G^1 A). \tag{1.5}$$

Just as nilpotency generalized commutativity $(c = 1)$, so here the case $c = 1$ is the case of *trivial* action of G on A. This brings the ideas of nilpotency and G-nilpotency very close, since a commutative group is precisely a group N such that the action of N on itself by conjugation is trivial.

As we shall see in Section 2, the two concepts of nilpotent group and G-nilpotent module are the essential ingredients in our *application* of nilpotency to topology. Here we pursue our programme of generalizing our concepts, here motivated by the desire to find a useful *common* generalization of the two concepts just mentioned; of course, we should select such a generalization to include also relative nilpotency as expressed in Definition 1.1.

This last remark provides the clue. For if $N' \triangleleft N$ then N operates on N' by conjugation; thus we may hope to find a fruitful generalization of Definition 1.1 and of the lower central G-series of a G-module by supposing N to be a G-group, that is, a group on which the group G acts, and defining a lower central G-series of N.

Definition 1.2. Let N be a G-group. We define the *lower central G-series* of N by the rule

$$(1.6)\quad \Gamma_G^1 N = N,\ \Gamma_G^{i+1} N = gp(a.xb.a^{-1}b^{-1}),\ a \in N,\ b \in \Gamma_G^i N, x \in G,\ i \geq 1;$$

and we say that N is *G-nilpotent of class* $\leq c$, written $\mathrm{nil}_G N \leq c$, if $\Gamma_G^{c+1} N = \{1\}$.

It is immediately obvious that this definition coincides with (1.4) above if N is commutative. It is also easy to see that Definition 1.2 generalizes Definition 1.1. For suppose $N \triangleleft G$ and let G operate on N by conjugation. We then have, from Definitions 1.1 and 1.2, two definitions of $\Gamma_G^i N$, and to see that they coincide, it suffices to verify that if $M \triangleleft G$ and if $K = gp(axbx^{-1}a^{-1}b^{-1})$, $a \in N$, $b \in M$, $x \in G$, then $K = [G,M]$.

The following remarks are also pertinent: (a) if N is G-nilpotent it is certainly nilpotent (as a group) and $\mathrm{nil}\, N \leq \mathrm{nil}_G N$; (b) each $\Gamma_G^i N$, in (1.6), is a normal G-closed subgroup of N; (c) $\Gamma_G^{i+1} N = gp([N,\Gamma_G^i N],\ xb.b^{-1})$, $x \in G$, $b \in \Gamma_G^i N$; (d) (1.5) does *not* generalize—on the other hand, we still have the inequality $\Gamma_G^i(\Gamma_G^1 N) \subseteq \Gamma_G^{i+1} N$, so that we may infer

(1.7) $$\mathrm{nil}_G \Gamma_G^1 N \le \mathrm{nil}_G N - 1.$$

The relation (1.7) is, naturally, very usefil in fashioning proofs by induction on G-nilpotency class.

A more surprising observation is that not only can Definition 1.1 be subsumed under Definition 1.2, but also the other way round! For let N be a G-group. We form the *semidirect product* of N and G; thus $P = N \rtimes G$ is defined as follows. The underlying set of P is the cartesian product of the underlying sets of N and G, and the group operation in P is given by

(1.8) $$(a_1,x_1)(a_2,x_2) = (a_1.x_1a_2, x_1x_2).$$

There is an obvious embedding $N \rightarrowtail P$ and a projection $P \twoheadrightarrow G$, giving rise to a group extension

(1.9) $$N \rightarrowtail P \twoheadrightarrow G$$

which *splits on the right* in the sense that there is a section homomorphism $G \to P$ (the obvious embedding). We then have

<u>Theorem 1.3</u>. $$\Gamma_P^i N = \Gamma_G^i N.$$

<u>Proof</u>. We argue by induction on i, the case $i = 1$ being trivial. Now if we conjugate $a \in N$ by $(c,x) \in P$, $c \in N$, $x \in G$, we obtain $c.xa.c^{-1}$. Thus, let us assume $\Gamma_P^i N = \Gamma_G^i N$, for some $i \ge 1$. The preceding remark, together with Definition 1.1, immediately shows that a system of generators of $\Gamma_P^{i+1} N$ consists of elements of the form $a.xb.a^{-1}b^{-1}$, $a \in N$, $b \in \Gamma_G^i N$, $x \in G$, establishing the inductive hypothesis and the theorem.

We next draw an immediate consequence from Theorems 1.1g and 1.3.

Corollary 1.4. *Let* N *be a G-group and let* P *be the semidirect product of* N *and* G. *Then* P *is nilpotent if and only if* G *is nilpotent and* N *is G-nilpotent. Indeed,*

$$\max(\operatorname{nil} G, \operatorname{nil}_G N) \le \operatorname{nil} P \le \operatorname{nil} G + \operatorname{nil}_G N.$$

We close this section with a theorem which shows that the G-nilpotence of a group N is, in a very strict sense, determined by the nilpotence of N and the G-nilpotence of N_{ab}, the abelianization of N. It is thus a generalization of a very apt kind of the two nilpotency concepts which led to its formulation.

Theorem 1.5. *Let* N *be a G-group. Then* N *is G-nilpotent if and only if* N *is a nilpotent group and* N_{ab} *is G-nilpotent.*

Proof. The entire argument is essentially due to Derek Robinson [R], although he considered a very slightly different situation. Certainly if N is G-nilpotent it is nilpotent; and, just as certainly, if N is G-nilpotent then any G-homomorphic image, and so in particular N_{ab}, is G-nilpotent, Conversely, let N_{ab} be G-nilpotent. It is easy to see that then $\otimes^i N_{ab}$, the i-fold tensor power of N_{ab}, with diagonal action, is also G-nilpotent. But $\Gamma^i N/\Gamma^{i+1}N$, as a G-homomorphic image of $\otimes^i N_{ab}$, is also G-nilpotent. Since N is nilpotent and

$$(1.10) \qquad \Gamma^i N/\Gamma^{i+1}N \rightarrowtail N/\Gamma^{i+1}N \twoheadrightarrow N/\Gamma^i N$$

is a central extension of G-groups, Theorem 1.5 follows from (1.10) by induction on i, using the following easy *generalization* of Theorem 1.1.

Theorem 1.1g'. *Let* $N' \rightarrowtail N \twoheadrightarrow N''$ *be a central extension of G-groups. Then*

$$\max(\mathrm{nil}_G N'', \mathrm{nil}_G N') \le \mathrm{nil}_G N \le \mathrm{nil}_G N'' + \mathrm{nil}_G N'.$$

The reader should now formulate a common generalization of Theorems 1.1g and 1.1g'.

2. Concept of nilpotence in topology

The concept of a *nilpotent space* is due to E. Dror. We consider the *homotopy* category of (based) spaces of the homotopy type of a connected polyhedron and (based) homotopy classes of continuous maps. Then a space X is said to be *nilpotent* if its fundamental group $\pi_1 X$ is a nilpotent group and operates nilpotently on the higher homotopy groups $\pi_n X$, $n \geq 2$. Notice that this definition involves precisely the first two definitions of Section 1, since $\pi_n X$ is abelian for $n \geq 2$.

Examples (a) If X is simply-connected it is, of course, nilpotent.

(b) A connected topological group is nilpotent; indeed, any *simple* space (in the classical sense) is nilpotent.

(c) Let G be a nilpotent Lie group (not necessarily connected) and let BG be its classifying space. Then BG is a nilpotent space.

(d) Let X be nilpotent, let Y be compact, and let C be a component of the function space X^Y. Then C is nilpotent. Note, in this example, that, even if we took X to be simply-connected, we could still only conclude that C is nilpotent.

Nilpotent spaces are of *interest* in topology because they include important classes of spaces, as the examples show (in particular, they constitute the smallest class containing the simply-connected spaces and closed under the construction of function spaces).

Since the notions of nilpotent group and nilpotent π-module turn out to be relevantin topology, it is to be expected that their common generalization, embodied in Definition 1.2, will also be relevant. Indeed, it is precisely what we need to *relativize* the definition of a nilpotent space. Since this latter concept is a homotopy concept (rather than a homology concept), it is to be expected that the relativization will be to a *fibre map*, rather than to a pair of spaces. Now if $f: E \to B$ is a fibre map with fibre F, then $\pi_1 E$ operates on the homotopy groups of F. This relativizes the absolute situation, to be interpreted as $X \to o$, where the 'fibre' is again X and we have the operation of $\pi_1 X$ on itself by conjugation and the usual operation of $\pi_1 X$ on the higher homotopy groups $\pi_i X$, $i \geq 2$.

<u>Definition 2.1</u>. The fibre map $f: E \to B$, with fibre F, is *nilpotent* if all spaces are connected and if $\pi_1 E$ operates nilpotently on $\pi_i F$, $i \geq 1$.

Note that the operation of $\pi_1 E$ on $\pi_1 F$ is to be nilpotent in precisely the sense of Definition 1.2. To demonstrate the 'correctness' of Definition 2.1 we enunciate two theorems, of which the second is the relativization of the first. For an interpretation and proof of these theorems see [HMR;Corollary II.2.12 and Theorem II.2.14].

<u>Theorem 2.1</u>. *The space* X *is nilpotent if and only if its Postnikov system admits a principal refinement.*

<u>Theorem 2.1r</u>. *The map* f *is nilpotent if and only if its Moore-Postnikov system admits a principal refinement.*

The following result may be regarded as in analogy with Theorem 1.1.

<u>Theorem 2.2</u>. *Let* $f: E \to B$ *be a fibre map with fibre* F, *all spaces being connected. Then* F *is nilpotent if* E *is nilpotent. Indeed,*

$$\operatorname{nil}_{\pi_1 F} \pi_i F \leq \operatorname{nil}_{\pi_1 E} \pi_i E + 1.$$

The analogy is very close; for, in the 'fibration'

$$\Omega B \to F \to E, \tag{2.1}$$

where ΩB is the space of loops on B, the image of $\pi_1 \Omega B$ is the center of $\pi_1 F$ so we may regard (2.1) as a 'central fibration'.

We turn now to Theorem 1.5. Let us recall that we may associate with a group N the Eilenberg-MacLane space $K(N,1)$, characterized by the property that

$$\pi_1 K(N,1) = N, \ \pi_i K(N,1) = 0, \ i \geq 2. \tag{2.2}$$

It is easy to see that $K(N,1)$ is nilpotent if and only if N is nilpotent. Moreover, if G operates on N, then G operates as a group of homotopy classes of self-homotopy equivalences on $K(N,1)$. Then Theorem 1.5 asserts that, if $X = K(N,1)$, then G operates nilpotently on $\pi_1 X$ if and only if X is nilpotent and G operates nilpotently on[1] $H_1 X$. Now it is plainly out of the question that, for an arbitrary connected space X on which G operates, the fact that G operates nilpotently on $\pi_1 X$, or indeed on $\pi_i X$, $1 \leq i \leq n$, where $n \leq \infty$, could imply the nilpotence of X. For, if G operates trivially, it operates nilpotently on $\pi_i X$, $i \geq 2$, and it operates nilpotently on $\pi_1 X$ provided only that $\pi_1 X$ is nilpotent. Thus if we seek to generalize Theorem 1.5 in the direction of nilpotent spaces it will be necessary to *postulate* the nilpotence of our space X. Indeed, we then find the following theorem [H]:

Theorem 2.3. *Let* X *be a nilpotent space on which* G *operates, and let* $n \geq 2$. *Then the following statements are equivalent:*

[1]Recall that, for any connected space X, $H_1 X$ is obtained by abelianizing $\pi_1 X$.

(i) G *operates nilpotently on* $\pi_i X$, $1 \le i < n$;

(ii) G *operates nilpotently on* $H_i X$, $1 \le i < n$.

We point out that not only does Theorem 2.3 *generalize* Theorem 1.5; it also *applies* it. For we are not going to be able to pass from statement (ii) to statement (i) without using Theorem 1.5 to infer from (ii) that G operates nilpotently on $\pi_1 X$.

There is another essential ingredient in the proof of Theorem 2.3. We have the group G acting (as a group of self-homeomorphisms or as a group of homotopy classes of self-homotopy equivalences) on the space X. Then, for a given $i \ge 1$, G acts on $\pi_1 X$ and on $\pi_i X$, and $\pi_1 X$ acts on $\pi_i X$. It is important to ask how these actions are related. It turns out that, writing K for $\pi_1 X$ and N for $\pi_i X$, the single controlling relation is

$$x(a.b) = xa.xb, x \in G, a \in K, b \in N. \tag{2.3}$$

We are thus led to make the following definition.

<u>Definition 2.2</u>. The G-group K *operates* on the G-group N if K operates as a group on the group N, subject to the relation (2.3).

We will study some consequences of this definition in the next section. For now we note that, whereas Theorem 2.3 involves an application of group theory to topology, Definition 2.2 makes a movement in the *opposite* direction, from topology back to group theory.

Our first observation in this section is that Theorem 2.3 may be applied to the study of nilpotent fibrations (Definition 2.1). In any fibration $F \to E \to B$ we have, classically, an action of $\pi_1 B$ on the *homology* groups of F. We may the prove:

<u>Corollary 2.4</u>. *Let* $F \to E \to B$ *be a fibration with all spaces connected. Then the fibration is nilpotent if and only if* F *is nilpotent and* $\pi_1 B$ *operates nilpotently on* $H_i F$, $i \geq 1$.

Note that it makes no difference here whether we speak of the operation of $\pi_1 B$ on $H_i F$ or of the operation of $\pi_1 E$ on $H_i F$. The former formulation is better since we normally have better control on the operation of $\pi_1 B$; but the latter sets in better evidence the resemblance of Corollary 2.4 to Theorem 1.5. Note also that the condition that $\pi_1 B$ operate nilpotently on $H_i F$ appears to be precisely the condition enabling us to apply the classical spectral sequence methods of Serre and Eilenberg-Moore to the study of the homology groups associated with a fibration; see, for example, [HR2].

3. Further results

Let us return now to group theory. We place ourselves in the context of relation (2.3), that is, we have the G-group K acting on the G-group N in such a way that (2.3) is satisfied. Form, as in Section 1, the semidirect product $P = K\rtimes G$ and define an operation of P on N by the rule

$$(a,x)b = a.xb,\ x \in G,\ a \in K,\ b \in N. \tag{3.1}$$

Proposition 3.1. *The rule (3.1) defines a group action of* P *on* N *if and only if (2.3) holds. The rule (3.1) then gives the unique action of* P *on* N *extending the given actions of* K *and* G.

We may now *generalize* the results of Section 1 from groups to G-groups. Theorem 1.1g' was already such a generalization, namely of Theorem 1.1—but we may now generalize Theorem 1.1.g! First note that if $N \triangleleft K$ in the category of G-groups, then conjugation is an action of the G-group K on the G-group N in the sense that (2.3) holds,

Theorem 3.2. *Let* $N' \rightarrowtail N \twoheadrightarrow N''$ *be an extension of G-groups and let* $P = N\rtimes G$. *Then*

$$\max(\mathrm{nil}_G N'', \mathrm{nil}_P N') \le \mathrm{nil}_G N \le \mathrm{nil}_G N'' + \mathrm{nil}_P N'.$$

Again, we may generalize Corollary 1.6 to

Theorem 3.3. *Let* N, K *be G-groups with* $N \triangleleft K$. *Then* K *is G-nilpotent if and only if* N *is nilpotent and* $K/[N,N]$ *is G-nilpotent.*

Note that we do not need to postulate N to be G-nilpotent in Theorem 3.3. For if $K/[N,N]$ is G-nilpotent, so too is N_{ab}. Thus, by Theorem 1.5, if also N is nilpotent, then N is G-nilpotent.

Reverting to (3.1) one may prove the following result.

Theorem 3.4. *Let the* G*-group* K *act on the* G*-group* N *so that (2.3) is satisfied and let* $P = K \rtimes G$ *act on* N *by (3.1). Then* N *is* P*-nilpotent if and only if* N *is* K*-nilpotent and* G*-nilpotent.*

Indeed, Stammbach has proved that

$$\mathrm{nil}_P N \leq (\mathrm{nil}_K N)(\mathrm{nil}_G N) \tag{3.2}$$

From this and Theorem 3.2 we infer

Corollary 3.5. *Let* $N' \rightarrowtail N \twoheadrightarrow N''$ *be an extension of* G*-groups. Then* N *is* G*-nilpotent if and only if* N', N'' *are* G*-nilpotent and* N' *is* N*-nilpotent.*

We may improve this statement by observing that, given any extension $N' \rightarrowtail N \xrightarrow{\varepsilon} N''$ there is always an induced action of N'' on N'_{ab}, given by

$$\varepsilon a . b'[N',N'] = ab'a^{-1}[N',N'], \ a \in N, \ b' \in N'.$$

Then Theorem 1.5 enables us to deduce, from Corollary 3.5,

Corollary 3.6. *Let* $N' \rightarrowtail N \twoheadrightarrow N''$ *be an extension of* G*-groups. Then* N *is* G*-nilpotent if and only if* N', N'' *are* G*-nilpotent and* N'_{ab} *is* N''*-nilpotent.*

The advantage of this formulation is that the condition given for N to be G-nilpotent makes no mention of G. Thus we may regard the statement of Corollary 3.6 as follows. Suppose given a nilpotent action of N'' on N'_{ab} and let $N' \rightarrowtail N \twoheadrightarrow N''$ be any extension of G-groups compatible with this action. Then N is G-nilpotent if and only if N' and N'' are G-nilpotent. Manifestly, the assertion in this form is a 'G-generalization'.

Finally let us return to topology. Theorems of the type of Theorem 2.3, asserting that the homotopy groups of a space have a certain property up to a given dimension if and only if the homology groups have the same property up to the same dimension, are familiar in homotopy theory. The classical result of this kind is due to Hurewicz who proved that the homotopy groups of a simply-connected space X vanish in dimensions $i < n$ if and only if the homology groups vanish in the same dimensions. Moreover, as Hurewicz proved, the so-called Hurewicz homomorphism $\underline{h}: \pi_n X \to H_n X$ is then an isomorphism. Serre [S] derived a beautiful *generalization* of the Hurewicz Theorem. He defined, axiomatically, a *class* of abelian groups C (the trivial groups form a class) and proved that the homotopy groups of a simply-connected space X belong to C in dimensions $i < n$ if and only if the homology groups of X, in the same dimensions, belong to C, and that then the Hurewicz homomorphism $h: \pi_n X \to H_n X$ is a C-isomorphism, meaning that the kernel and cokernel of h are groups in C. A nilpotent version of this result was proved in [ED] and [HR1]: given a suitable definition of a class of nilpotent groups, then it may be proved that the homotopy groups of a nilpotent space belong to C, in dimensions $i < n$, if and only if the homology groups of X, in the same dimensions, belong to C, and that then the Hurewicz homomorphism h induces a C-isomorphism $h': \pi_n' X \to H_n X$, where $\pi_n' X$ is $\pi_n X$ with the operators from $\pi_1 X$ killed; thus $\pi_n' X = \pi_n X / \Gamma_\pi^2 \pi_n X$, where $\pi = \pi_1 X$. The formulation of Theorem 2.3 suggests the following:

(a) It should be possible to formulate axiomatically a definition of a class C of nilpotent G-groups;

(b) The G-nilpotent G-groups should then satisfy the axioms for such a class;

(c) There should be a theorem of Hurewicz type for nilpotent G-spaces, that is, nilpotent spaces X on which G acts, asserting that the homotopy groups of such a space X belong to the class C, in dimensions $i < n$, if and only if the homology groups of X belong to C in the same dimensions, and $h': \pi_n'X \to H_nX$ is then a C-isomorphism.

It turns out that this programme can be carried out, so that Theorem 2.3 is exhibited as a partial statement of a special case of a general *Hurewicz Theorem* mod C *for nilpotent G-spaces*. Crucial to this programme is the observation that our condition (2.3)—in the case that N is commutative—is precisely the condition needed to ensure that G acts on the homology groups $H_m(K,N)$. Thus we may build into our theory of G-groups the usual homology theory of groups.

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