Séminaire Jean Leray.
Sur les équations aux dérivées partielles

Fritz John

Error estimates in the derivation of the non-linear equations for elastic plates

Séminaire Jean Leray (1962-1963), p. 169-172
<http://www.numdam.org/item?id=SJL_1962-1963____169_0>

ERROR ESTIMATES IN THE DERIVATION OF THE NON-LINEAR EQUATIONS FOR ELASTIC PLATES

by

Fritz JOHN

We consider the simplest mathematical model for an isotropic, homogenous, elastic body permitting finite deformations. Let x_i be the coordinates of a particle in the unstrained state, $\bar{x}_i$ those in the strained state. Let $d\bar{s}^2 = g_{ik}\,dx_i\,dx_k$ be the line element in the strained state. The internal energy per unit volume is taken to be

$$W = \frac{E}{8(1+\sigma)}\left[(g_{ik}-\delta_{ik})(g_{ik}-\delta_{ik}) + \frac{\sigma}{1-2\sigma}(g_{ii}-\delta_{ii})(g_{kk}-\delta_{kk})\right.$$

(E = Young's modulus, σ = Poisson's ratio, $0 < \sigma < \frac{1}{2}$).

We make W dimensionless by choosing units of force for which $E/2(1+\sigma) = 1$. The stress matrix (t_{ik}) is defined by

$$t_{ik} = 2\frac{\partial W}{\partial g_{ik}} = g_{ik} - \delta_{ik} + \frac{\sigma}{1-2\sigma}(g_{jj}-\delta_{jj})\,\delta_{ik} \quad .$$

These are the stress-strain relations. The equations of equilibrium in the absence of body forces are then

$$(t_{ik} + P_{ik})_{,k} = 0$$

where

$$P_{ik} = (g_{ir} - \delta_{ir})t_{rk} - W\delta_{ik} \; .$$

(Here $f_{,k}$ stands for $\partial f/\partial x_k$). In addition we have 6 compatibility equations for the g_{ik} corresponding to the fact that the Riemann curvature tensor of the g_{ik} vanishes. Expressing everywhere the g_{ik} in terms of the t_{rs} we have a system of 6 second order and 3 first order equations for the t_{ik}, which are, of course, not independent from each other. Additional conditions consist in either prescribing the displacements or the stresses on the boundary.

Existence and uniqueness of solutions has been established for "sufficiently small" boundary data. (For references, in particular to the work of F. Stopelli, see G. Grioli : Mathematical Theory of Elastic Equilibrium, Ergebnisse der Angewandten Mathematik, 1962).

Of particular interest is the problem of obtaining from the full 3-dimensional theory contained in the equations above two-dimensional equations for thin plates and shells. A rigorous derivation encounters the difficulty that the inequalities used in establishing existence and uniqueness of solutions are likely to become invalid in the case of thin objects. This is indicated by the possible occurence of buckling and of boundary layers. Usually equations for plates and shells are derived from asymptotic expansions or from ad hoc assumptions on the relative orders of magnitude of the various stress components.

A different approach will be indicated here for the case of a thin plate, leading to the classical equations of v. Karman and Föppl. No forces shall act on the plane parallel faces $x_3 = \pm h$ of the plate, corresponding to the boundary conditions $t_{13} = t_{23} = t_{33} = 0$. Instead of describing the solution completely in terms of data on the lateral faces of the plate we start with an <u>a priori</u> inequality $|t_{ik}| \leqq \varepsilon$ for the maximum of all stress components in the plate. (The difficult problem how such an estimate can be deduced from prescribed boundary data on the lateral faces of the plate is not considered). The aim is to get "concrete" estimates for the error terms in the v. Karman-Föppl equations in terms of ε , h and of the distance a of the points considered from the lateral boundary of the plate, without having to pass to the limit

$h \to 0$. This is achieved by estimating all derivatives of the t_{ik} in terms of ε, h, a on the basis of the 3-dimensional differential equations which form an elliptic system. The main tool here are inequalities of the type of Gårding and Sobolev. (For details see the speaker's report : estimates for the error in the equations of non-linear plate theory, New York University, Courant Institute of Mathematical Sciences, IMM-NYU 308, 1963). The non-linearity of the differen equations forces us to restrict ourselves to cases where $\varepsilon a^2 h^{-2}$ lies below a certain universal constant ε_0 . The main result of the estimates is that any n-th derivative of the t_{ik} does not exceed the value $K\varepsilon/a^{n-1}h$, where K only depends on n and σ . An n-th derivative with respect to x_1, x_2 alone can even be estimated by $K\varepsilon/a^n$. As a consequence one can justify some of the statements usually assumed without proof. It turns out, for example, that some of the stress components are automatically of higher order :

$$t_{33} = O(\varepsilon h^2/a^2), \quad t_{3\alpha} = O(\varepsilon h/a) \quad \text{for } \alpha = 1,2 \quad .$$

The Karman-Föppl equations introduce the quantities

$$\tau_{\alpha\beta} = \frac{1}{2h} \int_{-h}^{h} (t_{\alpha\beta} + P_{\alpha\beta})dx_3 \quad , \quad (\alpha, \beta = 1,2)$$

and $w = \bar{x}_3(x_1, x_2, 0)$. The $\tau_{\alpha\beta}$ can be expressed as second derivatives of an "Airy-function" $\phi(x_1, x_2)$. From the differential equations and estimates for the t_{ik} one finds after a suitable rigid motion of the deformed body the relations

$$\phi_{\alpha\alpha,\beta\beta} = 2(1+\sigma)\,(w_{,12}w_{,12} - w_{,11}w_{,22}) + R_1$$

$$hw_{\alpha\alpha,\beta\beta} = \frac{3}{2}(1-\sigma)h^{-1}(w_{,11}\phi_{,22} - 2w_{,12}\phi_{,12} + w_{,22}\phi_{,11}) + R_2$$

$$(\alpha, \beta = 1,2) \quad ,$$

where $|R_\alpha| \leqq Kh^2/a^4$, and K depends only on σ. These are the v. Karman-Föppl equations with universal estimates for the error terms. One convinces oneself that the quantities R_α are generally smaller than the other terms occuring in the equations. The validity of the equations is restricted to a portion of the plate of size $O(h\varepsilon^{-1/2})$; for larger portions one may not be able to get along with a fixed rigid motion.