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Holomorphic semi-groups

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HOLOMORPHIC SEMI-GROUPS

By Kôsaku Yosida (University of Tokyo)

§ 1. <u>The Theorem</u>. Let X be a locally convex, sequentially complete, linear topological space, and let L(X,X) be the set of all linear continuous mappings defined on X with values in X .

A system $\{T_t \,;\, t \geqslant 0\}$ of mappings $T_t \in L(X,X)$ is called an equi-continuous semi-group of class (C_0) if :

(1) $T_t T_s = T_{t+s}$, $T_0 = I$ (the identity),

(2) $\lim\limits_{t \to t_0} T_t x = T_{t_0} x$ for every $t_0 \geqslant 0$ and $x \in X$,

(3) for every continuous semi-norm p in X, there exists a continuous semi-norm q in X such that

$$p(T_t x) \leqslant q(x) \text{ for all } t \geqslant 0 \text{ and } x \in X$$

We shall prove the following

<u>Theorem</u> . The three propositions (I), (II) and (III) given below are mutually equivalent :

(I) For every $t > 0$ and $x \in X$, $T_t' x = \lim\limits_{h \to 0} \dfrac{T_{t+h} - T_t}{h} x$ exists, and, for a suitable positive constant C , the system of mappings

$$\{(C\, t\, T_t')^n \,;\, 1 \geqslant t > 0,\ n = 0, 1, 2, \ldots\}$$

is equi-continuous.

(II) The Taylor expansion

$$T_\lambda x = \sum_{n=0}^{\infty} \frac{|\lambda - t|^n}{\underline{|n}} T_t^{(n)} x$$

converges for every $x \in X$ and every complex number λ with $|\arg \lambda| < \tan^{-1}(Ce^{-1})$, in such a way that the system of mappings $\left\{ e^{-\lambda} T_\lambda ; |\arg \lambda| < \tan^{-1} \left(\frac{Ce^{-1}}{2^k}\right)\right\}$ is, for a certain $k > 0$, equi-continuous.

(III) The infinitesimal generator A of T_t defined by

$$Ax = \lim_{t \downarrow 0} \frac{T_t - I}{t} x$$

satisfies the condition that the resolvent $(\lambda I - A)^{-1}$ exists as a mapping $\in L(X, X)$ for $\mathrm{Re}(\lambda) > 0$ and the system of mappings

$$\left\{ [C_1 \lambda(\lambda I - A)^{-1}]^n ; \mathrm{Re}(\lambda) \geqslant 1, \quad n = 0, 1, 2, \ldots \right\}$$

is equi-continuous for a certain positive constant C_1.

§ 2. <u>The Sketch of the Proof of the Theorem</u>. We first recall known facts concerning equi-continuous semi-groups T_t of class (C_0) :

The domain $D(A)$ of A is dense in X ; for every λ with $\mathrm{Re}(\lambda) > 0$, the resolvent $(\lambda I - A)^{-1}$ exists and $\in L(X,X)$; $(\lambda I - A)^{-1} x = \int_0^\infty e^{-\lambda t} T_t x dt$ for every $x \in X$ and λ with $\mathrm{Re}(\lambda) > 0$; the system of operators

$$\left\{ [\sigma(\sigma I - A)^{-1}]^m ; \sigma > 0, \quad m = 0, 1, 2, \ldots \right\}$$

is equi-continuous ; for every $x \in D(A)$, we have

$$\text{(4)} \qquad \frac{dT_t x}{dt} = AT_t x = T_t Ax , \qquad t \geqslant 0 ,$$

concerning $T_t^{(n)} = (T_t^{(n-1)})'$, we have the

<u>Lemma</u>. Let $T_t x \in D(A)$ for every $t > 0$ and $x \in X$. Then $T_t x$ is infinitely differentiable in t and

(5) $$T_t^{(n)} = (T'_{t/n})^n .$$

Proof. For any $t_0 > 0$ with $t > t_0$, we have, by (1) and (4),

$$T'_t x = AT_t x = T_{t-t_0} AT_{t_0} x .$$

Hence

$$T''_t x = T'_{t-t_0} AT_{t_0} x = AT_{t-t_0} AT_{t_0} x = AT_{t/2} AT_{t/2} x = (T'_{t/2})^2 x .$$

We have only to repeat the same reasoning to obtain (5).

Proof of the Theorem

(I) implies (II). By the equi-continuity of $\left\{[CtT'_t]^n \ ; \ n \geqslant 0, \ 1 \geqslant t > 0\right\}$, we obtain, remembering (5),

$$p\left(\frac{|\lambda-t|^n}{\lfloor n} T_t^{(n)} x\right) \leqslant \frac{|\lambda-t|^n}{t^n} \frac{n^n}{\lfloor n} \frac{1}{C^n} p\left(\left(\frac{t}{n} CT'_{t/n}\right)^n x\right) \leqslant \left(\frac{|\lambda-t|}{t} C^{-1} e\right)^n q(x) .$$

Hence the first part of (II) is proved.

Next consider the semi-group $\{S_t\}$ defined by

$$S_t = e^{-t} T_t .$$

Then

$$tS'_t = te^{-t} T'_t - te^{-t} T_t .$$

Thus remembering that $0 \leqslant te^{-t} \leqslant 1$ for $t \geqslant 0$, we easily see that

$$\left\{(2^{-k} tS'_t)^n \ ; \ t > 0, \ n = 0,1,2,\ldots\right\}$$

is with a certain $k > 0$, equi-continuous. Hence, as above, we see that $e^{-\lambda} T_\lambda$, which is an holomorphic extension of S_t, satisfies the condition that $\left\{e^{-\lambda} T_\lambda \ ; |\arg\lambda| < \tan^{-1}\left(\frac{Ce^{-1}}{2^k}\right)\right\}$ is equi-continuous.

(II) implies (III). Differentiating $(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} T_t dt$ with respect to λ, we obtain, for $\lambda = \sigma + 1 + i\tau$ with $\sigma \geqslant 0$,

$$[(\sigma + 1 + i\tau)(\sigma + 1 + i\tau)I - A)^{-1}]^{n+1} x$$

$$= \frac{(\sigma+1+i\tau)^{n+1}}{\lfloor n} \int_0^\infty e^{-(\sigma+1+i\tau)} t^n S_t x dt , \quad x \in X .$$

Let $\tau < 0$. Then, by Cauchy's integral theorem, we obtain, for $0 < \theta < \tan^{-1}(\frac{Ce^{-1}}{2})$,

$$\left[(\sigma+1+i\tau)((\sigma+1+i\tau)I-A)^{-1}\right]^{n+1}x = \frac{(\sigma+1+i\tau)^{n+1}}{\lfloor n} \int_0^\infty e^{-(\sigma+i\tau)re^{i\theta}} r^n S_{re^{i\theta}} x . e^{i\theta} dr .$$

Hence, by the equi-continuity of $\{S_{re^{i\theta}} ; r \geqslant 0\}$; we have

$$p\left(\left[(\sigma+1+i\tau)(\sigma+1+i\tau)I-A)^{-1}\right]^{n+1}x\right)$$

$$\leqslant q(x) . \frac{|\sigma+1+i\tau|^{n+1}}{\lfloor n} \int_0^\infty e^{(-\sigma\cos\theta+\tau\sin\theta)r} r^n dr$$

$$= q(x). \frac{|\sigma+1+i\tau|^{n+1}}{|\tau\sin\theta-\sigma\cos\theta|^{n+1}}$$

since $0 < \theta < \frac{\pi}{2}$, $\tau < 0$ and $\sigma \geqslant 0$, we easily see that the second factor on the right is $\leqslant C_1^n$.

We also obtain similar estimate for the case $\tau > 0$.

Thus (II) implies (III).

(III) implies (I). We have, from

$$p\left(\left[C_1\lambda_0(\lambda_0 I-A)^{-1}\right]^n x\right) \leqslant q(x) \text{ with } \mathrm{Re}(\lambda_0) \geqslant 1 ,$$

the inequality

$$p\left(\left[(\lambda-\lambda_0)(\lambda_0 I-A)^{-1}\right]^n x\right) \leqslant \frac{|\lambda-\lambda_0|^n}{C_1^n |\lambda_0|^n} q(x)$$

Thus, if $\mathrm{Re}(\lambda_0) \geqslant 1$ and $\frac{|\lambda-\lambda_0|}{C_1|\lambda_0|} < 1$, the series

$$\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I-A)^{-(n+1)}x$$

converges and represents the resolvent $(\lambda_0 \, I-A)^{-1}$ in such a way that

$$(6) \qquad p((\lambda I-A)^{-1}x) \leq \left(1-\frac{|\lambda - \lambda_0|}{C_1 \, |\lambda_0|}\right)^{-1} q(x) = 0 \quad \frac{1}{|\lambda|}$$

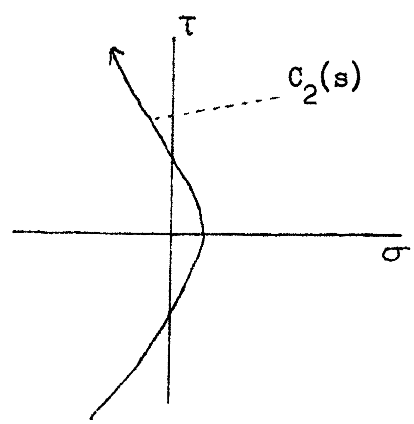
when $|\lambda| \to \infty$ in the domain of the complex λ-plane which lies on the right of a oriented path (see the figure)

$C_2(s) = \sigma(s)+i \, \tau(s) \qquad (-\infty < s < \infty)$

such that

$\lim_{s\uparrow\infty} \tau(s) = \infty$, $\lim_{s\downarrow -\infty} \tau(s) = -\infty$,

$\lim_{s\uparrow\infty} \frac{\sigma(s)}{\tau(s)} < 0$, $\lim_{s\downarrow -\infty} \sigma(s)/\tau(s) > 0$



We thus can define, for $t > 0$,

$$(7) \quad \hat{T}_t x = \frac{1}{2\pi i}\int_{C_2(s)} e^{\lambda t} (\lambda \, I-A)^{-1}x \, d\lambda$$

If we are able to show that

$$(8) \quad \hat{T}_t = T_t \quad ,$$

then, by

$$T_t^{(n)}x = \frac{1}{2\pi i}\int_{C_2(s)} e^{\lambda t} \lambda^n (\lambda \, I-A)^{-1}x d\lambda = (T'_{t/n})^n x,$$

we obtain

$$(tT'_t)^n x = \frac{1}{2\pi i}\int_{C_2(s)} e^{\lambda t} (t\lambda)^n (\lambda \, I-A)^{-1} x d\lambda$$

which implies (I).

We shall prove (8). To this purpose, we first prove (9), (10) and (11) :

(9) $\lim_{t \downarrow o} \hat{T}_t x_o = x_o$ for every $x_o \in D(A)$,

(10) $\hat{T}_t' x = A\hat{T}_t x$ for every $x \in X$ and $t > 0$,

(11) $\hat{T}_t x$ is of exponential growth when $t \uparrow \infty$,

(11) is clear from (7). (10) is proved from

$$\hat{T}_t' x - A\hat{T}_t x = \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} \left\{ \lambda(\lambda I-A)^{-1}x - A(\lambda I-A)^{-1}x \right\} d\lambda$$

$$= \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} dt$$

by shifting the path of integration $C_2(s)$ to the left.

To prove (9), we take a λ_o with Re $(\lambda_o) > 0$ on the right of the path $C_2(s)$ and take $y_o \in X$ such that $x_o = (\lambda_o I-A)^{-1} y_o$.

Then

$$\hat{T}_t x_o = \hat{T}_t (\lambda_o I-A)^{-1} y_o = \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} (\lambda I-A)^{-1} (\lambda_o I-A)^{-1} y_o d\lambda$$

$$= \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} \frac{1}{\lambda_o - \lambda} (\lambda I-A)^{-1} y_o d\lambda$$

$$- \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} \frac{1}{\lambda_o - \lambda} (\lambda_o I-A)^{-1} y_o d\lambda$$

The second integral on the right is o as may be seen by shifting the path of integration C_2 (s) to the left. Hence, by (6),

$$\lim_{t \downarrow o} \hat{T}_t x_o = \frac{1}{2\pi i} \int_{C_2(s)} \frac{1}{\lambda_o - \lambda} (\lambda I-A)^{-1} y_o d\lambda$$

$$= (\lambda_o I-A)^{-1} y_o \quad \text{(the residue at } \lambda = \lambda_o\text{)}.$$

We are now ready to prove (8). Put

$$y_t = \hat{T}_t x_0 - T_t x_0 \quad .$$

Then $\lim_{t \downarrow 0} y_t = 0$, $y'_t = Ay_t \ (t > 0)$ and y_t is of exponential growth as $t \uparrow \infty$. Hence, for sufficiently large $\mathrm{Re}(\lambda)$, we obtain

$$A \int_0^\infty e^{-\lambda t} y_t dt = \int_0^\infty e^{-\lambda t} Ay_t dt = \int_0^\infty e^{-\lambda t} y'_t dt$$

$$= \lambda \int_0^\infty e^{-\lambda t} y_t dt$$

by partial integration. But, since every λ with $\mathrm{Re}(\lambda) > 0$ is in the resolvent set of A, we must have

$$\int_0^\infty e^{-\lambda t} y_t dt = 0 \quad \text{for all } \lambda \text{ if } \mathrm{Re}(\lambda) \text{ is sufficiently large.}$$

This proves that $y_t = 0$, i.e., $\hat{T}_t x_0 = T_t x_0$ for every $x_0 \in D(A)$. As $D(A)$ is dense in X, we obtain $\hat{T}_t = T_t$.

<u>Remark</u>. In the case when X is a Banach space, the equivalence of (II) and (III) is proved by E. Hille and R. S. Phillips [1]. The condition (I) was observed by K. Yosida [1], [2]. The theorem given in the present note is adapted from K. Yosida [3].

In the case when X is a Banach space, we can construct, from any equi-continuous semi-group T_t of class (C_0), a holomorphic semi-group $\tilde{T}_{t,\alpha} = \tilde{T}_t$ as follows K. Yosida [4], V. Balakrisnan [5] and T. Kato [6] ; Consider

$$(12) \qquad f_{t,\alpha}(\lambda) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{z\lambda - z^{\alpha} t} \, dz$$

Where $\sigma > 0$, $t > 0$, $\lambda \geq 0$ and $0 < \alpha < 1$; we here take the branch of the function Z^{α} in such a way that

$\mathrm{Re}(Z^{\alpha}) > 0$ for $\mathrm{Re}(Z) > 0$. Then

$$(13) \quad \tilde{T}_{t,\alpha} x = \tilde{T}_t x = \int_0^{\infty} f_{t,\alpha}(s) \, T_s x ds.$$

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