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N. N. Bogoliubov
B. I. Khatset
D. Ya. Petrina

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MATHEMATICAL DESCRIPTION OF EQUILIBRIUM STATE OF CLASSICAL SYSTEMS BASED ON THE CANONICAL FORMALISM

by N.N. BOGOLIUBOV, B.I. KHATSET, D.Ya. PETRINA

Introduction

To derive thermodynamic relations on the basis of statistical mechanics, one has to study systems with infinite number of degrees of freedom. These systems arise from finite ones when the number of particles N tends to infinity, which is followed by a proportional increase of the volume $V_N (V_N = vN, v = \text{const.})$. This creates rather difficult mathematical problems connected with the justification of the limiting procedure as $N \to \infty$. To solve these problems the formalisms of either canonical or grand canonical ensemble as well as that of partition functions are used.

During the last two decades a considerable advance has been observed in this field. In 1946 one of the authors [1] suggested some ways of the rigorous mathematical justification of the limiting procedure in statistical mechanics, using the formalism of the Gibbs canonical ensemble, and developed a general method of determining the limiting distribution functions as formal series in powers of the density $\frac{1}{v}$. In 1949 two of the authors [2] worked out the foundations of the rigorous mathematical description of infinite systems in statistical mechanics. A comprehensive presentation of the results obtained has been given in [3] in 1956. The works [2,3] comprise the complete solution of mathematical problems arising when performing the limiting process as $N \to \infty$ in the systems described by canonical ensemble for the case of particle interaction through a positive two-body potential and sufficiently small densities. It should be noted that the system of equations for the distribution functions was essentially treated as an operator equation in a Banach space. However, the methods developed in the works mentioned above did not call the attention of the scientists.

In 1963 Ruelle [4] again suggested a similar approach to studying the sets of equations for the distribution functions. He used the grand canonical formalism, which led him, in our opinion, to simpler problems involving the justification of the limiting procedure. At the same time Ruelle succeeded in extending the class of potential functions in question by exploiting the rather ingenious idea of symmetrization of the starting equations for the distribution functions.

The purpose of the present paper is to give, on the basis of the theory of canonical ensemble, a rigorous mathematical description of the equilibrium state (for small densities) of infinite systems of particles whose interaction potential obeys Ruelle's condition [4] and is not required to be positive. The methods developed in [2,3] and Ruelle's method of symmetrization are mainly employed.

In Section 1 the problem is stated and the relations for the distribution functions in the case of a finite volume are derived, which after taking the limit become the well-known Kirkwood-Salsburg equations. Contrary to the grand canonical formalism, there are no equations whatever for the distribution functions in the case of Gibbs ensemble in a finite volume. The appropriate equations appear only after taking the infinite-volume limit. This brings about new problems in comparison with the grand canonical formalism.

In Section 2 the theorem of existence and uniqueness of the solution of the Kirkwood-Salsburg equations for the potentials obeying Ruelle's condition is proved, and the densities are explicitly evaluated, for which the solution can be represented by iterations. The theorem on the analytical character of the density dependence of the limiting distribution functions is proved as well.

Section 3 presents the proof of existence of the limiting distribution functions when the number of particles tends to infinity.

Finally, in Section 4 the uniqueness of these limiting functions is proved.

1. Statement of the Problem

1. Consider a system consisting of N identical particles enclosed in a three-dimensional macroscopic volume V_N and interacting through central forces which are characterized by the mutual potential $\Phi(q)$. It is assumed that the position of each particle is determined completely by its three cartesian coordinates q^α $(\alpha = 1,2,3)$, $q = (q^1,q^2,q^3)$.

Our point of departure is the common theory of equilibrium states based on the Gibbs canonical distribution ; and in the presentation of the subject we follows the works [1,2,3].

Introduce the probability distribution functions for the positions of all particles with the density

$$D_N = D_N(q_1,\dots,q_N) = Q^{-1}(N,V_N)\exp\{-\frac{U_N}{\theta}\} ,$$

where

$$U_N = \sum_{1\leq i<j\leq N} \Phi(q_i-q_j),\ \Phi(q_i-q_j) = \Phi(|q_i-q_j|) \quad ,$$

is the potential energy of the system,

$$Q(N,V_N) = \int_{V_N} \cdots \int_{V_N} \exp\{-\frac{U_N}{\theta}\} . dq_1 \ldots dq_N , dq_i = dq_i^1 dq_i^2 dq_i^3 ,$$

is the configurational integral. The physical system considered is a Gibbs canonical ensemble. Introduce now a sequence of distribution functions [1]

$$(1.1) \qquad F_S^{(N)}(q_1, \ldots, q_S; V_N) = V_N^S \int_{V_N} \cdots \int_{V_N} D_N(q_1, \ldots, q_S, q_{S+1}; \ldots, q_N) dq_{S+1} \ldots dq_N .$$

As usual, V_N is assumed to be a sphere whose volume is also denoted by $V_N; V_N = vN$, where v is the volume per particle, $\frac{1}{v}$ is the density of particles. The principal object of investigation in statistical physics is presented by the limiting functions

$$F_S(q, \ldots, q_S; v) = \lim_{N\to\infty} F_S^{(N)}(q_1, \ldots, q_S; V_N) .$$

In the limit of an infinite volume the density $\frac{1}{v}$ is assumed to be constant. When studying the problem associated with the limiting procedure mentioned above we employ the set of equations which the limiting functions $F_S(q_1, \ldots, q_S; v)$ obey. Below we shall derive some relations which are necessary for obtaining the appropriate equations.

2. Consider the expression $\exp\{-\frac{U_N}{\theta}\}$ and transform it as follows

$$(2.1) \; \exp\{-\frac{1}{\theta} \sum_{1\leq i<j\leq N} \Phi(q_i - q_j)\} = \exp\{-\frac{1}{\theta} \sum_{i=2}^{S} \Phi(q_1 - q_i)\} \times$$

$$\times \exp\{-\frac{1}{\theta} \sum_{2\leq i<j\leq N} \Phi(q_i - q_j)\} \times \prod_{i=s+1}^{N} [\varphi_{q_1}(q_i) + 1] .$$

where

$$\varphi_q(q_i) = \exp\{-\frac{1}{\theta} \Phi(q - q_i)\} - 1 .$$

Substituting (2.1) into (1.1) one obtains

$$(3.1) \qquad F_S^{(N)}(q_1, \ldots, q_S; V_N) = V_N^S \frac{Q(N-1, V_N)}{Q(N, V_N)} \exp\{-\frac{1}{\theta} \sum_{i=2}^{S} \Phi(q_1 - q_i)\} \times$$

$$\times \int_{V_N} \cdots \int_{V_N} D_{N-1}(q_2, \ldots, q_s, q_1^*, \ldots, q_{N-S}^*)$$

$$[1 + \sum_{K=1}^{N-S} \frac{(N-S)(N-S-1)\ldots(N-S-K+1)}{K!} \prod_{i=1}^{K} \varphi_{q_1}(q^*)] \times dq_1^* \ldots dq_{N-S}^* ,$$

where

$$D_M(q_1,\ldots,q_M) = Q^{-1}(M,V_N)\exp\{-\frac{1}{\theta}\sum_{1\le i<j\le M}\Phi(q_i-q_j)\},$$

$$Q(M,V_N) = \int_{V_N}\cdots\int_{V_N}\exp\{-\frac{1}{\theta}\sum_{1\le i<j\le M}\Phi(q_i-q_j)\}\,dq_1\ldots dq_M .$$

Introduce also the distribution functions

$$F_K^{(N-\ell)}(q_1,\ldots,q_K;V_N) = V_N^K\int_{V_N}\cdots\int_{V_N} D_{N-\ell}(q_1,\ldots,q_K,q_{K-1},\ldots,q_{N-\ell})\,dq_{K+1}\ldots dq_{N-\ell}$$

and quantities

$$(4.1)\qquad a_M(V_N) = V^M\frac{Q(M-1,V_N)}{Q(M,V_N)}$$

with

$$F_S^{(N)}(q_1,\ldots,q_S;V_N) = F_S^{(N-0)}(q_1,\ldots,q_S;V_N),$$

in terms of which (3.1) is written as

$$(5.1)\qquad \begin{aligned} F_S^{(N)}(q_1,\ldots,q_S;V_N) &= a_N(V_N)\exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[F_{S-1}^{(N-1)}(q_2,\ldots,q_S;V_N) + \\ &+\sum_{K=1}^{N-S}\frac{(1-\frac{S}{N})(1-\frac{S+1}{N})\ldots(1-\frac{S+K-1}{N})}{K!\;v^K}\int_{V_N}\cdots\int_{V_N}F_{S+K-1}^{(N-1)}(q_2,\ldots,q_S,q_1^*,\ldots,q_K^*;V_N)\times \\ &\times\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)\,dq_1^*\ldots dq_K^*]; \qquad 1<S<N; \end{aligned}$$

$$F_N^{(N)}(q_1,\ldots,q_N;V_N) = a_N(V_N)\exp\{-\frac{1}{\theta}\sum_{i=2}^{N}\Phi(q_1-q_i)\}\cdot F_{N-1}^{(N-1)}(q_2,\ldots,q_N;V_N);$$

$$F_1^N(q_1;V_N) = a_N(V_N)[1+\sum_{K=1}^{N-1}\frac{(1-\frac{1}{N})(1-\frac{2}{N})\ldots(1-\frac{K}{N})}{K!\;v^K}\times$$

$$\times\int_{V_N}\cdots\int_{V_N}F_K^{(N-1)}(q_2,\ldots,q_S;V_N)\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)\,dq_1^*\ldots dq_K^*].$$

Similarly, for the functions $F_S^{(N-\ell)}(q_1,\ldots,q_S;V_N)$ one obtains the relations

$$F_S^{(N-\ell)}(q_1,\dots,q_S;V_N) = \frac{N}{N-\ell}\, a_{N-\ell}(V_N)\,\exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[F_{S-1}^{(N-\ell-1)}(q_2,\dots,q_S;V_N) +$$

(6.1)

$$+\sum_{K=1}^{N-\ell-S}\frac{((1-\frac{\ell+S}{N})\dots(1-\frac{\ell+S+K-1}{N})}{K!\,v^K}\int_{V_N}\dots\int_{V_N}F_{S+K-1}^{(N-\ell-1}(q_2,\dots,q_S,q_1^*,\dots,q_K^*;V_N)\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)\times$$

$$\times\, dq_1^*\dots dq_K^*], \qquad (1 < S < N-\ell)\ ;$$

$$F_{N-\ell}^{(N-\ell)}(q_1,\dots,q_{N-\ell};V_N) = \frac{N}{N-\ell}\, a_{N-\ell}(V_N)\exp\{-\frac{1}{\theta}\sum_{i=2}^{N-\ell}\Phi(q_1-q_i)\}F_{N-\ell-1}^{(N-\ell-1)}(q_1,\dots,q_{N-\ell};V_N);$$

$$F_1^{(N-\ell)}(q_1;V_N) = \frac{N}{N-\ell}\, a_{N-\ell}(V_N)[1+\sum_{K=1}^{N-\ell-1}\frac{(1-\frac{\ell+1}{N})\dots(1-\frac{\ell+K}{N})}{K!\,v^K}$$

$$\int_{V_N}\dots\int_{V_N}F_K^{(N-\ell-1)}(q_1^*,\dots,q_K^*;V_N)\times\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*]\ .$$

3. Suppose for a while that the limits

$$F_S(q_1,\dots,q_S;v) = \lim_{N\to\infty}F_S^{(N)}(q_1,\dots,q_S;V_N)\ , \qquad S = 1,2,\dots$$

(7.1)
$$F_S^{\ell}(q_1,\dots,q_S;v) = \lim_{N\to\infty}F_S^{(N-\ell)}(q_1,\dots,q_S;V_N)\ , \qquad S = 1,2,\dots$$

$$a(v) = a_o(v) = \lim_{N\to\infty}a_N(V_N);\ a_\ell(v) = \lim_{N\to\infty}a_{N-\ell}(V_N)$$

exist in some sense and write the relations (5.1) and (6.1) after performing the limit in a formal way. It is easily seen that the relations (5.1) become

$$F_S(q_1,\dots,q_S,v) = a(v)\,\exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[F^1_{S-1}(q_2,\dots,q_S;v) +$$

(8.1)

$$+\sum_{K=1}^{\infty}\frac{1}{K!\,v^K}\int\dots\int F^1_{S+K-1}(q_2,\dots,q_S,q_1^*,\dots,q_K^*;v)\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*]\ .$$

(We do not indicate the integration limits when the integration is carried out over the whole three-dimensional space). After taking the limit, the relation (6.1) will be of the form

$$F_S^{\ell}(q_1,\dots,q_S;v) = a_\ell(v)\ \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[F_{S-1}^{\ell+1}(q_2,\dots,q_S;v) +$$

(9.1)

$$+\sum_{K=1}^{\infty}\frac{1}{K!\,v^K}\int\dots\int F_{S+K-1}^{\ell+1}(q_2,\dots,q_S,q_1^*,\dots,q_K^*;v)\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*].$$

Should

$$F_S^{\ell}(q_1,\dots,q_S;v) = F_S(q_1,\dots,q_S;v); a_{\ell}(v) = a(v) \tag{10.1}$$

be valid for any $\ell \geqq 1$, the relations (8.1), (9.1) will become the well-known Kirkwood-Salsburg equations [5]

$$F_S(q_1,\dots,q_S;v) = a(v)\exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[F_{S-1}(q_2,\dots,q_S;v) + \\ + \sum_{K=1}^{\infty}\frac{1}{K!v^K}\int\dots\int F_{S+K-1}(q_2,\dots,q_S;q_1^*,\dots,q_K^*;v)\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*]\,. \tag{11.1}$$

To complete (11.1) for the case of $S = 1$, we put $F_0^{\ell} = 1$. It follows from the definitions that all $F_S(q_1,\dots,q_S;v)$ are symmetric functions of variables q_i.

4. We are now in a position to discuss the problems arising in the mathematical description of the system in equilibrium state. To present a complete justification for the mathematical description of such systems, based on the Gibbs canonical distribution and using sequences of distribution functions, we think it necessary to solve the following three problems :

1) to prove that the limits (7.1) exist in a certain sense ;

2) to prove that the limiting distribution functions do not depend on the way of taking the limit (that is, (10.1) holds) ;

3) to prove that for sufficiently small densities $\frac{1}{v}$ there exists a unique solution of the Kirkwood-Salsburg set of equations.

We shall solve the above mentioned problems in the following order : first, problem 3 ; and then 1 and 2.

2. The existence theorem for the solution of the Kirkwood-Salsburg set of equations

1. In the present section we shall consider the Kirkwood-Salsburg equations (11.1) for the distribution functions $F_S(q_1,\dots,q_S;v)$ and prove that for sufficiently small densities $\frac{1}{v}$ they possess a unique solution.

We have the set of equations

$$F_S(q_1,\dots q_S;v) = a(v)\exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[F_{S-1}(q_2,\dots,q_S;v) + \\ + \sum_{K=1}^{\infty}\frac{1}{K!v^K}\int\dots\int F_{S+K-1}(q_2,\dots,q_S;q_1^*,\dots,q_K^*;v)\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*]\,,\quad F_0 = 1\,. \tag{1.2}$$

The solution of this set will be determined in a Banach space [2,3,4] which is introduced as follows. Consider the linear space whose elements are the columns of measurable bounded functions

$$f = \begin{pmatrix} f_1(q_1) \\ f_2(q_1, q_2) \\ f_3(q_1, q_2, q_3) \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

with ordinary operations of addition and multiplication by a number. This linear space becomes a Banach space B for the following norm of the element

(2.2)
$$\|f\| = \sup_S \Big[\frac{1}{A^S} \sup_{q_1, \ldots, q_S} |f_S(q_1, \ldots, q_S)|\Big] ,$$

where A is a positive constant which is to be determined.

In [2,3] the equivalent norm

$$\|f\| = \sup_S \Big[\frac{1}{SA^S} \sup_{q_1, \ldots, q_S} |f_S(q_1, \ldots, q_S|\Big]$$

was introduced, which was more convenient when studying the Mayer-Montroll equations for the distribution functions ([1], p. 23).

The operator K is formally defined in the space B through the relation

(3.2)
$$(Kf)_S(q_1, \ldots, q_S) = \exp\Big\{-\frac{1}{\theta} \sum_{i=2}^{S} \Phi(q_1 - q_i)\Big\}\Big[f_{S-1}(q_2, \ldots, q_S) +$$
$$+ \sum_{K=1}^{\infty} \frac{1}{K!v^K} \int \cdots \int f_{S+K-1}(q_2, \ldots, q_S, q_1^*, \ldots, q_K^*) \prod_{i=1}^{K} \varphi_{q_1}(q_i^*) dq_1^* \ldots dq_K^*\Big] ; \quad f_0 = 0.$$

The set of equations (1.2) can now be written down in the form

(4.2)
$$F = a(v)KF + a(v)F^0 ,$$

where

(5.2)
$$F = \begin{pmatrix} F_1(q_1, v) \\ F_2(q_1, q_2; v) \\ F_3(q_1, q_2, q_3; v) \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} ; \quad F^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

Below we shall prove the existence and uniqueness of the solution of the operator equation (4.2) in the space B under some assumptions concerning the potential $\Phi(q)$ and for sufficiently small $\frac{1}{v}$.

As it is known, in order that equation (4.2) possess a unique solution in B it is sufficient that $a(v)F^{o}\in B$, the operator $a(v)K$ be defined in the whole space B and its norm be less that unity. The solution F is represented by the series

$$F = \sum_{n=0}^{\infty} (a(v)K)^{n} a(v)F^{o} , \tag{6.2}$$

which converges in the norm of the space B . It is easily seen [2,3] that these conditions are valid for sufficiently small $\frac{1}{v}$ in the case of non-negative potential $\Phi(q)$ which is assumed to be a real-valued function in E_3 such that

$$I = \int |\exp\{-\frac{1}{\theta}\Phi(q)\} - 1|dq < \infty . \tag{7.2}$$

We shall assume first that $a(v)$ and v are independent (generally speaking, complex) parameters with

$$|a(v)| < 2 \tag{8.2}$$

$$|\frac{1}{v}| < \frac{1}{2eI} , \tag{9.2}$$

and evaluate the norm of the operator $a(v)K$.

It should be noted that the condition $\Phi(q) \geqq 0$ implies that

$$\exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1 - q_i)\} \leqq 1 . \tag{10.2}$$

Then it follows from (3.2) that

$$\sup_{q_1,\dots,q_S} |a(v)(Kf)_S(q_1,\dots,q_S)| \leqq |a(v)|\Big[\sup_{q_2,\dots,q_S} |f_{S-1}(q_2,\dots,q_S)|\frac{A^{S-1}}{A^{S-1}} +$$

$$+ \sum_{K=1}^{\infty}\frac{1}{K!|v|^{K}}\int\dots\int \sup_{q_2,\dots,q_S,q_1^*,\dots,q_K^*}|f_{S+K-1}(q_1,\dots,q_S,q_1^*,\dots,q_K^*)|\frac{A^{S+K-1}}{A^{S+K-1}}\prod_{i=1}^{K}|\varphi_{q_1}(q_i^*)| \times \tag{11.2}$$

$$dq_1^* \dots \times dq_K^* \leqq |a(v)| e^{\frac{A}{|v|}I} A^{S-1} \|f\| .$$

Hence,

$$\|a(v)Kf\| = \sup_S \{\frac{1}{A^S}\sup_{q_1,\dots,q_S}|(a(v)Kf)_S(q_1,\dots,q_S)| \leqq \frac{|a(v)|}{A} e^{\frac{A}{|v|}I}\|f\| . \tag{12.2}$$

If now we put $A = 2e$ and take into account the inequalities (8.2) and (9.2), we get

$$\|K\| < \frac{|a(v)| e^{\frac{2eI}{v}}}{2e} < 1 \quad .$$

At the same time we see that the operator K is defined in the whole space B . It remains to check only the condition $F^{O} \epsilon B$. The latter immediately follows from the definitions (5.2) and (2.2) since

$$\|a(v)F^{O}\| = \|a(v)F^{O}\| = \frac{|a(v)|}{A} < 1 \quad .$$

It is thus shown that for $\Phi(q) \geqq 0$ and with the assumptions (7.2) and (8.2), the Kirkwood-Salsburg set of equations (1.2) has a unique solution in a neighbourhood of the point $\frac{1}{v} = 0$.

The method described above was developed by two of the authors in 1949 [2,3]. We shall show now that it may also be applied to the more general case when the requirement that the potential $\Phi(q)$ be nonnegative is replaced by the following Ruelle's condition [4] :

<u>There exists a positive constant</u> b <u>such that for all</u> S <u>and any</u> $q_1, \dots, q_S \epsilon E_3$ the following inequality is valid :

(13.2) $$\frac{1}{\theta} U_S(q_1, \dots, q_S) \geqq - Sb \ .$$

From this condition it results that for any point $(q_1, \dots, q_S)$ belonging to E_{3S} there exists at least one index i such that

(14.2) $$\frac{1}{\theta} \sum_{j \neq i} \Phi(q_i - q_j) > - 2b \ .$$

In view of the symmetry of the functions F_S , one may, after Ruelle [4], write equations (1.2) in a symmetric form. Let π_ℓ denote the operator acting on the function $f_S(q_1, \dots q_S)$ through the formula

$$\pi_\ell f_S(q_1, q_2, \dots, q_{\ell-1}, q_\ell, q_{\ell+1}, \dots, q_S) = f_S(q_\ell, q_2, \dots, q_{\ell-1}, q_1, q_{\ell+1}, \dots, q_S) \ .$$

It follows from (14.2) that there exist measurable functions $\nu_i(q_1, \dots, q_S)$, invariant under the rotation group, having the values in the interval [0,1], and such that

$$\sum_{i=1}^{S} \nu_i(q_1, \dots, q_S) = 1 \ ; \ \pi_K \nu(q_1, \dots, q_S) = \nu_K(q_1, \dots, q_S) \ ,$$

with the inequality (14.2) being valid if $\nu_i(q_1, \dots, q_S) \neq 0$. The set of functions ν_i can be considered as the partition of the unity.

Now we define the operator π by the relation

$$\pi f_S(q_1,\dots,q_S) = \sum_{\ell=1}^{S} \pi_\ell[\nu_1(q_1,\dots,q_S) f_S(q_1,\dots,q_S)] .$$

In view of the symmetry of the functions $F_S(q_1,\dots,q_S;v)$, equations (1.2) may be represented in the following form

$$(15.2)\quad \begin{aligned} F_S(q_1,\dots,q_S;v) &= a(v)\pi \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[F_{S-1}(q_2,\dots,q_S;v) + \\ &+ \sum_{K=1}^{\infty}\frac{1}{K!v^K}\int\dots\int F_{S+K-1}(q_2,\dots,q_S,q_1,\dots,q_K^*;v)\prod_{i=1}^{K}\varphi_{q_i^*}(q_i^*)dq_1^*\dots dq_K^*] .\end{aligned}$$

With the assumption that $\Phi(q)$ obeys the conditions (8.2) and (13.2), we shall establish the existence and uniqueness of the solution to the set (15.2) for sufficiently small $\frac{1}{v}$. As above, $a(v)$ and v are considered as independent complex parameters and the inequality (8.2) is assumed to be valid.

The set (15.2) may be written in the form (4.2) provided that the operator K is now determined by the symmetrized relations (3.2) :

$$(16.2)\quad \begin{aligned} (Kf)_S(q_1,\dots,q_S) &= \pi \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[f_{S-1}(q_2,\dots,q_S) + \\ &+ \sum_{K=1}^{\infty}\frac{1}{K!v^K}\int\dots\int f_{S+K-1}(q_2,\dots,q_S,q_1^*,\dots,q_K^*)\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*] , \quad f_0 = 0 .\end{aligned}$$

THEOREM I. <u>The set of equations</u> (15.2) <u>possesses a unique solution in the Banach space</u> B <u>for</u>

$$(17.2)\qquad |a(v)| < 2 \; , \; \frac{1}{|v|} < \frac{1}{2e^{2b+1}I} .$$

<u>This solution is a holomorphic function of</u> $a(v)$ <u>and</u> $\frac{1}{v}$ <u>in the domain</u> (17.2).

To prove the first assertion, it suffices to show that with the conditions (17.2) the norm of the operator K defined by (16.2) is less than 1.

Bearing in mind that the estimate (10.2) is now replaced by

$$|\pi \exp\{-\frac{1}{\theta}\sum_{i=1}^{\infty}\Phi(q_1-q_i)\}| < e^{2b} ,$$

which follows from (14.2), we obtain from (15.2) :

$$\sup_{q_1,\dots,q_S} |a(v)(Kf)_S(q_1,\dots,q_S)| \leq |a(v)| e^{2b} . e^{\frac{A}{|v|}I} A^{S-1} ||f|| ;$$

(18.2)
$$||a(v)Kf|| \leq \frac{|a(v)|}{A} e^{2b} e^{\frac{A}{|v|}I} ||f|| .$$

Putting $A = 2e^{2b+1}$ and taking into account (17.2) we obtain from (18.2) the estimate required :

$$||a(v)K|| < \frac{|a(v)|}{2e^{2b+1}} \exp\{2b + \frac{2e^{2b+1}I}{|v|}\} \leq k < 1 .$$

We have thus proved the existence and uniqueness of the solution of the set (15.2), or, which is the same, of the equation (4.2) in the domain (17.2). This solution is a holomorphic function of $a(v)$ and v in this domain since it may be represented by the series (6.2) which converges uniformly with regard to $a(v)$ and v in each closed domain contained in (17.2).

4. Turning now to our case of interest of the limiting distribution functions $F_S(q_1,\dots,q_S;v)$, we have to consider v as a real (positive) variable and $a(v)$ as a function of v. Let us show that in this case the condition $|a(v)| < 2$ of (17.2) may be omitted. To this end, we shall find the estimate for $a_\ell(v)$.

LEMMA I. <u>The numbers</u> $a_\ell(v)$ <u>obey the inequality</u>

(19.2)
$$a_\ell(v) \leq \frac{1}{1 - \frac{I}{v}} .$$

<u>Proof</u> [3]. Consider the quantity

$$\frac{Q(M,V_N)}{Q(M-1,V_N)} = \frac{1}{Q(M-1,V_N)} \int_{V_N} \dots \int_{V_N} \exp\{-\frac{1}{\theta} \sum_{1\leq i<j\leq M} \Phi(q_i-q_j)\} dq_1 \dots dq_M =$$

(20.2)

$$= \int_{V_N} dq [\int_{V_N} \dots \int_{V_N} \frac{1}{Q(M-1,V_N)} \exp\{-\frac{1}{\theta} \sum_{1\leq i<j\leq M-1} \Phi(q_i-q_j)\} \prod_{i=1}^{M-1} \{\varphi_q(q_i)+1\} dq_1 \dots dq_{M-1}] .$$

We shall use the following elementary inequality

(21.2)
$$\prod_{i=1}^{M-1} [\varphi_q(q_i)+1] \geq 1 - \sum_{i=1}^{M-1} |\varphi_q(q_i)| ,$$

which holds not only for our special functions $\varphi_q(q_i)$, but also for arbitrary quantities a_i such that $1+a > 0$.

By making use of (21.2) one obtains from (20.2) the following estimate

$$\frac{Q(M,V_N)}{Q(M-1,V_N)} \geqq \int_{V_N} dq[\int_{V_N} \dots \int_{V_N} (1-\sum_{i=1}^{N-1} |\varphi_q(q_i)|) \frac{1}{Q(M-1,V_N)} \times$$

$$\times \exp\{-\frac{1}{\theta} \sum_{1\leqq i<j\leqq M-1} \Phi(q_i-q_j)\} dq_1 \dots dq_{M-1} \geqq V_N - (M-1)I ,$$

whence for any $M \leqq N$

$$a_M(V_N) = v \frac{M Q(M-1,V_N)}{Q(M,V_N)} \leqq v \frac{1}{V_N-(M-1)I} < \frac{1}{1-\frac{I}{v}} .$$

Since, by definition, $a_\ell(v) = \lim_{N\to\infty} a_{N-\ell}(V_N)$ for any ℓ , then $a_\ell(v)$ also satisfy the inequality (19.2).

From Lemma I it results that for

$$\frac{1}{v} < \frac{1}{2e^{2b+1}I} < \frac{1}{2I}$$

the inequality $a_\ell(v) < 2$ holds, that is, the second condition of (17.2) implies the first one.

Finally, consider the character of the density dependence of the limiting distribution functions.

THEOREM II. <u>The distribution functions</u> $F_S(q_1,\dots,q_S;v)$ <u>are holomorphic with respect to</u> $\frac{1}{v}$ <u>in a neighbourhood of the zero point.</u>

<u>Proof</u> [3]. By Theorem I, $F_S(q_1,\dots,q_S;v)$ are holomorphic functions of $a(v)$ and $\frac{1}{v}$ in the domain (17.2), that is, by virtue of Lemma I, for

$$\frac{1}{v} < \frac{1}{2e^{2b+1}I} .$$

It is thus sufficient to show that $a(v)$ is a holomorphic function of $\frac{1}{v}$ in some neighbourhood of the point $\frac{1}{v} = 0$.

The solution of Equation (4.2) is translation-invariant since each term of the series (6.2) possesses this property. Therefore, $F_1(q_1;v) = \text{const.}$ As it will be demonstrated in Section 4,

$$(22.2) \quad \lim_{N\to\infty} \frac{1}{V_N} \int_{V_N} F_1(q_1;v)dq_1 = \lim_{N\to\infty} \frac{1}{V_N} \int_{V_N} F_1^{(N)}(q_1;v)dq_1 = 1$$

whence $F_1(q_1;v) = 1$.

Now, by making use of the Kirkwood-Salsburg equations, we obtain

$$(23.2)\quad 1 = F_1(q_1; v) = a(v)[1 + \sum_{K=1}^{\infty} \frac{1}{K!v^K} \int\dots\int F_K(q_1^*,\dots,q_K^*; v) \times \prod_{i=1}^{K} \varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*] \equiv \chi(a,v)$$

where the function χ is holomorphic in a and $\frac{1}{v}$ in the domain (17.2). Let us show that in some neighbourhood of the point $\frac{1}{v} = 0$ for $|a| < 2$ the partial derivative $\frac{\partial\chi}{\partial a} \neq 0$. The latter condition implies that the relation (23.2) can be solved with respect to a and the function $a(v)$ is holomorphic with respect to $\frac{1}{v}$ in a neighbourhood of the point $\frac{1}{v} = 0$.

The following expression is obtained for $\frac{\partial\chi}{\partial a}$:

$$(24.2)\quad \frac{\partial\chi}{\partial a} = 1 + \sum_{K=1}^{\infty} \frac{1}{K!v^K} \int\dots\int F_K(q_1^*,\dots,q_K^*; v) \prod_{i=1}^{K} \varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^* + a\frac{1}{v}\frac{\partial}{\partial a}[\sum_{K=1}^{\infty} \frac{1}{K!v^{K-1}} \int\dots\int F_K(q_1^*,\dots,q_K^*; v) \prod_{i=1}^{K} \varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*] .$$

It follows from (28.2) that $\frac{\partial\chi}{\partial a} = 1$ at the point $\frac{1}{v} = 0$ and, as a consequence, $\frac{\partial\chi}{\partial a} \neq 0$ in some neighbourhood of the zero point. This completes the proof.

3. Existence of the limiting distribution functions

1. In the present section we are studying the problem I stated in Section 1. When solving this problem, it is convenient to use the relations (5.1), (6.1) which are the basis for deriving the Kirkwood-Salsburg equations.

Consider thus the relations

$$(1.3)\quad F_S^{(N-\ell)}(q_1,\dots,q_S; V_N) = \frac{N}{N-\ell} a_{N-\ell}(V_N)\pi \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[F_{S-1}^{(N-\ell-1)}(q_2,\dots,q_S; V_N) + \sum_{K=1}^{N-\ell-S} \frac{(1-\frac{\ell+S}{N})\dots(1-\frac{\ell+S+K-1}{N})}{K!v^K} \int_{V_N}\dots\int_{V_N} F_{S+K-1}^{(N-\ell-1)}(q_2,\dots,q_S;q_1^*,\dots,q_K^*; V_N) \times \prod_{i=1}^{K} \varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*] ,$$

$$F_{N-\ell}^{(N-\ell-1)}(q_1,\dots,q_{N-\ell}; V_N) = \frac{N}{N-\ell} a_{N-\ell}(V_N)\pi \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i\} \times F_{N-\ell-1}^{(N-\ell-1)}(q_2,\dots,q_{N-\ell}, V_N) \qquad F_0^{(N-\ell-1)} = 1$$

which follow from (6.1) due to the functions $F_S^{(N-\ell)}(q_1, \dots, q_S; V_N)$ being symmetric (see Sec. 2, (15.2)). These relations will be assumed valid in the whole 3S-dimensional Euclidean space E_{3S}. The functions $F_S^{(N-\ell)}(q_1, \dots, q_S; V_N)$ are defined in the whole E_{3S} by the formulas (1.1), (4.1).

The following notations are introduced

$$(2.3)\quad F^{(N-\ell)} = \begin{pmatrix} F_1^{(N-\ell)}(q_1; V_N) \\ F_2^{(N-\ell)}(q_1, q_2; V_N) \\ \vdots \\ F_{N-\ell}^{(N-\ell)}(q_1, q_2, \dots, q_{N-\ell}; V_N) \\ 0 \\ \vdots \end{pmatrix} \qquad F^{o(N-\ell)} = \begin{pmatrix} \frac{N}{N-\ell} a_{N-\ell}(V_N) \\ 0 \\ \vdots \end{pmatrix}$$

Let $K_S^{(N-\ell)}$ denote the operator acting in the Banach space B on an arbitrary column f by the formulas

$$(3.3)\quad (K_S^{(N-\ell)} f)_S(q_1, \dots, q_S) = \frac{N}{N-\ell} a_{N-\ell}(V_N) \pi \exp\{-\frac{1}{\theta} \sum_{i=2}^{S} \Phi(q_1 - q_i)\}[f_{S-1}(q_2, \dots, q_S) +$$

$$+ \sum_{K=1}^{N-\ell-S} \frac{(1-\frac{\ell+S}{N})\dots(1-\frac{\ell+S+K-1}{N})}{K!\, v^K} \int_{V_N} \dots \int_{V_N} f_{S+K-1}(q_2, \dots, q_S, q_1^*, \dots, q_K^*) \prod_{i=1}^{K} \varphi_{q_1}(q_i^*) \times$$

$$\times dq_1^* \dots dq_K^*] \quad , \qquad (K_S^{(N-\ell)} f)_{i \neq S}(q_1, \dots, q_i) = 0 \quad , \qquad S \neq N-\ell$$

$$(K_{N-\ell}^{(N-\ell)} f)_{N-\ell}(q_1, \dots, q_{N-\ell}) = \frac{N}{N-\ell} a_{N-\ell}(V_N) \pi \exp\{-\frac{1}{\theta} \sum_{i=2}^{N-\ell} \Phi(q_1 - q_i)\} f_{N-\ell-1} \times$$

$$\times (q_2, \dots, q_{N-\ell}) \quad , \qquad (K_{N-\ell}^{(N-\ell)} f)_{i \neq N-\ell}(q_1, \dots, q_i) = 0 \quad , \qquad f_o = 0 .$$

Here the expressions $(K_S^{(N-\ell)} f)_S(q_1, \dots, q_S)$ are also defined in the whole E_{3S}. Denote by $K^{(N-\ell)}$ the following operator

$$K^{(N-\ell)} = K_1^{(N-\ell)} + K_2^{(N-\ell)} + \dots + K_{N-\ell}^{(N-\ell)} .$$

With the operators $K^{(N-\ell)}$ and columns $F^{(N-\ell)}$ the relations (1.3) are represented in the following compact form

$$(4.3)\qquad F^{(N-\ell)} = K^{(N-\ell)} F^{(N-\ell-1)} + F^{o(N-\ell)} .$$

2. We shall investigate now the properties of the operator $K^{(N-\ell)}$.

One can show that the norms of the operators $K^{(N-\ell)}$ and $K_S^{(N-\ell)}$ $(1 \leq S \leq N-\ell)$ for $\frac{1}{v} < \frac{1}{2e^{2\lambda+1}I}$ are less than unity, $\|K^{(N-\ell)}\| \leq k < 1$, $\|K_S^{(N-\ell)}\| \leq k < 1$. The argument is similar to that employed in the proof of Theorem I if one takes into account that the quantities $\frac{N}{N-\ell} a_{N-\ell} (V_N)$ obey the inequalities

$$\frac{N}{N-\ell} a_{N-\ell}(V_N) < 2 \qquad \text{and} \qquad (1-\frac{\ell+S}{N})\ldots(1-\frac{\ell+S+K-1}{N}) < 1$$

for $\frac{1}{v} < \frac{1}{2I}$. The number λ appearing in the norm definition is subjected to the same restriction as in Section 2. The positive sequences

$$a_N(V_N) \;,\; \frac{N}{N-1} a_{N-1}(V_N),\ldots,\frac{N}{N-\ell} a_{N-\ell}(V_N),\ldots; \qquad N = 3,4,\ldots$$

are bounded from above by 2 for $\frac{1}{v} < \frac{1}{2I}$ *). Therefore, using the diagonal process, one can separate any finite number $\ell+1$ of converging subsequences

$$a_{N_i}(V_{N_i}),\; \frac{N_i}{N_i-1} a_{N_i-1}(V_{N_i}),\ldots,\; \frac{N_i}{N_i-\ell} a_{N_i-\ell}(V_{N_i}) \;,$$

whose limits as $N_i \to \infty$ are denoted by $a(v)$, $a_1(v),\ldots,a_\ell(v)$;

$$\lim_{N_i\to\infty} a_{N_i}(V_{N_i}) = a(v);\; \lim_{N_i\to\infty} \frac{N_i}{N_i-1} a_{N_i-1}(V_{N_i}) = a_1(v),\ldots$$

$$\ldots,\; \lim_{N_i\to\infty} \frac{N_i}{N_i-\ell} a_{N_i-\ell}(V_{N_i}) = a(v) \;.$$

We shall see below (Section 4) that for each sequence $a_N(V_N)$, $\frac{N}{N-1}a_{N-1}(V_N),\ldots$ $\frac{N}{N-\ell}a_{N-\ell}(V_N)$ there exists only one limiting point, that is, the sequences

$$a_N(V_N),\; \frac{N}{N-1} a_{N-1}(V_N)\ldots,\frac{N}{N-\ell} a_{N-\ell}(V_N)$$

themselves converge as $N \to \infty$. Moreover, it will be shown that $a(v) = a_1(v) = \ldots$ $\ldots = a_\ell(v)$. For the time being, we shall mean under the sequences

$$a_N(V_N),\frac{N}{N-1} a_{N-1}(V_N),\ldots,\frac{N}{N-\ell} a_{N-\ell}(V_N)$$

some of their converging subsequences and the limiting process will be performed over these subsequences as $N \to \infty$, although we shall write $N \to \infty$.

Let K_S denote the operator acting on an arbitrary element f of the Banach space B through the formula

$$(K_S f)_S(q_1,\ldots,q_S) = (Kf)_S(q_1,\ldots,q_S),(K_S f)_{i\neq S}(q_1,\ldots,q_i) = 0 \;.$$

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*) We put $a_{N-\ell}(V_N) = 0$ for $\ell > N-3$.

Let $\psi^{(R)}(q)$ be the characteristic function of the sphere V_N with radius

$$R_N = \sqrt[3]{\frac{3}{4\pi} vN}$$

and the origin as a centre and V_N' the sphere with radius $R_N - r_N$ and characteristic function $\psi^{(R-r)}(q)$. The function r_N is required to possess the following properties: $r_N \to \infty$ as $N \to \infty$, $r_N/R_N \to 0$ as $N \to \infty$.

Let $\psi^{(R-r)}$ be the operator acting on an arbitrary element $f \in B$ by the formula

$$(\psi^{(R-r)}f)_S(q_1,\dots,q_S) = \psi^{(R-r)}(q_1)\dots\psi^{(R-r)}(q_S)f_S(q_1,\dots,q_S) .$$

One more property of the operator $K_S^{(N-\ell)}$ is given by the following lemma.

LEMMA 2. If $\frac{1}{v} < \frac{1}{2e^{2b+1}I}$, then the sequence of operators $\psi^{(R-r)}(a_\ell(v)K_S - K_S^{(N-\ell)})$, ℓ being fixed, converges, relative to the norm, to zero as $N \to \infty$.

Proof. Consider for any f the expression

$$\psi^{(R-r)}(a_\ell(v)K_S - K_S^{(N-\ell)})f = a_\ell(v)\psi^{(R-r)}\pi \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\} \times$$

$$\times [f_{S-1}(q_2,\dots,q_S) + \sum_{K=1}^{\infty}\frac{1}{K!v^K}\int\dots\int f_{S+K-1}(q_2,\dots,q_S,q_1^*,\dots,q_K^*) \times$$

$$\times \prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*] -$$

$$-\frac{N}{N-\ell}a_{N-\ell}(V_N)\psi^{(R-r)}\pi \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[f_{S-1}(q_2,\dots,q_S) +$$

$$+\sum_{K=1}^{N-\ell-S}\frac{(1-\frac{\ell+S}{N})\dots(1-\frac{\ell+S+K-1}{N})}{K!\,v^K}\int_{V_N}\dots\int_{V_N} f_{S+K-1}(q_2,\dots,q_K,q_1^*,\dots,q_K^*) \times$$

$$\times \prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*] =$$

$$= a_\ell(v)\psi^{(R-r)}\pi \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[f_{S-1}(q_2,\dots,q_S) +$$

$$+\sum_{K=1}^{N_0}\frac{1}{K!\,v^K}\int\dots\int f_{S+K=1}(q_2,\dots,q_S,q_1^*,\dots,q_K^*)\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*] -$$

$$-\frac{N}{N-\ell}a_{N-\ell}(V_N)\psi^{(R-r)}\pi \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[f_{S-1}(q_2,\dots,q_S) +$$

$$+\sum_{K=1}^{N_0}\frac{(1-\frac{\ell+S}{N})\dots(1-\frac{\ell+S+K-1}{N})}{K!\,v^K}\int\dots\int f_{S+K-1}(q_2,\dots,q_S,q_1^*,\dots,q_K^*) \times$$

$$\times \prod_{i=1}^{K} \varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*] + a_\ell(v)\Psi^{(R-r)}\pi \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\} \times$$

$$\times [\sum_{K=N_0+1}^{\infty} \frac{1}{K!\, v^K}\int\dots\int f_{S+K-1}(q_2,\dots,q_S;q_1^*,\dots,q_K^*)\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*] -$$

$$-\frac{N}{N-\ell}a_{N-\ell}(V_N)\Psi^{(\ell-r)}\pi \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[\sum_{K=N_0+1}^{N-\ell-S}\frac{(1-\frac{\ell+S}{N})\dots(1-\frac{\ell+S+K-1}{N})}{K!\, v^K}\times$$

$$\times \int_{V_N}\dots\int_{V_N} f_{S+K-1}(q_2,\dots,q_S,q_1^*,\dots,q_K^*)\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^*] + \frac{N}{N-\ell}a_{N-\ell}(V_N)\times$$

$$\times \Psi^{(R-r)}\pi \exp\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\}[\sum_{K=1}^{N_0}\frac{(1-\frac{\ell+S}{N})\dots(1-\frac{\ell+S+K-1}{N})}{K!\, v^K}\times$$

$$\times \int\dots\int f_{S+K-1}(q_2,\dots,q_K;q_1^*,\dots,q_K^*)[1-\prod_{i=1}^{K}\Psi^{(R)}(q_i^*)]\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)\times dq_1^*\dots dq_K^*],$$

where N_0 is as yet an arbitrary finite number.

Let β, γ, denote the last three terms respectively. The following estimates hold

$$|\beta| \le A^{S-1}a(v)e^{2b}\|f\|\sum_{K=N_0+1}^{\infty}\frac{1}{K!\, v^K}A^K I^K = A^S\varepsilon_1(N_0)\|f\|$$

$$|\gamma| \le A^{S-1}\frac{N}{N-\ell}a_{N-\ell}(V_N)e^{2b}\|f\|\sum_{K=N_0+1}^{\infty}\frac{1}{K!v^K}A^K I^K = A^S\varepsilon_1(N_0)\|f\| ,$$

$$|\delta| \le A^{S-1}{}_{N-\ell}(V_N)e^{2b}\|f\|\sum_{K=1}^{\infty}\frac{1}{K!\, v^K}\varepsilon_2(r_N)KI^{K-1}A^K \le A^S\varepsilon(r_N)\|f\| ,$$

where

$$\varepsilon_2(r_N) = \int |e^{-\frac{1}{\theta}\Phi(q)} - 1|(1-\Psi^{(r)}(q))dq .$$

Being the remainder of the absolutely converging series, the quantity $\varepsilon_1(N_0)$ can be made arbitrarily small for sufficiently large but finite N_0 independently of N. The quantity $\varepsilon_2(r_N)$ can be made arbitrarily small for sufficiently large N due to the integral

$$\int |e^{-\frac{1}{\theta}\Phi(q)} - 1|dq$$

being absolutely convergent. The difference of the first two expressions, denoted by α , obeys the following estimate :

$$|\alpha| \leq A^{S-1}e^{2b}\|f\| \sum_{K=0}^{N_0} |a_\ell(v) - \frac{N}{N-\ell}\, a_{N-\ell}(V_N)(1-\frac{\ell+S}{N})\dots(1-\frac{\ell+S+K-1}{N})| \times \frac{A^K I^K}{K!v^K} =$$

$$= A^S \varepsilon_3(N)\|f\|$$

with $\varepsilon_3(N)$ being arbitrarily small for large enough N since $a_{N-\ell}(V_N) \to a_\ell(v)$ (here N_0 is a fixed finite number).

Thus we obtain finally

$$\sup|\Psi^{(R-r)}[(a_\ell(v)K_S - K_S^{(N-\ell)})f]_S(q_1,\dots,q_S)| \leq A^S(2\varepsilon_1(N_0) + \varepsilon_2(r_N) + \varepsilon_3(N)\|f\| ,$$

which is equivalent to the following estimate

$$\|\Psi^{(R-r)}(a_\ell(v)K_S - K_S^{(N-\ell)})f\| \leq (2\varepsilon_1(N_0) + \varepsilon_2(r_N) + \varepsilon_3(N))\|f\| .$$

This immediately implies

(5.3) $$\|\Psi^{(R-r)}(a_\ell(v)K_S - K_S^{(N-\ell)})\| \leq 2\varepsilon_1(N_0) + \varepsilon_2(r_N) + \varepsilon_3(N) = \varepsilon(r_N, N) , \quad \lim_{N\to\infty} \varepsilon(r_N,N) = 0$$

which means that the sequence of operators $\Psi^{(R-r)}(a_\ell(v)K_S - K_S^{(N-\ell)})$ converges to zero as $N \to \infty$ relative to the norm. Lemma is proved.

Let B_n denote the space consisting of columns f with $f_{n+1} = f_{n+2} = \dots = 0$ and with the norm given by

$$\|f\| = \sup_{1\leq S\leq n} \{\frac{1}{A^S} \sup_{q_1\dots q_S} |f_S(q_1,\dots,q_S)|\} .$$

Consider the operators from B into B_n :

$$K_{[n]}^{(N-\ell)} = \sum_{1\leq S\leq n} K_S^{(N-\ell)} , \quad K_{[n]} = \sum_{1\leq S\leq n} K_S .$$

The norms of these operators are less than unity.

The operators

$$\Psi^{(R-r)}(K^{(N-\ell)} - a_\ell(v)K_{[n]})$$

are the sums of a finite number of operators

$$\Psi^{(R-r)}(K_S^{(N-\ell)} - a_\ell(v)K_S)$$

which converge to zero relative to the norm. Therefore the operators

$$\Psi^{(R-r)}(K_{[n]}^{(N-\ell)} - a_\ell(v)K_{[n]})$$

also converge to zero relative to the norm. It is easily seen that the following estimate is valid :

$$\|\Psi^{(R-r)}(K^{(N-\ell)}_{[n]} - a_\ell(v)K_{[n]})\| \leqq \varepsilon(r_N,N) .$$

Below we shall also need the following inequalities

$$(6.3)\quad \|\Psi^{(R-r+\frac{j}{n}r)}a_\ell(v)K_{[n]} - \Psi^{(R-r+\frac{j}{n}r)}a_\ell(v)K_{[n]}\Psi^{(R-r+\frac{j+1}{n}r)}\| \leqq \varepsilon(\tfrac{1}{n}r_N,N)$$

$$\|\Psi^{(R-r+\frac{j}{n}r)}K^{(n-\ell)}_{[n]} - \Psi^{(R-r+\frac{j}{n}r)}K^{(N-\ell)}_{[n]}\Psi^{(R-r+\frac{j+1}{n}r)}\| \leqq \varepsilon(\tfrac{1}{n}r_N,N)$$

where n,j are integers, $j < n$. These inequalities are verified in the same manner as (5.3) in Lemma 2.

Remark. The operator sequence $\Psi^{(R-r)}(K^{(N-\ell)} - a_\ell(v)K)$ does not converge to zero relative to the norm. Indeed, the operator $\Psi^{(R-r)}K^{(N-\ell)}$ acts from B into $B_{N-\ell}$, and $\Psi^{(R-r)}R^{N-\ell}f = 0$ for the elements $f\in B$ with $f_S = 0$ for $S \leqq N-\ell$. Therefore, for any arbitrarily large N , there is always an element f such that $\Psi^{(R-r)}K^{(N-\ell)}f = 0$ and $\Psi^{(R-r)}a_\ell(v)Kf \neq 0$, and the norm of the element $\Psi^{(R-r)}a_\ell(v)Kf$ is finite.

3. Consider now the relation (4.3). Applying it repeatedly we get

$$(7.3)\quad F^{(N-\ell)} = K^{(N-\ell)}K^{(N-\ell-1)}\dots K^{(3)}F^{(2)} + K^{(N-\ell)}K^{(N-\ell-1)}\dots K^{(4)}F^{o(3)}+\dots$$

$$+ K^{(N-\ell)}K^{(N-\ell-1)}\dots K^{(i)}F^{o(i-1)}+\dots+ K^{(N-\ell)}F^{o(N-\ell-1)} + F^{o(N-\ell)} =$$

$$= K^{(N-\ell)}K^{(N-\ell-1)}\dots K^{(3)}F^{(2)} + \sum_{i=0}^{N-\ell-4}\Big(\prod_{j=0}^{i} K^{(N-\ell-j)}\Big)F^{o(N-\ell-i-1)} + F^{o(N-\ell)} ,$$

where the operator $K^{(N-\ell-j)}$ acts after the operator $K^{(N-\ell-j-1)}$ and, according to the definition*,

*) The columns $F^{(N_1-\ell_1)}$ and $F^{(N_2-\ell_2)}$ differ from each other for $N_1 \neq N_2$, $N_1 \neq N_2$, $\ell_1 \neq \ell_2$, $N_1 - \ell_1 = N_2 - \ell_2$, by definition.

$$F^{(2)} = \begin{pmatrix} F_1^{(2)}(q_1; V_N) \\ F_2(q_1, q_2; V_N) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \qquad F_1^{(2)}(q_1; V_N) = \frac{V_N \int_{V_N} \exp\{-\frac{1}{\theta}\Phi(q_1-q_2)\} dq_2}{\int_{V_N}\int_{V_N} \exp\{-\frac{1}{\theta}\Phi(q_1-q_2)\} dq_1\, dq_2}$$

$$F_2^{(2)}(q_1, q_2; V_N) = \frac{V_N^2 \exp\{-\frac{1}{\theta}\Phi(q_1-q_2)\}}{\int_{V_N}\int_{V_N} \exp\{-\frac{1}{\theta}\Phi(q_1-q_2)\} dq_1\, dq_2}$$

We have further

$$(8.3) \qquad \|F^{(2)}\| \leq \max\left\{\frac{1}{A}\, \frac{(V_N+I_1)V_N}{V_N(V_N+I_2)},\ \frac{V_N^2\, e^{2b}}{A^2 V_N(V_N+I_2)}\right\}$$

where use has been made of the following formulas

$$\int_{V_N} \exp\{-\frac{1}{\theta}\Phi(q_1-q_2)\} dq_2 = \int_{V_N} \varphi_{q_1}(q_2) dq_2 + V_N = I_1 + V_N$$

$$(9.3) \qquad |I_1| \leq \sup_{q_1} \left| \int_{V_N} \varphi_{q_1}(q_2) dq_2 \right| \leq I \quad ,$$

$$\int_{V_N}\int_{V_N} \exp\{-\frac{1}{\theta}\Phi(q_1-q_2)\} dq_1\, dq_2 = \int_{V_N}\int_{V_N} \varphi_{q_1}(q_2) dq_1\, dq_2 + V_N^2 = V_N I_2 + V_N^2 .$$

$$|I_2| \leq I .$$

It follows from (8.3), (9.3) that for large enough N (which will be assumed in what follows) the inequality $\|F^{(2)}\| < 1$ holds.

Bearing in mind that $\|F^{(2)}\| < 1$, $\|F^{o(N-\ell)}\| < 1, \dots, \|F^{o(3)}\| < 1$ we obtain from (7.3) the following estimate for $\|F^{(N-\ell)}\|$

$$(10.3) \qquad \|F^{(N-\ell)}\| \leq \sum_{i=0}^{N-\ell-3} \prod_{j=0}^{i} \|K^{(N-\ell-j)}\| + 1 < \frac{1}{1 - \frac{2e^{2b} e^{\frac{IA}{v}}}{A}} = \frac{1}{1-k} .$$

The latter inequality implies that the columns $F^{(N-\ell)}$ belong to the Banach space B for

$$\frac{1}{v} < \frac{1}{2e^{2b+1} I} \quad ,$$

and their norms are bounded uniformly with regard to N and ℓ .

Consider the columns

(11.3) $$F^{\ell} = \sum_{i=0}^{\infty} \prod_{j=0}^{i} a_{\ell+j}(v)K^{i}F^{o},$$

which appear when iterating the relations (see (9.1))

(12.3) $$F^{\ell} = a_{\ell}(v)KF^{\ell+1} + a_{\ell}(v)F^{o} ,$$

and belong to the space B as it follows from the estimate

(13.3) $$\|F^{\ell}\| \leq \sum_{i=0}^{\infty} \prod_{j=0}^{i} a_{\ell+j}(v)\|K\|^{i}\|F^{o}\| \leq \sum_{i=0}^{\infty} k^{i} = \frac{1}{1-k}$$

$$\text{for } \frac{1}{v} < \frac{1}{2e^{2b+1}I} .$$

THEOREM III. <u>For any fixed</u> ℓ <u>and</u> $\frac{1}{v} < \frac{1}{2e^{2b+1}I}$ <u>the sequence</u>

$$\Psi^{(R-r)}(F^{(N-\ell)} - F^{\ell})$$

<u>tends to zero in the space</u> B <u>as</u> $N \to \infty$.

<u>Proof.</u> Let us represent the difference $\Psi^{(R-r)}(F^{(N-\ell)} - F^{\ell})$ in the form

(14.3) $$\Psi^{(R-r)}(F^{(N-\ell)}-F^{\ell}) = \Psi^{(R-r)}[K^{(N-\ell)}K^{(N-\ell-1)}\ldots K^{(3)}K^{(2)} +$$
$$+ \sum_{i=0}^{N-\ell-4} \prod_{j=0}^{i} K^{(N-\ell-j)}F^{o(N-\ell-i-1)} + F^{o(N-\ell)} - \sum_{i=0}^{\infty} \prod_{j=0}^{i} a_{\ell+j}(v)K^{i}F^{o}] =$$
$$= \Psi^{(R-r)}[\sum_{i=0}^{n} \prod_{j=0}^{i} K^{(N-\ell-j)}F^{o(N-\ell-i-1)} + F^{o(N-\ell)} - \sum_{i=0}^{n+1} \prod_{j=0}^{i} a_{\ell+j}(v)K^{i}F^{o}] + \eta(n) .$$

The norm of the column $\eta(n)$ does not exceed $2\frac{k^{n}}{1-k}$ and can be made arbitrarily small for large enough n. By virtue of the definitions of the operators $K^{(N-\ell)}$, K and the columns $F^{o(N-\ell)}$, F^{o}, one has

$$\sum_{i=0}^{n} \prod_{j=0}^{i} K^{(N-\ell-j)}F^{o(N-\ell-i-1)} + F^{o(N-\ell)} = \sum_{i=0}^{n} \prod_{j=0}^{i} K^{(N-\ell-j)}_{[n+2]}F^{o(N-\ell-i-1)} + F^{o(N-\ell)},$$

(15.3) $$\sum_{i=0}^{n+1} \prod_{j=0}^{i} a_{\ell+j}(v)K^{i}F^{o} = \sum_{i=0}^{n+1} \prod_{j=0}^{i} a_{\ell+j}(v)\, K^{i}_{[n+2]}\, F^{o} .$$

The inequalities (5.3), (6.3) imply

$$\|\Psi^{(R-r)}\prod_{j=0}^{i}K_{[n+2]}^{(N-\ell-j)}-\Psi^{(-r)}K_{[n+2}^{(N-\ell)}\Psi^{(R-r+\frac{1}{n}r)}K_{[n+2]}^{(N-\ell-1)}\ldots\Psi^{(R+r+\frac{i-1}{n}r)}\times$$

$$\times K_{[n+2]}^{(N-\ell-i+1)}\|\leqq(i-1)k^{i-1}\varepsilon(\tfrac{1}{n}r_N,N),$$

$$\|\Psi^{(R-r)}\prod_{j=0}^{i}a_{\ell+j}(v)K_{[n+2]}^{i}-\Psi^{(R-r)}a_{\ell}(v)K_{[n+2]}\Psi^{(R-r+\frac{1}{n}r)}\times$$

(16.3)

$$\times a_{\ell+1}(v)K_{[n+2]}\ldots\Psi^{(R-r+\frac{i-1}{n}r)}a_{\ell+i-1}(v)K_{[n+2]}\|\leqq(i-1)k^{i-1}\varepsilon(\tfrac{1}{n}r_N,N),$$

$$\|\Psi^{(R-r)}K_{[n+2]}^{(N-\ell)}\Psi^{(R-r+\frac{1}{n}r)}K_{[n+2]}^{(N-\ell-1)}\ldots\Psi^{(R-r+\frac{i-1}{n}r)}K_{[n+2]}^{(N-\ell-i+1)}-$$

$$-\Psi^{(R-r)}a_{\ell}(v)K_{[n+2]}\Psi^{(R-r+\frac{1}{n}r)}a_{\ell+1}(v)K_{[n+2]}\ldots\Psi^{(R-r+\frac{i-1}{n}r)}\times a_{\ell+i-1}(v)K_{[n+2]}\|$$

$$\leqq ik^{i-1}\varepsilon(\tfrac{1}{n}r_N,N)$$

Hence

$$\|\Psi^{(R-r)}(\prod_{j=0}^{i}a_{\ell+j}(v)K_{[n+2]}^{i}-\prod_{j=0}^{i-1}K_{[n+2]}^{(N-\ell-j)}\|\leqq 3ik^{i-1}\varepsilon(\tfrac{1}{n}r_N,N)\,. \tag{17.3}$$

It is easily verified that

$$\|F^{o(N-\ell-j)}-a_{\ell+j}(v)F^{o}\|\leqq\varepsilon(\tfrac{1}{n}r_N,N)\,. \tag{18.3}$$

Using the above inequalities we obtain

$$\|\Psi^{(R-r)}[\sum_{i=0}^{n}\prod_{j=0}^{i}K_{[n+2]}^{(N-\ell-j)}F^{o(N-\ell-i-1)}+F^{o(N-\ell)}-\sum_{i=0}^{n+1}\prod_{j=0}^{i}a_{\ell+j}(v)K_{[n+2]}^{i}F^{o}\|\leqq$$

(19.3)

$$\leqq 3\varepsilon(\tfrac{1}{n}r_N,N)\sum_{i=0}^{n}(i+1)k^{i}+\varepsilon(\tfrac{1}{n}r_N,N)\sum_{i=0}^{n+1}k^{i}\leqq 3\varepsilon(\tfrac{1}{n}r_N,N)\frac{1}{(1-k)^2}+\varepsilon(\tfrac{1}{n}r_N,N)\frac{1}{1-k}\leqq$$

$$\leqq 4\varepsilon(\tfrac{1}{n}r_N,N)\frac{1}{(1-k)^2}\,.$$

The final estimate is given by

$$\|\Psi^{(R-r)}(F^{(N-\ell)}-F^{\ell})\|\leqq 4\varepsilon(\tfrac{1}{n}r_N,N)\frac{1}{(1-k)^2}+2\frac{k^n}{1-k}=\tilde{\varepsilon}(r_N,N)\,. \tag{20.3}$$

Choosing n and N sufficiently large, one can make the right hand side of the inequality (20.3) arbitrarily small. This means that

$$\lim_{N\to\infty} \|\Psi^{(R-r)}(F^{(N-\ell)} - F^{\ell})\| = 0 .$$

Theorem is proved.

As a consequence of Theorem III we obtain that

$$\lim_{N\to\infty} \|\Psi^{(R-r)}(K^{(N-\ell)}F^{(N-\ell-1)} + F^{o(N-\ell)} - a_{\ell}(v)KF^{\ell+1} + a_{\ell}(v)F^{o})\| = 0 .$$

4. Uniqueness of the limiting distribution functions

1. In this Section we shall prove the uniqueness of the limiting distribution functions, i.e., we shall prove that $F^{\ell} = F$ and

$$a_{\ell}(v) = a(v) , \qquad \ell = 1,2,... \tag{1.4}$$

THEOREM IV. For small enough $\frac{1}{v}$, the limiting distribution functions are identical and the relations (1.4) are valid.

Proof. According to (12.3) we have

$$F^{\ell} = a_{\ell}(v)KF^{\ell+1} + a_{\ell}(v)F^{o} ,$$

$$F^{\ell+1} = a_{\ell+1}(v)KF^{\ell+2} + a_{\ell+1}(v)F^{o} . \tag{2.4}$$

Consider the difference

$$F^{\ell}-F^{\ell+1} = a_{\ell}(v)K(F^{\ell+1}-F^{\ell+2}) - (a_{\ell+1}(v)-a_{\ell}(v))KF^{\ell+2}+(a_{\ell}(v)-a_{\ell+1}(v))F^{o} . \tag{3.4}$$

It follows from (3.4) that

$$\|F^{\ell}-F^{\ell+1}\| \leq a_{\ell}(v)\|K\|\|F^{\ell+1}-F^{\ell+2}\| + |a_{\ell+1}(v)-a_{\ell}(v)|\,\|K\|\|F^{\ell+2}\| + |a_{\ell+1}(v)-a_{\ell}(v)|\,\|F^{o}\| \tag{4.4}$$

$$\leq \frac{a_{\ell}(v)}{A} e^{2b}e^{\frac{A_I}{v}}\|F^{\ell+1}-F^{\ell+2}\| + |a_{\ell+1}(v)-a_{\ell}(v)|\frac{1}{A}e^{2b}e^{\frac{A_I}{v}}\|F^{\ell+2}\| + \frac{|a_{\ell+1}(v)-a_{\ell}(v)|}{A} .$$

Below we shall establish the following estimate :

$$|a_{\ell}(v)-a_{\ell+1}(v)| \leq a_{\ell}(v)a_{\ell+1}(v)e^{\frac{A_I}{v}}\|F^{\ell+1}-F^{\ell+2}\| . \tag{5.4}$$

Bearing in mind (5.4) one gets from (4.4)

$$(6.4)\qquad \|F^{\ell}-F^{\ell+1}\| \leqq \Big[\frac{a_{\ell}(v)}{A}\, e^{2b} e^{\frac{A}{v}I} + a_{\ell}(v)a_{\ell+1}(v) e^{2\frac{A}{v}I}\frac{1}{A}\, e^{2b}\|F^{\ell+2}\| + $$

$$+ \frac{a_{\ell}(v)a_{\ell+1}(v)}{A}\, e^{\frac{A}{v}I}\Big]\|F^{\ell+1}-F^{\ell+2}\| = \delta(v)\|F^{\ell+1}-F^{\ell+2}\| .$$

According to Lemma 1, the quantities $a_{\ell}(v)$ obey the inequality

$$a_{\ell}(v) < 2 \qquad \text{for} \qquad \frac{1}{v} < \frac{1}{2I} .$$

It follows from (13.3) that for the norm of $F^{\ell+2}$ the following estimate holds

$$(7.4)\qquad \|F^{\ell+2}\| \leqq \frac{1}{1-\frac{2e^{2b}e^{\frac{A}{v}I}}{A}} , \qquad A > 2e^{2b+1} \quad \text{for} \quad \frac{1}{v} < \frac{1}{2Ie^{2b+1}} .$$

Choosing A sufficiently large, one can easily show that for small enough $\frac{1}{v}$ the quantity $\delta(v)$ in (6.4) can be made less than unity. The inequality (6.4) yields

$$(8.4)\qquad \|F^{\ell}-F^{\ell+1}\| \leqq \delta^{i}(v)\|F^{\ell+i} - F^{\ell+i+1}\| \leqq \delta^{i}(v)\, \frac{2}{1-\frac{2e^{2b}e^{\frac{A}{v}I}}{A}} .$$

Since $\delta(v) < 1$ and the inequality (8.4) is valid for any i, the norm $\|F^{\ell}-F^{\ell+1}\|$ is arbitrarily small, that is, $F^{\ell} = F^{\ell+1}$ and, in general, $F = F^{1} = F^{2} = \dots = F^{\ell+1} = \dots$.

Now the inequalities (5.4) imply that for small enough $\frac{1}{v}$ the quantities $a_{\ell}(v)$ and $a_{\ell+1}(v)$ are identical, and, in general, $a(v) = a_1(v) = a_2(v)\dots$.

Theorem is proved.

It follows from Theorem IV that the column F satisfies the Kirkwood-Salsburg equation

$$F = a(v)KF + a(v)F^{0} .$$

2. The functions $F_S^{(N-\ell)}(q_1,\dots,q_S;V_N)$ satisfy the relations

$$(9.4)\qquad \frac{1}{V_N^S}\int_{V_N}\dots\int_{V_N} F_S^{(N-\ell)}(q_1,\dots,q_S;V_N)dq_1\dots dq_S = 1 .$$

Let us take the limit in (9.4) and show that the following relation holds

$$(10.4)\quad \lim_{N\to\infty}\frac{1}{V_N^S}\int_{V_N}\dots\int_{V_N} F_S^{\ell}(q_1,\dots,q_S,v)dq_1\dots dq_S =$$

$$= \lim_{N\to\infty}\frac{1}{V_N^S}\int_{V_N}\dots\int_{V_N} F_S^{(N-\ell)}(q_1,\dots,q_S;V_N) \times dq_1\dots dq_S .$$

To this end, consider the identity

$$(11.4)\quad \frac{1}{V_N^S}\int_{V_N}\cdots\int_{V_N}(F_S^{\ell}(q_1,\dots,q_S,v) - F_S^{(N-\ell)}(q_1,\dots,q_S,V_N))dq_1\dots dq_S =$$

$$= \frac{1}{V_N^S}\int_{V_N}\cdots\int_{V_N}[\Psi^{(R-r)}(F_S^{\ell}(q_1,\dots,q_S;v) - F_S^{(N-\ell)}(q_1,\dots,q_S;V_N) +$$

$$+ (1-\Psi^{(R-r)})(F_S^{\ell}(q_1,\dots,q_S;v) - F_S^{(N-\ell)}(q_1,\dots,q_S;V_N))]dq_1\dots dq_S$$

and evaluate the first and the second terms of the right-hand side of (11.4). The second term obeys the following estimate

$$\left|\frac{1}{V_N^S}\int_{V_N}\cdots\int_{V_N}(1-\Psi^{(R-r)})(F_S^{\ell}(q_1,\dots,q_S;v) - F_S^{(N-\ell)}(q_1,\dots,q_S;V_N))dq_1\dots dq_S\right| \leqq$$

$$\leqq \frac{2}{1-k}A^S S\frac{R_N^3-(R_N-r_N)^3}{R_N^3} = \frac{2}{1-k}A^S S[1-(1-\frac{r_N}{R_N})^3] .$$

The inequality above implies that the second term of (11.4) tends to zero as $N\to\infty$.

The first term of (11.4) obeys the following estimate

$$\left|\frac{1}{V_N^S}\int_{V_N}\cdots\int_{V_N}\Psi^{(R-r)}(F_S^{\ell}(q_1,\dots,q_S;v) - F_S^{(N-\ell)}(q_1,\dots,q_S;V_N))dq_1\dots dq_S\right|\leqq$$

$$\leqq \frac{(R_N-r_N)^{3S}}{R_N^{3S}}A^S\tilde{\varepsilon}(r_N,N) = (1-\frac{r_N}{R_N})^{3S}A^S\tilde{\varepsilon}(r_N,N);\ \lim_{N\to\infty}\tilde{\varepsilon}(r_N,N) = 0 .$$

The relation (10.4) is thus proved. As it is easily seen, the functions $F_S^{\ell}(q_1,\dots,q_S;v)$ are translation-invariant and therefore the functions $F_1^{\ell}(q_1;v)$ are constant. The relation (10.4) implies that $F_1^{\ell}(q_1;v) = 1$, $\ell \geqq 0$.

Proceed now to proving the inequality (5.4). From (12.3) it follows that

$$\frac{1}{a_\ell(v)} = 1 + \sum_{K=1}^{\infty}\frac{1}{K!v^K}\int\cdots\int F_K^{\ell+1}(q_1^*,\dots,q_K^*;v)\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^* .$$

Using also the formula

$$a_\ell(v) - a_{\ell+1}(v) = a_\ell(v)a_{\ell+1}(v)(\frac{1}{a_{\ell+1}(v)} - \frac{1}{a_\ell(v)})=$$

$$= a_\ell(v)a_{\ell+1}(v)\sum_{K=1}^{\infty}\frac{1}{K!v^K}\int\dots\int [F_K^{\ell+2}(q_1^*,\dots,q_K^*;v) - F_K^{\ell+1}(q_1^*,\dots,q_K^*;v)\times$$

$$\times\prod_{i=1}^{K}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_K^* ,$$

we obtain the required estimate (5.4) :

$$|a_{\ell}(v)-a_{\ell+1}(v)| \leqq a_{\ell}(v)a_{\ell+1}(v)e^{\frac{IA}{v}}\|F^{\ell+1} - F^{\ell+2}\| \quad .$$

3. We shall show now that the sequence $a_N(V_N)$ has the unique limiting point. Indeed, if there were two subsequences converging to $a(v)$ and $a^1(v)$ respectively, then corresponding to them there would be two columns F and F' obeying, by Theorem IV, the equations

$$F = a(v)KF + a(v)F^0 \ ,$$

$$F^1 = a^1(v)KF^1 + a^1(v)F^0 \ .$$

Bearing in mind that $F_1(q;v) = 1$ and $F_1^1(q;v) = 1$, we obtain for the difference $a(v) - a^1(v)$ the following estimate

(12.4) $$|a(v) - a^1(v)| \leqq a(v)a^1(v)e^{\frac{IA}{v}}\|F - F^1\| \quad .$$

This, in the same way as in Theorem IV, implies the validity of the inequality

(13.4) $$\|F - F^1\| \leqq \delta(v)\|F - F^1\|$$

where $\delta(v) < 1$ for sufficiently small $\frac{1}{v}$. It follows from (13.4) that $F = F^1$ and from (15.4) that $a(v) = a^1(v)$.

Bearing this in mind, we finally obtain from (10.4)

$$\lim_{N\to\infty} \frac{1}{V_N^S}\int_{V_N}\cdots\int_{V_N} F_S^{(N-\ell)}(q_1,\dots,q_S;V_N)dq_1\dots dq_S =$$

$$= \lim_{N\to\infty} \frac{1}{V_N^S}\int_{V_N}\cdots\int_{V_N} F_S(q_1,\dots,q_S;v)dq_1\dots dq_S \quad .$$

The uniqueness of the limiting distribution functions being taken into account, Theorem III (Section 3) can be formulated as follows.

For any fixed ℓ and $\frac{1}{v} < \frac{1}{2\ell}$ the sequence

$$V^{(R-r)}[F_S^{(N-\ell)}(q_1,\dots,q_S)V_N - F_S(q_1,\dots,q_S;v)]$$

tends to zero as $N\to\infty$ uniformly with regard to $q_1,\dots,q_S$. Observe that the limiting distribution functions are holomorphic functions of $\frac{1}{v}$ in some neighbourhood of the point $\frac{1}{v} = 0$.

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