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ON THE IDEAL STRUCTURE OF CERTAIN BANACH ALGEBRAS

by Yngve DOMAR

Let A be a commutative Banach algebra with unit e and the generators $\alpha_1 , \alpha_2 , \dots , \alpha_n$. We assume that

(1) $$\|\alpha_j^m\|^{1/m} \to 0 \quad , \quad \text{when } m \to \infty \quad ,$$

for every j . We put for simplicity

$$\alpha^p = \alpha_1^{p_1} \cdot \alpha_2^{p_2} \cdot \dots \cdot \alpha_n^{p_n} \quad ,$$

where p stands for $(p_1 , p_2 , \dots , p_n)$ with non-negative integers as components. We also define $|p| = p_1 + p_2 + \dots + p_n$.

We say that an element b in the dual space B has the degree q if $b(\alpha^p) = 0$, whenever $|p| > q$, and if q is the smallest non-negative integer with this property. A closed ideal in A is said to have the co-degree q if its annihilator subspace contains only elements of degree $\leqslant q$ and at least some element of degree q . Infinite degree and co-degree are defined in a similar way.

For every α^p and every $b \in B$, $b(\alpha^p a)$ defines a bounded linear functional on A , that is to say an element in B . We denote it $\alpha^p \circ b$, and call it a translation of b . Linear combinations of translates are defined in a natural way.

The algebra A can be thought of as a completion of the algebra of polynomials in e and the generators, under the topology which is determined by the norm. We say that a Banach algebra A^0 is larger than A if it can be interpreted as a similar completion, but using a smaller norm. The dual space B^0 of A^0 can then in an obvious way be interpreted as a subspace of B equipped with a larger norm. We shall in the following denote the norms in B and B^0 by $\| \|^*$ and $\| \|^{0*}$, respectively.

We now make the following assumption : suppose that for every $q \geqslant 0$ and for every closed ideal of infinite co-degree there exist an element β in the annihilator subspace, a larger Banach algebra A^0 with dual space B^0 and a constant C , such that

1° the degree of β is finite $\geqslant q$ or infinite,

2° $\beta \in B^0$

3° for every linear combination b of translates of β

$$\|b + b'\|^* \leqslant C \sum_{p_0 \,|\, |p_0| = q} \|\alpha^{p_0} \circ b\|^{0*} ,$$

for some $b' \in B$ of degree $\leqslant q - 1$.

We then have the following result :

THEOREM. - Suppose that I is a proper closed ideal in A with co-degree finite $\geqslant q$ or infinite. Then there exists a proper closed ideal I_q with co-degree q , such that $I \subset I_q$.

We shall only indicate the proof, which is based on the fact that if an element belongs to the annihilator subspace, then the same is true for its translates.

The analogue of (1) for A^0 shows that there exists a sequence of elements β_n in the annihilator subspace, fulfilling the requirements on β in the assumption, and such that

$$\sup_{|p|=q} \|\alpha^p \circ \beta_n\|^{0*} = 1$$

and

$$\sup_{|p|=q+1} \|\alpha^p \circ \beta_n\|^{0*} \to 0 \qquad , \text{ when } \quad n \to \infty \quad .$$

Using the Hahn-Banach theorem it can be shown that we can find linear combinations b_n of β_n and its translates, such that the same relations are fulfilled with β_n changed to b_n , and, moreover,

$$\varliminf_{n\to\infty} \sup_{|p|=q} |b_n(\alpha^p)| > 0 \quad .$$

By the assumption we have a sequence of elements b'_n of degree $\leqslant q - 1$ such that $\|b_n + b'_n\|^*$ is bounded. It is possible to show that these elements b'_n may be chosen in the annihilator subspace. We can extract from $\{b_n + b'_n\}$ a weakly convergent subsequence. The limit element is then an element of degree q in the annihilator subspace. From this element it is easy to construct the desired ideal.

In the applications of the theorem, the principal difficulty is to verify that the assumption is fulfilled. We shall only discuss a very special case, connected with earlier investigations in [1] and [2].

Let α satisfy $0 < \alpha < 1$ and form the class of all Lebesgue measurable complex-valued functions f on R^n such that

$$\|f\| = \int_{R^n} e^{|x|^2} |f(x)| \, dx < \infty \quad ,$$

where $|x|^2 = x_1^2 + \dots + x_n^2$, and $x_1, \dots, x_n$ are Cartesian coordinates in R^n. It is easy to see that this is a commutative Banach algebra K under convolution. Algebras of this kind have been studied in [1]. We form the ideal I which is the closure of the ideal of all functions $f \in K$ such that their Fourier transforms $\hat{f}$ vanish in some neighborhood of the origin of the dual n-dimensional space. Put $A = K/I$. It can be shown that there are functions in K such that their Fourier transforms are $\equiv it_j$ in a neighborhood of the origin, where t_j denotes the coordinate which corresponds to x_j. Form the corresponding elements α_j in A. It is possible to show that A has a unit and that α_j are generators, which fulfil (1). B can be interpreted as the class of Lebesgue measurable complex-valued functions b such that

(2) $$\text{ess. sup} \frac{|b(x)|}{e^{|x|^\alpha}} < \infty \quad ,$$

and which have their spectrum in the sense of [1] empty or consisting of one point, namely the origin. This means that the functions coincide almost everywhere with the restrictions to real variables of the class of entire functions of n variables, of exponential type and minimal type, and satisfying (2). With this choice of representing functions and with a proper normalization of the Fourier transform it can be shown that $\alpha^p \circ b$ is the corresponding derivative $D^p b$. Hence an element of degree q is simply a polynomial of degree q. It is now possible to show that the assumption is fulfilled if A^0 is the algebra which we obtain if the "weight-function" $e^{|x|^\alpha}$ is changed to $(1 + |x|)^{-q} e^{|x|^\alpha}$. 3° in the assumptions is then an easy consequence of 2°, and 1° and 2° follow from the fact that if $b \in B$, then $D^p b \in B^0$ if $|p|$ is sufficiently large. The proof of this must be omitted here.

For $n = 1$, this case of the theorem was treated in [2], but the proof in that paper does not extend to the case when $n > 1$. As in [2] it is possible to treat a larger class of weight-functions with the same method.

We end by formulating the result in our special case as a theorem on integral equations.

THEOREM. - Let $\{f_i\}$ be a class of functions in K, such that the origin is the only common zero of the Fourier transforms $\hat{f}_i$. Form the system of integral equations

$$\varphi * f_i = 0 \quad ,$$

where φ belongs to the dual class of K, that is the class of functions, satisfying (2). Then all solutions φ are polynomials of degree $\leqslant q - 1$, if (and only if) no solution is a polynomial of degree q.

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