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THE APPLICATION OF THE METHOD OF DIFFERENTIAL RELATIONS TO THE EQUATIONS OF CONTINUUM MECHANICS

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The method of differential relations consists in the following. To a given system of differential equations

$$(S) \qquad \Phi_\nu \left(x_q, u_r, \frac{\partial u_r}{\partial x_q}, \dots, \frac{\partial^k u_r}{\partial x_1^{q_1} \partial x_2^{q_2} \dots \partial x_n^{q_n}} \right) = 0$$

$$(\nu, r = 1, \dots, z; \quad q = 1, \dots, n; \quad q_1 + q_2 + \dots + q_n = k) \qquad (1)$$

it is adjoined a system of additional differential relations

$$(D) \quad \Psi_{\lambda_\alpha} \left(x_q, u_r, \frac{\partial u_r}{\partial x_q}, \dots, \frac{\partial^{j_\alpha} u_r}{\partial x_1^{q_1} \partial x_2^{q_2} \dots \partial x_n^{q_n}} \right) = 0 ,$$

$$(\alpha = 1, 2, \dots, m; \quad \lambda_\alpha = 1, 2, \dots, \nu_\alpha; \quad q = 1, \dots, n; \quad r = 1, \dots, z; \quad q_1 + q_2 + \dots + q_n = j_\alpha) , \qquad (2)$$

that consists of a set $\sum_{\alpha=1}^{m} \nu_\alpha$ of differential equations of the order j_α .

Henceforth the additional differential relations will be called the differential relations. Let us denote their set by a letter $\mathfrak{D}$ and a combined system S and $\mathfrak{D}$ by letters $S\mathfrak{D}$. In a general case the system $S\mathfrak{D}$ is assumed not to be in the involution, and it is necessary to study it on the compatibility. If $S\mathfrak{D}$ is not in the involution, then in the course of analysis on the compatibility it is complemented with the new equations that its solution should satisfy. If $S\mathfrak{D}$ is in the involution, then it is possible to search for its solutions that will be the partial solutions of the system S , The essence of the method of differential relations can be stated as follows: the solutions of the system $S\mathfrak{D}$ are searched for easier than the solutions of the system S , since in the first case the arbitrariness of a general solution is narrower than in the second one.

It is possible not to set a priori a form of differential relations $\Psi_{\lambda\alpha}$ and equations Φ_μ of a subsystem $S' \subset S$, but to determine it a posteriori when solving a converse problem of the compatibility theory for a system $S\mathfrak{A}$ that can be formulated as follows:

What a form should have the differential relations $\Psi_{\lambda\alpha}$ and the equations Φ_μ of the subsystem $S' \subset S$ in order that an over-determined system $S\mathfrak{A}$ has a given arbitrariness in its solution?

The introduced definitions can be formulated geometrically by using a notion of an extended system. Let us denote

$$\frac{\partial^k u_\gamma}{\partial x_1^{p_1} \partial x_2^{p_2} \dots \partial x_n^{p_n}} = \mathcal{P}_{\gamma p_1 \dots p_n}, \quad u_\gamma = \mathcal{P}_{\gamma 0},$$

$$p_1 + p_2 + \dots + p_n = k \leq j, \qquad j = \max\{h, j_\alpha\} \tag{3}$$

For a value $\mathcal{P}_{\gamma p_1 \dots p_n}$ we obtain a set of equations

$$\frac{\partial \mathcal{P}_{\gamma p_1 \dots p_n}}{\partial x_l} = \mathcal{P}_{\gamma p_1 \dots p_l + 1 \dots p_n} \tag{4}$$

and the finite relations

$$\chi_\xi \left(x_q, \mathcal{P}_{\gamma 0}, \dots, \mathcal{P}_{\gamma q_1 \dots q_n}, \dots \right) = 0. \tag{5}$$

The relations (5) are obtained from (I) when differentiating with respect to the variables $x_1, \dots, x_n$ up to the obtaining of the derivatives of the order j. The system (3)-(5) is called an extended one. Then in a space of variables $\mathcal{P}_{\gamma q_1 \dots q_n}$ of the extended system, the differential relations (2) determine some surface $\mathfrak{S}$.

The problem is formulated as follows:

Determine the conditions that the surface $\mathfrak{S}$ should satisfy in order that to hold the integral variety of the extended system with a given arbitrariness.

In the first stage of the solution of the converse problem of the compatibility theory one must search for a set of conditions of the solutions existence of this problem. Generally, it is a system of partial differential equations relative to the functions $\Psi_{\lambda\alpha}$ and Φ_{μ} :

$$\Omega_{\gamma}\left(x_q, u_r, \Psi_{\lambda\alpha}, \Phi_{\mu}, \frac{\partial\Psi_{\lambda\alpha}}{\partial x_q}, \ldots, \frac{\partial\Phi_{\mu}}{\partial u_{\gamma}}, \ldots\right) = 0. \qquad (6)$$

In the second stage of the converse problem solution of the compatibility theory, it is necessary to solve the equations Ω_{γ} , i.e. to obtain a form $\Psi_{\lambda\alpha}$ and Φ_{μ} . By using such a method the partial classes of the system (I) solution are singled and one can obtain some idea about the structure of a general solution. However, the obtained solutions do not satisfy any boundary-value conditions that a general system (I) solution can satisfy, i.e. the obtained solutions are the formal ones. Nevertheless, the solutions obtained using a method of differential relations can be employed when solving the specific boundary-value problems. In connection with the fact that the analysis on the compatibility requires to carry out excessive analytical calculations, it arose an actual problem to employ an electronic computer for these purposes. An algorithm of the analysis on the compatibility according to Cartan of Pfaff system of the form

$$\begin{aligned} du^{i_1} &= a^{i_1}_{\tau}(x, u, \varphi)\, dx^{\tau}, \\ du^{i_2} &= a^{i_2}_{\tau}(x, u, \varphi)\, dx^{\tau} + b^{i_2}_{\alpha}(x, u, \varphi)\, du^{\alpha}, \end{aligned}$$
$$(\varphi^{\lambda} = \varphi^{\lambda}(x_1, \ldots, x_n, u^1, \ldots, u^{s'}), \quad \alpha = s+1, \ldots, z;$$
$$\tau = 1, 2, \ldots, n; \quad \lambda = 1, 2, \ldots, m). \qquad (7)$$

was realized by using an electronic computer in [I]. An external differential from the Pfaff equations with an index i_2 is assumed to be identically equal to zero. An analysis on the compatibility of any Pfaff system can be reduced to an analysis on the compatibility of the system under consideration.

It is possible not to fix the functions $\varphi^{\lambda}(x_1,\dots,x_n,u^1,\dots,u^s)$ a priori, but to determine them in the course of the solution of the converse problem of the compatibility theory.

Cartan's method as an electronic computer algorithm can be presented in a form of a block-diagram (Fig. I) in which the following sequence of operations is realized:

1) Obtain a system of equations defining the most general integral element.
2) Choose a complete, independent and noncontradictory subsystem in the system of defining equations.
3) Lower the rank of the defining system by putting conditions on the functions φ^{λ}.
4) Construct a chain of integral elements, compute the characteristics and verify Cartan's criterion.
5) Extend the Pfaff system.
6) Differentiate the system of finite relations and choose from among them a complete, independent and noncontradictory subsystem.
7) Complement the Pfaff system and verify the noncontradictoriness of the complemented system.
8) Reduce a complemented system to the initial form (7).

The algorithm can operate in regimen I when the finite relations obtained are adjoined to the initial system, or in regimen II when the finite relations are satisfied by choice of the functions φ^{λ}. The item 3 is included in the operation of the algorithm only in regimen II.

By using an electronic computer BESM-6 a series of test direct and converse problems of the compatibility theory was calculated. Thus, it is shown a principal possibility to use an electronic computer for realization of so complicated algorithm as it is Cartan's one.

The progressive waves of the system of quasilinear equations [2] can serve as an example of solutions being characterized by the finite relations

$$a_{ijk}(u_1, u_2, \ldots, u_m)\frac{\partial u_j}{\partial x_k} = 0\,, \quad i, j, k = 1, \ldots, m; \qquad (8)$$

A progressive wave of the rank r is called a solution $u_i = u_i(x_1, \ldots, x_m)$ of the system (8) that satisfies the $m-r$ functional dependencies $\varphi_\alpha(u_1, \ldots, u_m) = 0$.

In the case of gas dynamics equations, the equations of double and triple waves were obtained [3,4]. In particular, the problems on the gas motion behind a piston consisting of two and three planes (in Fig.2 it is presented a configuration of subregions in a case of two pistons), the problem on the gas escape into the vacuum on an oblique wall (Fig.3) [5] and a number of other problems were solved in terms of the double and triple waves.

The new explicit solutions [6] were obtained for the equations of one-dimensional gas dynamics with the help of the method of differential relations, some of them generalize the solutions obtained earlier, in particular, the results by Martin, Ludford and Zav'yalov. For some equations of state, the problems on the motion of a shock wave and the interference of the Riemann wave and a contact zone (Fig.4) were solved analytically. Moreover, these solutions had served as a test for the numerical methods.

The method of differential relations was applied also to the one-dimensional equations of dynamics of non-elastic continuous media. In this case it was solved a converse problem of the compatibility theory: what must the equation of state be like in order that the initial system has one or another class of solutions being characterized by the differential relations? The obtained solutions were employed to the problem on the dynamic deformation of a non-elastic core one end of which is rigidly fastened, and another one is moving according to a given law [7]. In Fig.5, it is presented a pattern of typical subregions arising in a plane of independent derivatives (t, x) and separated one from another by the characteristics of the initial system. In subregions I and 2, the partial solutions characterized by one and two differential relations, respectively, take place.

It is instructive to note finally, that the method of differential relations generalizes such classical methods as a method of intermediate integral, self-similar solutions and functionally invariant solutions.

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