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A THEOREM ON CARTAN PSEUDOGROUPS

par A. KUMPERA

The purpose of this note is to give a proof of the well-known third fundamental theorem in the theory of continuous (involutive) pseudogroups for the non transitive case. E. Cartan gave two proofs of the theorem in the transitive case ([1],[2]). For the sake of brevity we assume all standard facts about exterior differential systems and «derived spaces». References are given in each particular case and we follow strictly the definitions and notations given in the references. We also omit all straightforward computations though in some cases they are not very short. Throughout the paper all manifolds, maps, forms, etc. are assumed real analytic.

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Let M and N be domains in real euclidean space and (M, N, π) a fibered manifold, i.e., $\pi : M \to N$ is a projection of M onto N whose rank at every point is equal to $n = dim\ N$. Moreover, we assume there are coordinate systems (x^j) on N and (x^j, x^{n+k}) on M such that the fibers with respect to π are defined by $x^j = cte$ $(1 \leq j \leq n)$.

A Cartan pseudogroup on M with invariants π (or invariant functions x^j) is a pseudogroup of transformations Γ operating on M and satisfying the following conditions :

1) The orbits of Γ are the fibers of the projection π
2) There are linear forms $(\omega^i, \tilde{\omega}^\lambda)$, $1 \leq i \leq r$, $1 \leq \lambda \leq p$, $n \leq r$, $r + p = dim\ M$ defined on M such that
   a) $(\omega^i, \tilde{\omega}^\lambda)$ is a basis of the space of covectors at every point of M

b) $d\omega^i = \frac{1}{2} c^i_{jk}\, \omega^j \wedge \omega^k + a^i_{j\lambda}\, \omega^j \wedge \tilde{\omega}^\lambda$ where the c^i_{jk} and $a^i_{j\lambda}$ are analytic functions on M which are constant on the fibers of π and $c^i_{jk} + c^i_{kj} = 0$

c) $\omega^j = dx^j \qquad 1 \leq j \leq n$

d) The matrices $a_\lambda = \| a^i_{j\lambda} \|$ are linearly independent at every point of M.

3) A local transformation φ of M belongs to Γ if and only if φ preserves the forms ω^i, i.e. $\varphi^*\omega^i = \omega^i$.

Let π_1 and π_2 be the natural projections of $M \times M$ onto the first and second factors respectively. Consider the fibered manifold $(M \times M, M, \pi_1)$ and let $(\Sigma, (M \times M, M, \pi_1))$ be the closed exterior differential system on $M \times M$ with independent variables in M generated by the forms (cf [3] and [5])

$$x^j \circ \pi_1 - x^j \circ \pi_2 \qquad 1 \leq j \leq n$$

$$\pi_1^* \omega^i - \pi_2^* \omega^i \qquad 1 \leq i \leq r .$$

The Cartan pseudogroup Γ is said to be involutive if $(\Sigma, (M \times M, M, \pi_1))$ is involutive at every integral point. Equation (b) will be called the structure equation of Γ. The system $(\omega^i, \tilde{\omega}^\lambda)$ as in (2) is called a Cartan system for Γ. Next we state, without proof, several facts which shall be used later. Write

$$d\tilde{\omega}^\lambda = \frac{1}{2} \varepsilon^\lambda_{\eta\mu}\, \tilde{\omega}^\eta \wedge \tilde{\omega}^\mu + \frac{1}{2} \nu^\lambda_{ij}\, \omega^i \wedge \omega^j + \xi^\lambda_{i\mu}\, \omega^i \wedge \tilde{\omega}^\mu$$

where the $\varepsilon^\lambda_{\eta\mu}$ and ν^λ_{ij} are skew-symmetric in the lower indices.

Differentiating equation (b) and replacing $d\tilde{\omega}^\lambda$ by its value above, we obtain the following equations :

$$C_1) \qquad a^i_{j\eta} a^j_{l\mu} - a^i_{j\mu} a^j_{l\eta} = a^i_{l\lambda}\, \varepsilon^\lambda_{\eta\mu}$$

$$C_2) \qquad a^j_{l\lambda} c^i_{kj} + a^j_{k\lambda} c^i_{jl} + a^i_{j\lambda} c^j_{lk} + \left(\frac{\partial a^i_{l\lambda}}{\partial x^k} - \frac{\partial a^i_{k\lambda}}{\partial x^l}\right) = a^i_{l\rho}\, \xi^\rho_{k\lambda} - a^i_{k\rho}\, \xi^\rho_{l\lambda}$$

$$C_3) \quad c^i_{jk} c^j_{lm} + c^i_{jl} c^j_{mk} + c^i_{jm} c^j_{kl} - \left(\frac{\partial c^i_{km}}{\partial x^l} + \frac{\partial c^i_{lk}}{\partial x^m} + \frac{\partial c^i_{ml}}{\partial x^k}\right) = a^i_{k\lambda} \nu^\lambda_{lm} + a^i_{l\lambda} \nu^\lambda_{mk} + a^i_{m\lambda} \nu^\lambda_{kl}.$$

Remark that $c^i_{jk} = a^i_{j\lambda} = 0$ for $i \leq n$ and $\dfrac{\partial a^i_{k\lambda}}{\partial x^l} = \dfrac{\partial c^i_{km}}{\partial x^l} = 0$ for $l > n$.

Equations C_1, C_2 and C_3 are linear equations in the unknowns ε, ν and ξ. The compatibility of these equations at every point of M is a necessary condition for the functions $a^i_{j\lambda}$ and c^i_{jk} to be the structure functions of a Cartan pseudogroup Γ. We shall refer to them as the «Cartan conditions».

PROPOSITION 1. *Let Γ be an involutive Cartan pseudogroup and $(\omega^i, \tilde{\omega}^\lambda)$ a Cartan system for Γ. A linear differential form ω defined on M is invariant under all transformations of Γ if and only if ω is a linear combination of the forms ω^i the coefficients being functions of x^j, $1 \leq j \leq n$, only.*

For a proof cf. [7].

In what follows we shall consider only involutive pseudogroups. The above proposition implies the following corollaries:

COROLLARY 1. *Let* $(\omega^i, \tilde{\omega}^\lambda)$ *and* $(\omega^i, {}^*\tilde{\omega}^\lambda)$ *be two Cartan systems for* Γ. *Then* ${}^*\tilde{\omega}^\lambda = b^\lambda_\mu \tilde{\omega}^\mu + b^\lambda_i \omega^i$ *where* b^λ_μ *are functions which depend only upon the coordinates* x^j, $1 \le j \le n$. *If* c^i_{jk}, $a^i_{j\lambda}$ *and* ${}^*c^i_{jk}$, ${}^*a^i_{j\lambda}$ *are the respective structure functions then*

$$a^i_{j\mu} = {}^*a^i_{j\lambda} b^\lambda_\mu \qquad c^i_{jk} - {}^*c^i_{jk} = a^i_{j\lambda} {}^*b^\lambda_k - {}^*a^i_{k\lambda} b^\lambda_j .$$

COROLLARY 2. *Let* $(\omega^i, \tilde{\omega}^\lambda)$ *and* $({}^*\omega^i, \tilde{\omega}^\lambda)$ *be two Cartan systems for* Γ *and* c^i_{jk}, $a^i_{j\lambda}$, ${}^*c^i_{jk}$, ${}^*a^i_{j\lambda}$ *the respective structure functions. Then* ${}^*a^i_{j\lambda} = g^i_k a^k_{l\lambda} (g^{-1})^l_j$ *where* ${}^*\omega^i = g^i_j \omega^j$.

Denote by $\mathfrak{G}(x)$ the space of endomorphisms of $\mathbf{R}^r$ spanned by the matrices $a_\lambda(x) = \| a^i_{j\lambda}(x) \|$. We have $dim\, \mathfrak{G}(x) = p$ and equation C_1 implies that $\mathfrak{G}(x)$ is a linear Lie algebra. Denote by $\mathfrak{D}(x)$ the derived space (*) of $\mathfrak{G}(x)$. $\mathfrak{D}(x)$ is also a linear Lie algebra. The following propositions are easily proved:

PROPOSITION 2. *If* $\mathfrak{G}(x)$ *is involutive then* $\mathfrak{D}(x)$ *is also involutive.*

Set $d(x) = dim\, \mathfrak{D}(x)$, $\delta(x) = codim\, \mathfrak{D}(x)$ and denote by $\tau_k(x)$ the characters of $\mathfrak{G}(x)$ and by $\tau'_k(x)$ the characters of $\mathfrak{D}(x)$.

PROPOSITION 3. *The exterior differential system* Σ *is in involution at every integral point if and only if*

a) $d(x)$ *is constant*

b) $\mathfrak{G}(x)$ *is involutive for every* $x \in M$.

If Σ is involutive the characters $\tau_k(x)$ and $\tau'_k(x)$ are also constant. The purpose of this note is to prove that the Cartan conditions are also sufficient conditions for the existence of a Cartan pseudogroup with prescribed structure functions. We proceed to the precise formulation of the problem.

In $N = \mathbf{R}^n$ we are given functions c^i_{jk} and $a^i_{j\lambda}$, $1 \le i, j, k \le r$, $n \le r$, $1 \le \lambda \le p$ which satisfy the following conditions:

1) $c^i_{jk} = 0 \quad a^i_{j\lambda} = 0$ for $1 \le i \le n$

2) $c^i_{jk} + c^i_{kj} = 0$

3) (Cartan conditions) For each $x \in \mathbf{R}^n$ the linear equations C_1, C_2, C_3 in the unknowns ν^ρ_{ij}, $\varepsilon^\rho_{\mu\eta}$, $\xi^\rho_{i\mu}$ where $\nu^\rho_{ij} + \nu^\rho_{ji} = 0$ and $\varepsilon^\rho_{\mu\eta} + \varepsilon^\rho_{\eta\mu} = 0$, are compatible $(1 \le i, j \le r,\ 1 \le \rho, \eta, \mu \le p$ and $\dfrac{\partial a^i_{k\lambda}}{\partial x^l} = \dfrac{\partial c^i_{km}}{\partial x^l} = 0$ for $l > n)$.

(*) For the notion of derived space (espace déduit), the definition of characters and the notion of involutive space of homomorphisms we refer the reader to [4] and [6].

4) For each $x \in \mathbf{R}^n$ the matrices $a_\lambda(x) = \| a^i_{j\lambda}(x) \|$ are linearly independent.

5) Denote by $\mathfrak{G}(x)$ the linear Lie algebra spanned by the matrices $a_\lambda(x)$, by $\mathfrak{D}(x)$ the derived algebra of $\mathfrak{G}(x)$. Set $d(x) = dim\ \mathfrak{D}(x)$ and $\delta(x) = codim\ \mathfrak{D}(x)$ $(d + \delta = rp)$ and denote by $\tau_k(x)$ and $\tau'_k(x)$ the characters of $\mathfrak{G}(x)$ and $\mathfrak{D}(x)$ respectively. The condition states that $\delta(x)$ is constant and $\mathfrak{G}(x)$ is involutive for every $x \in \mathbf{R}^n$. This implies in particular that $\tau_k(x)$ and $\tau'_k(x)$ are constant [cf. 6]. The relations between τ_k and τ'_k are given on pg. 9.

Given any point $x_o \in \mathbf{R}^n$ we can find a matrix $g \in GL(n, \mathbf{R})$ such that writing ${}^*a^i_{j\lambda}(x) = g^i_k\, a^k_{l\lambda}(g^{-1})^l_j$ we get (cf. [6])

$$\tau_k(x_o) = rank \left\| \begin{array}{c} {}^*a^i_{1\lambda}(x_o) \\ {}^*a^i_{2\lambda}(x_o) \\ \text{- - - - - - - -} \\ {}^*a^i_{k\lambda}(x_o) \end{array} \right\| \qquad 1 \leq k \leq r$$

Hence there is a neighborhood U of x_o such that for $x \in U$, $\tau_k(x)$ is also given by the rank of the above matrix computed at the point x. Since the problem we are considering is of local nature we shall assume that the functions $a^i_{j\lambda}$ satisfy the above rank property for any $x \in \mathbf{R}^n$. Corollary 2 justifies our procedure.

The first step is to determine differential forms $(\omega^i, \tilde{\omega}^\lambda)$ on $M = \mathbf{R}^m$, $m = r+p$, which satisfy condition 2) in the definition of a Cartan pseudogroup. In the linear group $GL(m, \mathbf{R})$ we introduce the coordinates p^i_s and q^ρ_s, $1 \leq i \leq r$, $1 \leq \rho \leq p$, $1 \leq s \leq m$, $m = r+p$, according to the following convention :

$$g = \left\| \begin{array}{c} p^i_s \\ \text{- - - - - - - - - -} \\ q^\rho_s \end{array} \right\| \begin{array}{l} \}\, r \\ \\ \}\, p \end{array}$$

We denote by $\overline{\mathfrak{M}}$ the submanifold of $GL(m, \mathbf{R})$ consisting of all matrices

$$n \{ \left\| \begin{array}{c|c} \overbrace{Id}^{n} & 0 \\ \hline * & ** \end{array} \right\|$$

$\overline{\mathfrak{M}}$ has a system of coordinates given by $\{ p^i_s, i > n, q^\rho_s \}$. Remark that on $\overline{\mathfrak{M}}$ $p^i_s = \delta^i_s$ for $i \leq n$. $dim\ \overline{\mathfrak{M}} = m(m-n)$.

Define $\mathfrak{M} = \overline{\mathfrak{M}} \times \mathbf{R}^m$ and let $\pi^1 : \mathfrak{M} \to \mathbf{R}^m$, $\pi^2 : \mathbf{R}^m \to \mathbf{R}^n$ be the natural projections. We obtain the following fibered manifolds $(\mathfrak{M}, \mathbf{R}^m, \pi^1)$, $(\mathbf{R}^m, \mathbf{R}^n, \pi^2)$, $(\mathfrak{M}, \mathbf{R}^n, \pi^2 \circ \pi^1)$. Denote the coordinates of $\mathbf{R}^m$ by $x^1, \ldots, x^m$.

On $\mathfrak{M}$ we define the following 1-forms :

$$\Omega^i = p^i_s\, dx^s \qquad \Pi^\rho = q^\rho_s\, dx^s .$$

Clearly the forms (Ω^i, Π^ρ) are linearly independent at every point of $\mathfrak{M}$ and $\Omega^i = dx^i$ for $1 \leq i \leq n$. Moreover $dx^s = F^s_i \Omega^i + F^s_{r+\rho} \Pi^\rho$ where $\| F^s_\sigma(g,x) \|_{1 \leq s, \sigma \leq m} = g^{-1}$.

We shall also denote by $a^i_{j\lambda}$ and c^i_{jk} the functions $a^i_{j\lambda} \circ \pi^2 \circ \pi^1$ and $c^i_{jk} \circ \pi^2 \circ \pi^1$ defined on $\mathfrak{M}$.

Let $(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1))$ be the closed exterior differential system on $\mathfrak{M}$ with indenpendent variables in $\mathbf{R}^m$ generated by

$$\Theta^i = d\Omega^i - \tfrac{1}{2} c^i_{jk}\, \Omega^j \wedge \Omega^k - a^i_{j\rho}\, \Omega^j \wedge \Pi^\rho \qquad (1 \leq i \leq r)$$

A basis for $\overline{\Sigma}$ is given by the forms $\{\Theta^i, d\Theta^i\}$.

Denote by $G^m(\mathfrak{M}, \mathbf{R}^m, \pi^1)$ the Grassmann manifold of admissible m-contact elements of $\mathfrak{M}$ for the given independent variables.

By definition $G^m(\mathfrak{M}, \mathbf{R}^m, \pi^1)$ is the subset of $G^m(\mathfrak{M})$ consisting of all m-contact elements E such that $\pi^1_* \mid E$ is an isomorphism onto the tangent space to $\mathbf{R}^m$. It is easy to see that $G^m(\mathfrak{M}, \mathbf{R}^m, \pi^1)$ is the set of m-contact elements on which either the linear forms (Ω^i, Π^ρ) or the linear forms $dx^1, \ldots, dx^m$ are linearly independent. The set $(\Omega^i, \Pi^\rho, dp^i_s, dq^\rho_s)$, $i > n$, is a basis of linear forms at every point of $\mathfrak{M}$. Denote by $(\Omega_i, \Pi_\rho, \delta p^i_s, \delta q^\rho_s)$ the dual basis. A system of coordinates for $G^m(\mathfrak{M}, \mathbf{R}^m, \pi^1)$ is given by $\{x^s, p^i_s, q^\rho_s, p^i_{sj}, \bar{p}^i_{s\mu}, q^\rho_{sj}, \bar{q}^\rho_{s\mu}\}$, $i > n$, where (on the admissible contact elements)

$$d p^i_s = p^i_{sj}\, \Omega^j + \bar{p}^i_{s\mu} \Pi^\mu$$
$$d q^\rho_s = q^\rho_{sj}\, \Omega^j + \bar{q}^\rho_{s\mu} \Pi^\mu .$$

The dual basis of (Ω^i, Π^ρ) on an admissible contact element is given by

$$L_j = \Omega_j + \sum_i p^i_{sj}\, \delta p^i_s + \sum_\rho q^\rho_{sj}\, \delta q^\rho_s$$
$$T_\mu = \Pi_\mu + \sum_i \bar{p}^i_{s\mu}\, \delta p^i_s + \sum_\rho \bar{q}^\rho_{s\mu}\, \delta q^\rho_s .$$

We shall now prove that $(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1))$ is in involution at every integral point (cf. [5] §1, n. 7 and 10). First we show that the set of admissible integral m-contact elements is a regularly imbedded submanifold and give a lower bound for its dimension.

Let $E^m(X)$, $X \in \mathfrak{M}$ be an admissible m-contact element and $\{L_\alpha, T_\mu\}$, $1 \leq \alpha \leq r$ its distinguished basis. Then

$$\Theta^i(L_\alpha, L_\beta) = dp^i_s(L_\alpha)\, dx^s(L_\beta) - dp^i_s(L_\beta)\, dx^s(L_\alpha) - c^i_{\alpha\beta} =$$
$$p^i_{s\alpha} F^s_\beta - p^i_{s\beta} F^s_\alpha - c^i_{\alpha\beta}$$

$$\Theta^i(L_\alpha, T_\mu) = dp^i_s(L_\alpha)\, dx^s(T_\mu) - dp^i_s(T_\mu)\, dx^s(L_\alpha) - a^i_{\alpha\mu} = p^i_{s\alpha} F^s_{r+\mu} - \bar{p}^i_{s\mu} F^s_\alpha - a^i_{\alpha\mu}$$

$$\Theta^i(T_\mu, T_\eta) = dp^i_s(T_\mu)\, dx^s(T_\eta) - dp^i_s(T_\eta)\, dx^s(T_\mu) = \bar{p}^i_{s\mu} F^s_{r+\eta} - \bar{p}^i_{s\eta} F^s_{r+\mu} .$$

We remark that the above expressions involve only the coordinates «p». Introducing the following transformation of coordinates :

$$\begin{aligned} \mathfrak{p}^i_{\alpha\beta} &= p^i_{s\beta} F^s_\alpha \\ \mathfrak{p}^i_{\alpha,\, r+\eta} &= \bar{p}^i_{s\eta} F^s_\alpha \\ \mathfrak{p}^i_{r+\mu,\, \beta} &= p^i_{s\beta} F^s_{r+\mu} \\ \mathfrak{p}^i_{r+\mu,\, r+\eta} &= \bar{p}^i_{s\eta} F^s_{r+\mu} \end{aligned}$$

we obtain the following set of equations in the coordinates «$\mathfrak{p}$» which define the set of elements $E^m(X)$ integral for the forms Θ^i :

$$I \quad \begin{cases} \mathfrak{p}^i_{\beta\alpha} - \mathfrak{p}^i_{\alpha\beta} - c^i_{\alpha\beta} = 0 & \alpha < \beta \\ \mathfrak{p}^i_{r+\mu,\, \alpha} - \mathfrak{p}^i_{\alpha,\, r+\mu} - a^i_{\alpha\mu} = 0 & \\ \mathfrak{p}^i_{r+\eta,\, r+\mu} - \mathfrak{p}^i_{r+\mu,\, r+\eta} = 0 . & \mu < \eta \end{cases}$$

It is easy to see that this set of linear equations is of maximum rank (all the equations are linearly independent) hence it admits solutions at each point $X \in \mathfrak{M}$. The set $\mathfrak{I}^m(\Theta)$ of admissible $E^m(X)$ which are integral for the forms Θ^i is a regularly imbedded submanifold of $G^m(\mathfrak{M}, \mathbf{R}^m, \pi^1)$ of dimension $m + m(m-n) + m^2 p + \frac{1}{2}(r-n)m(m+1)$. Moreover $(\mathfrak{I}^m(\Theta), \mathfrak{M}, \pi^1)$ is a fibered manifold with isomorphic fibers, each isomorphic to an affine hyperplane. Next we examine the condition for $E^m(X) \in \mathfrak{I}^m(\Theta)$ to be integral for $d\Theta^i$.

$$d\Theta^i(L_\alpha, L_\beta, L_\gamma) = a^i_{k\lambda} \Omega^k \wedge d\Pi^\lambda(L_\alpha, L_\beta, L_\gamma) + \Big\{ c^i_{j\alpha} c^j_{\gamma\beta} + c^i_{j\beta} c^j_{\alpha\gamma} + c^i_{j\gamma} c^j_{\beta\alpha} - \Big(\frac{\partial c^i_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial c^i_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial c^i_{\beta\gamma}}{\partial x^\alpha}\Big)\Big\}$$

$$d\Theta^i(L_\alpha, L_\beta, T_\mu) = a^i_{k\lambda} \Omega^k \wedge d\Pi^\lambda(L_\alpha, L_\beta, T_\mu) + \Big\{ a^j_{\alpha\mu} c^i_{\beta j} + a^j_{\beta\mu} c^i_{j\alpha} + a^i_{j\mu} c^j_{\alpha\beta} - \Big(\frac{\partial a^i_{\beta\mu}}{\partial x^\alpha} - \frac{\partial a^i_{\alpha\mu}}{\partial x^\beta}\Big)\Big\}$$

$$d\Theta^i(L_\alpha, T_\mu, T_\eta) = a^i_{k\lambda} \Omega^k \wedge d\Pi^\lambda(L_\alpha, T_\mu, T_\eta) + \{ a^i_{j\eta} a^j_{\alpha\mu} - a^i_{j\mu} a^j_{\alpha\eta} \}$$

$$d\Theta^i(T_\mu, T_\eta, T_\varphi) \equiv 0 \quad (\textit{identically})$$

Since $a^i_{k\lambda} \Omega^k \wedge d\Pi^\lambda = a^i_{k\lambda} \Omega^k \wedge dq^\lambda_s \wedge dx^s$ we have :

$$a^i_{k\lambda}\Omega^k \wedge d\Pi^\lambda(L_\alpha, L_\beta, L_\gamma) = a^i_{\alpha\lambda}(q^\lambda_{s\beta}F^s_\gamma - q^\lambda_{s\gamma}F^s_\beta) + a^i_{\beta\lambda}(q^\lambda_{s\gamma}F^s_\alpha - q^\lambda_{s\alpha}F^s_\gamma) + \\ + a^i_{\gamma\lambda}(q^\lambda_{s\alpha}F^s_\beta - q^\lambda_{s\beta}F^s_\alpha)$$

$$a^i_{k\lambda}\Omega^k \wedge d\Pi^\lambda(L_\alpha, L_\beta, T_\mu) = a^i_{\alpha\lambda}(q^\lambda_{s\beta}F^s_{r+\mu} - \bar{q}^\lambda_{s\mu}F^s_\beta) + a^i_{\beta\lambda}(\bar{q}^\lambda_{s\mu}F^s_\alpha - q^\lambda_{s\alpha}F^s_{r+\mu})$$

$$a^i_{k\lambda}\Omega^k \wedge d\Pi^\lambda(L_\alpha, T_\mu, T_\eta) = a^i_{\alpha\lambda}(\bar{q}^\lambda_{s\mu}F^s_{r+\eta} - \bar{q}^\lambda_{s\eta}F^s_{r+\mu})$$

Introducing the following transformation of coordinates :

$$\begin{aligned} q^\lambda_{\alpha\beta} &= q^\lambda_{s\beta}F^s_\alpha \\ q^\lambda_{\alpha,\, r+\eta} &= \bar{q}^\lambda_{s\eta}F^s_\alpha \\ q^\lambda_{r+\mu,\,\beta} &= q^\lambda_{s\beta}F^s_{r+\mu} \\ q^\lambda_{r+\mu,\, r+\eta} &= \bar{q}^\lambda_{s\eta}F^s_{r+\mu} \end{aligned}$$

we obtain the following set of linear equations in the coordinates « q» which define the set of elements $E^m(X) \in \mathfrak{J}^m(\Theta)$ integral for $d\Theta^i$.

$$II \begin{cases} a^i_{\alpha\lambda}(q^\lambda_{\alpha\beta} - q^\lambda_{\beta\gamma}) + a^i_{\beta\lambda}(q^\lambda_{\alpha\gamma} - q^\lambda_{\gamma\alpha}) + a^i_{\gamma\lambda}(q^\lambda_{\beta\alpha} - q^\lambda_{\alpha\beta}) + \Phi^i_{\alpha\beta\gamma}(x^1, \dots, x^n) = 0, \\ \hfill \alpha < \beta < \gamma \\ a^i_{\alpha\lambda}(q^\lambda_{r+\mu,\,\beta} - q^\lambda_{\beta,\, r+\mu}) + a^i_{\beta\lambda}(q^\lambda_{\alpha,\, r+\mu} - q^\lambda_{r+\mu,\,\alpha}) + \Phi^i_{\alpha,\,\beta,\, r+\mu}(x^1, \dots, x^n) = 0, \\ \hfill \alpha < \beta \\ a^i_{\alpha\lambda}(q^\lambda_{r+\mu,\, r+\eta} - q^\lambda_{r+\eta,\, r+\mu}) + \Phi^i_{\alpha,\, r+\mu,\, r+\eta}(x^1, \dots, x^n) = 0, \qquad \mu < \eta \end{cases}$$

Next we will show that for each $X \in \mathfrak{M}$ the system II (restricted to $\mathfrak{J}^m(\Theta)$) is compatible i.e., there exists an $E^m(X) \in \mathfrak{J}^m(\Theta)$ integral for $d\Theta^i$.

Let $\overset{\circ}{\nu}{}^\rho_{ij}(X)$, $\overset{\circ}{\varepsilon}{}^\rho_{\mu\eta}(X)$, $\overset{\circ}{\xi}{}^\rho_{i\mu}(X)$ be a fixed solution for the equations C_1, C_2 and C_3 at the point X. It is enough to determine an admissible $E^m(X)$ for which

$$d\Pi^\rho \mid E^m(X) = \tfrac{1}{2}\overset{\circ}{\nu}{}^\rho_{ij}(X)\,\Omega^i \mid E^m \wedge \Omega^j \mid E^m + \overset{\circ}{\xi}{}^\rho_{i\mu}(X)\,\Omega^i \mid E^m \wedge \Pi^\mu \mid E^m + \\ + \tfrac{1}{2}\overset{\circ}{\varepsilon}{}^\rho_{\mu\eta}(X)\,\Pi^\mu \mid E^m \wedge \Pi^\eta \mid E^m .$$

Such an element will be integral for $d\Theta^i$.

The preceding condition is given by the equations :

$$III \begin{cases} d\Pi^\rho(L_\alpha, L_\beta) = q^\rho_{\beta\,\alpha} - q^\rho_{\alpha\,\beta} = \overset{\circ}{\nu}{}^\rho_{\alpha\beta}(X) \\ d\Pi^\rho(L_\alpha, T_\mu) = q^\rho_{r+\mu,\,\alpha} - q^\rho_{\alpha,\, r+\mu} = \overset{\circ}{\xi}{}^\rho_{\alpha\mu}(X) \\ d\Pi^\rho(T_\mu, T_\eta) = q^\rho_{r+\eta,\, r+\mu} - q^\rho_{r+\mu,\, r+\eta} = \overset{\circ}{\varepsilon}{}^\rho_{\mu\eta}(X). \end{cases}$$

Equations III are a linear system of maximum rank hence compatible. Denote by $\mathfrak{J}^m(\bar{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1))$ the set of admissible integral elements and by $\mathfrak{J}^m_X(\bar{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1))$ the set of admissible integral elements with base point X.

LEMMA 1. *For each* $X \in \mathfrak{M}$, $\mathfrak{I}^m_X(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1)) \neq \phi$.

Remark now that the six sets of equations in I and II are mutually linearly independent. Moreover, it is easy to see that all the equations of the last set in II are linearly independent their rank being $\frac{1}{2} p^2 (p-1)$. Since $\mathfrak{G}(X)$ is involutive and $\delta(X)$ constant, it follows easily that the second set of equations in II is of constant rank $p\,\delta$.

Let V be the standard $\mathbf{R}$-vector space of dimension m and V^* its dual. Denote by $\{\omega^1, \dots, \omega^r, \tilde{\omega}^1, \dots, \tilde{\omega}^p\}$ the standard basis of V^* and set $W = \overset{2}{\wedge} V^* \times \dots \times \overset{2}{\wedge} V^*$ (p-fold product). For each $X \in \mathfrak{M}$ denote by $\mathfrak{W}(X)$ the affine subspace of W consisting of all $\psi = (\psi^\rho)_\rho$, $\psi^\rho = \frac{1}{2} \nu^\rho_{\alpha\beta}(\psi)\, \omega^\alpha \wedge \omega^\beta + \xi^\rho_{\alpha\mu}(\psi)\, \omega^\alpha \wedge \tilde{\omega}^\mu + \frac{1}{2} \varepsilon^\rho_{\mu\eta}(\psi)\, \tilde{\omega}^\mu \wedge \tilde{\omega}^\eta$, such that the components $\nu^\rho_{\alpha\beta}(\psi)$, $\xi^\rho_{\alpha\mu}(\psi)$ and $\varepsilon^\rho_{\mu\eta}(\psi)$ are solutions of equations C_1, C_2, C_3 at the point X.

Denote by $\overline{\mathfrak{W}}(X)$ the subspace of W consisting of elements $\psi = (\psi^\rho)_\rho$ such that $a^i_{j\rho}(X)\, \omega^j \wedge \psi^\rho = 0$. Then clearly $\mathfrak{W}(X) = \psi_o + \overline{\mathfrak{W}}(X)$ where ψ_o is any fixed element of $\mathfrak{W}(X)$. Write $\Delta(X) = dim\, \mathfrak{W}(X) = dim\, \overline{\mathfrak{W}}(X)$. For each $\psi \in \mathfrak{W}(X)$ denote by $\mathfrak{I}_\psi(X)$ the set of $E^m(X) \in \mathfrak{I}^m_X(\Theta)$ such that $d\Pi^\rho \mid E^m(X) = \psi^\rho$ (i.e., the components of $d\Pi^\rho \mid E^m(X)$ with respect to $\{\Omega^i \mid E^m(X), \Pi^\mu \mid E^m(X)\}$ are the same as the components of ψ^ρ with respect to $\{\omega^i, \tilde{\omega}^\mu\}$. By equations C_1, C_2, C_3 it follows that any $E^m(X) \in \mathfrak{I}_\psi(X)$ is integral for $d\Theta^i$. Conversely if $E^m(X) \in \mathfrak{I}^m_X(\Theta)$ is integral for $d\Theta^i$ then $E^m(X) \in \mathfrak{I}_\psi(X)$ where $\psi^\rho = d\Pi^\rho \mid E^m(X)$. Hence

$$\mathfrak{I}^m_X(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1)) = \bigcup_{\psi \in \mathfrak{W}(X)} \mathfrak{I}_\psi(X).$$

Clearly $\mathfrak{I}_{\psi_1}(X) \cap \mathfrak{I}_{\psi_2}(X) = \phi$ if $\psi_1 \neq \psi_2$. On the other hand each $\mathfrak{I}_\psi(X)$, is defined by the system of linear equations of maximum rank

$$IV \begin{cases} q^\rho_{\beta\alpha} - q^\rho_{\alpha\beta} = \nu^\rho_{\alpha\beta}(\psi) \\ q^\rho_{r+\mu,\alpha} - q^\rho_{\alpha,r+\mu} = \xi^\rho_{\alpha\mu}(\psi) \\ q^\rho_{r+\eta,r+\mu} - q^\rho_{r+\mu,r+\eta} = \varepsilon^\rho_{\mu\eta}(\psi). \end{cases}$$

The rank of system IV is $\frac{1}{2} p\, m(m-1)$. Now remark that the homogeneous part of equations IV is independent of ψ and denote by $\mathfrak{V}(X) \subset \mathfrak{I}^m_X(\Theta)$ the linear subspace of its solutions. It is now a straightforward matter to establish a diffeomorphism

$$\mathfrak{W}(X) \times \mathfrak{V}(X) \longleftrightarrow \mathfrak{I}^m_X(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1)).$$

Hence

$$dim\, \mathfrak{I}^m_X(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1)) = dim\, \mathfrak{W}(X) + dim\, \mathfrak{V}(X) = \Delta(X) + \tfrac{1}{2} m(m+1)(m-n).$$

Next we shall determine a lower bound for $\Delta(X)$. Consider $\mathfrak{X} = V^* \times \dots \times V^*$ (p-fold product). Denote by $\overline{\mathfrak{X}}(X)$ the subspace of $\mathfrak{X}$ composed of the vectors $\varphi = (\varphi^\rho)_\rho$

such that $a^i_{j\rho}(X)\,\omega^j \wedge \varphi^\rho = 0$. Write $\varphi^\rho = b^\rho_k\,\omega^k + c^\rho_\mu\,\tilde{\omega}^\mu$.
Then $\varphi \in \mathfrak{X}(X)$ if and only if

$$a^i_{j\rho}(X)\,b^\rho_k\,\omega^j \wedge \omega^k + a^i_{j\rho}(X)\,c^\rho_\mu\,\omega^j \wedge \tilde{\omega}^\mu = 0\,.$$

We obtain the equations

$$V \quad \begin{cases} a^i_{j\rho}(X)\,b^\rho_k - a^i_{k\rho}(X)\,b^\rho_j = 0 \\ a^i_{j\rho}\,c^\rho_\mu = 0\,. \end{cases}$$

The second set of equations in V together with condition 4) imply that $c^\rho_\mu = 0$. Hence $\varphi \in \mathfrak{X}(X)$ if and only if $\varphi^\rho = b^\rho_k\,\omega^k$ with $\| b^\rho_k \| \in \mathfrak{D}(X)$. Let $\| b^\rho_{k\varepsilon}(X) \|$ $1 \leq \varepsilon \leq d$ be a basis of $\mathfrak{D}(X)$. Then a basis of $\mathfrak{X}(X)$ is given by the vectors $\mathfrak{x}_\varepsilon(X) = (b^\rho_{k\varepsilon}(X)\,\omega^k)_\rho$. Next we consider the following bilinear forms :

$$\mathfrak{x}^a_\varepsilon(X) = \mathfrak{x}_\varepsilon(X) \wedge \omega^a = (b^\rho_{k\varepsilon}(X)\,\omega^k \wedge \omega^a)_\rho \qquad 1 \leq a \leq r$$

$$\mathfrak{x}^{r+\mu}_\varepsilon(X) = \mathfrak{x}_\varepsilon(X) \wedge \tilde{\omega}^\mu = (b^\rho_{k\varepsilon}(X)\,\omega^k \wedge \tilde{\omega}^\mu)_\rho \qquad 1 \leq \mu \leq p\,.$$

From the definition of $\mathfrak{X}(X)$ it follows immediately that $\mathfrak{X}^a_\varepsilon(X),\ \mathfrak{X}^{r+\mu}_\varepsilon(X) \in \overline{\mathfrak{W}}(X)$. On the other hand from the particular choice of the functions $a^i_{j\lambda}$ made in the outset we also have (cf. [6] p. 5):

$$\tau'_k = rank \left\| \begin{matrix} b^\rho_{1\varepsilon}(X) \\ b^\rho_{2\varepsilon}(X) \\ \text{-------} \\ b^\rho_{k\varepsilon}(X) \end{matrix} \right\| \qquad 1 \leq k \leq r\,.$$

In $\overset{2}{\wedge} V^*$ take the basis $(\omega^a \underset{a<\beta}{\wedge} \omega^\beta,\ \omega^a \wedge \tilde{\omega}^\mu,\ \tilde{\omega}^\mu \underset{\mu<\eta}{\wedge} \tilde{\omega}^\eta)$. Let B be the corresponding basis in W. Then from the preceding remark it follows that the matrix of the vectors $\mathfrak{x}^a_\varepsilon(X),\ \mathfrak{x}^{r+\mu}_\varepsilon(X)$ with respect to B has rank $\geq \tau'_1(x) + \dots + \tau'_{r-1} + pd$. Since $\mathfrak{G}(x)$ is involutive we have the following relations :

$$\begin{aligned} \tau'_1 &= p \\ \tau'_2 &= p + (p - \tau_1) \\ &\text{----------------} \\ \tau'_k &= p + (p - \tau_1) + \dots + (p - \tau_{k-1}) \\ &\text{------------------} \\ \tau'_{r-1} &= p + (p - \tau_1) + \dots + (p - \tau_{r-2}) \end{aligned}$$

hence $\Delta(X) \geq p(pr + \frac{r(r-1)}{2} - \delta) - [(r-2)\,\tau_1 + (r-3)\,\tau_2 + \dots + \tau_{r-2}]$.

We shall now compute the characters of $\overline{\Sigma}$. The results will show in particular that the equality holds in the above relations. Let $E^m(X) \in \mathfrak{J}^m(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1))$ and denote by (L_a, T_μ) the basis of $E^m(X)$ defined previously. We shall compute the reduced characters of $E^m(X)\ mod(\Omega^i, \Pi^\rho)$. We have

$$\Theta^i = d\Omega^i \ mod\ (\Omega^i, \Pi^\rho)$$

$$d\Theta^i = a^i_{j\rho}\, \Omega^j \wedge d\Pi^\rho \ mod(\Omega^i, \Pi^\rho, d\Omega^i)\ .$$

Since $\overline{\Sigma}$ is generated by forms of degree ≥ 2 we have $t_o(E^m(X)) = 0$.
Write $a^i_{j\rho}\, \Omega^j \wedge d\Pi^\rho = \Phi^i$

$$L_\alpha \lrcorner\, d\Omega^i \equiv - F^s_\alpha \quad dp^i_s \ mod\ (\Omega^i, \Pi^\rho)$$

$$T_\mu \lrcorner\, d\Omega^i \equiv - F^s_{r+\mu}\, dp^i_s \ mod\ (\Omega^i, \Pi^\rho)$$

$$L_\alpha \lrcorner\, d\Pi^\rho \equiv - F^s_\alpha \quad dq^\rho_s \ mod\ (\Omega^i; \Pi^\rho)$$

$$T_\mu \lrcorner\, d\Pi^\rho \equiv - F^s_{r+\mu}\, dq^\rho_s \ mod\ (\Omega^i; \Pi^\rho)$$

$$(L_\alpha \wedge L_\beta) \lrcorner\, \Phi^i \equiv - a^i_{\alpha\rho} F^s_\beta \quad dq^\rho_s + a^i_{\beta\rho} F^s_\alpha\, dq^\rho_s \ mod\ (\Omega^i, \Pi^\rho)$$

$$(L_\alpha \wedge T_\mu) \lrcorner\, \Phi^i \equiv - a^i_{\alpha\rho} F^s_{r+\mu}\, dq^\rho_s \qquad mod\ (\Omega^i, \Pi^\rho)$$

$$(T_\mu \wedge T_\eta) \lrcorner\, \Phi^i \equiv 0 \qquad mod\ (\Omega^i, \Pi^\rho)\ .$$

We introduce the forms :

$$\omega^i_\alpha = - F^s_\alpha \quad dp^i_s$$

$$\omega^i_{r+\mu} = - F^s_{r+\mu}\, dp^i_s$$

$$\pi^\rho_\alpha = - F^s_\alpha \quad dq^\rho_s$$

$$\pi^\rho_{r+\mu} = - F^s_{r+\mu}\, dq^\rho_s\ .$$

Clearly the ω^i_s, π^ρ_s are linearly independent and

$$(L_\alpha \wedge L_\beta) \lrcorner\, \Phi^i \equiv a^i_{\alpha\rho}\, \pi^\rho_\beta - a^i_{\beta\rho}\, \pi^\rho_\alpha \ mod\ (\Omega^i, \Pi^\rho)$$

$$(L_\alpha \wedge T_\mu) \lrcorner\, \Phi^i \equiv a^i_{\alpha\rho}\, \pi^\rho_{r+\mu} \qquad mod\ (\Omega^i, \Pi^\rho)$$

$$(T_\mu \wedge T_\eta) \lrcorner\, \Phi^i \equiv 0 \qquad mod\ (\Omega^i, \Pi^\rho)\ .$$

It is now easy to compute a lower bound for the characters.

$$L_1 \lrcorner\ d\Omega^i \equiv \omega^i_1 \ \text{ hence } \ t_1(E^m(X)) \geq r-n$$

$$L_1 \lrcorner\ d\Omega^i \equiv \omega^i_1$$

$$L_2 \lrcorner\ d\Omega^i \equiv \omega^i_2$$

$$(L_1 \wedge L_2) \lrcorner\, \Phi^i \equiv a^i_{1\rho}\, \pi^\rho_2 - a^i_{2\rho}\, \pi^\rho_1 \ \text{ hence } \ t_2(E^m(X)) \geq 2(r-n) + \tau_1\ .$$

In general, if we write explicitly $L_\alpha \lrcorner\, d\Omega^i$ and $(L_\alpha \wedge L_\beta) \lrcorner\, \Phi^i$, $\alpha \leq k$, $\beta \leq k$, $k \leq r$, it is readily seen that $t_k(E^m(X)) \geq k(r-n) + [\tau_1 + \ldots + \tau_{k-1}]$.

Next we set $\tau_k = p$ for $k \geq r$. Then

$$L_\alpha \lrcorner d\Omega^i \equiv \omega^i_\alpha \qquad 1 \le \alpha \le r$$
$$T_1 \lrcorner d\Omega^i \equiv \omega^i_{r+1}$$
$$(L_\alpha \wedge L_\beta) \lrcorner \Phi^i \equiv a^i_{\alpha\rho}\pi^\rho_\beta - a^i_{\beta\rho}\pi^\rho_\alpha \qquad 1 \le \alpha < \beta \le r$$
$$(L_\alpha \wedge T_1) \lrcorner \Phi^i \equiv a^i_{\alpha\rho}\pi^\rho_{r+1} \qquad 1 \le a \le r$$

hence $t_{r+1}(E^m(X)) \ge (r+1)(r-n) + [\tau_1 + \ldots + \tau_r]$.

In general, if we write explicitly the expression of

$$L_\alpha \lrcorner d\Omega^i \qquad 1 \le \alpha \le r$$
$$T_\mu \lrcorner d\Omega^i \qquad 1 \le \mu \le k \qquad k \le p$$
$$(L_\alpha \wedge L_\beta) \lrcorner \Phi^i \qquad 1 \le \alpha < \beta \le r$$
$$(L_\alpha \wedge T_\mu) \lrcorner \Phi^i \qquad 1 \le \alpha \le r,\ 1 \le \mu \le k$$

then we obtain $t_{r+k}(E^m(X)) \ge (r+k)(r-n) + [\tau_1 + \ldots + \tau_{r+k-1}]$.
Adding up the characters we get

$$(t_o + t_1 + \ldots + t_{m-1})(E^m(X)) \ge \tfrac{1}{2} m(m-1)(r-n) + p\left(\frac{p(p-1)}{2} + \delta\right) + $$
$$+ [(r-2)\tau_1 + (r-3)\tau_2 + \ldots + \tau_{r-2}].$$

From the general theory of exterior differential systems (cf. [3], p. 20 or [6], p. 4) we know that the rank of equations I and II at the point $E^m(X)$ is $\ge (t_o + \ldots + t_{m-1})(E^m(X))$ hence $dim\ \mathfrak{I}^m_X(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1)) \le dim\ G^m_X(\mathfrak{M}, \mathbf{R}^m, \pi^1) - \sum_{k=o}^{m-1} t_k(E^m(X)) \le m^2(m-n) - \tfrac{1}{2} m(m-1)(r-n) - p\left(\frac{p(p-1)}{2} + \delta\right) - [(r-2)\tau_1 + (r-3)\tau_2 + \ldots + \tau_{r-2}]$.

Earlier we have proved that

$$dim\ \mathfrak{I}^m_X(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1)) = \Delta(X) + \tfrac{1}{2} m(m+1)(m-n) \ge p\left(pr + \frac{r(r-1)}{2} - \delta\right) -$$
$$- [(r-2)\tau_1 + (r-3)\tau_2 + \ldots + \tau_{r-2}] + \tfrac{1}{2} m(m+1)(m-n).$$

Since $m^2(m-n) - \tfrac{1}{2} m(m-1)(r-n) - \tfrac{1}{2} p^2(p-1) = \tfrac{1}{2} m(m+1)(m-n) + p\left(pr + \frac{r(r-1)}{2}\right)$ it follows that all the above inequalities become equalities.

This implies the following results :

a) $\Delta(X)$ is constant and equal to

$$p\left(pr + \frac{r(r-1)}{2} - \delta\right) - [(r-2)\tau_1 + (r-3)\tau_2 + \ldots + \tau_{r-2}].$$

b) The set of equations I and II is of constant rank so that $\mathfrak{I}^m(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1))$ is a regularly imbedded submanifold in $G^m(\mathfrak{M}, \mathbf{R}^m, \pi^1)$ of dimension $m + m(m-n) + m^2(m-n) - (t_o + \ldots + t_{n-1})$ where $t_k = k(r-n) + [\tau_1 + \ldots + \tau_{k-1}]$. Moreover $(\mathfrak{I}^m(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1)), \mathfrak{M}, \Pi)$ is a fibered manifold where $\Pi : G^m \to \mathfrak{M}$ is the natural projection.

c) $(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1))$ is in involution at every point of $\mathfrak{M}$ (cf. [5] pg. 4).

Let $i: M \subset \mathbf{R}^m \to \mathfrak{M}$ be an integral cross section for $(\overline{\Sigma}, (\mathfrak{M}, \mathbf{R}^m, \pi^1))$, M a domain in $\mathbf{R}^m$. Setting $\omega^i = i^*\Omega^i$ and $\tilde{\omega}^\lambda = i^*\Pi^\lambda$ we obtain a Cartan system $(\omega^i, \tilde{\omega}^\lambda)$ defined on M with structure equation $d\omega^i = \frac{1}{2} c^i_{jk}\, \omega^j \wedge \omega^k + a^i_{j\lambda}\, \omega^j \wedge \tilde{\omega}^\lambda$ and $\omega^j = dx^j$ for $1 \leq j \leq n$.

Let Γ be the set of all local transformations φ of the domain M such that $\varphi^* \omega^i = \omega^i$ $(1 \leq i \leq r)$ and $x^j \circ \varphi = x^j$ $(1 \leq j \leq n)$. Then Γ is the set of transformations $\pi_2 \circ \Phi$ where Φ is an integral cross section of the exterior differential system $(\Sigma, (M \times M, M, \pi_1))$ with independent variables in M, Σ being generated by the forms $x^j \circ \pi_1 - x^j \circ \pi_2$ $(1 \leq j \leq n)$, $\pi_1^* \omega^i - \pi_2^* \omega^i$ $(1 \leq i \leq r)$, and π_1, π_2 the natural projections of $M \times M$ into its factors. By proposition 3 the system $(\Sigma, (M \times M, M, \pi_1))$ is involutive at every integral point, i,e. a point $(x, y) \in M \times M$ such that $x^j(x) = x^j(y)$, $1 \leq j \leq n$. Hence the orbits of Γ are the fibers $x^j = cte$ and Γ is a Cartan pseudogroup with invariants x^j. A Cartan system for Γ is given by $(\omega^i, \tilde{\omega}^\lambda)$ and the corresponding structure functions are those prescribed at the outset.

References.

[1] E. CARTAN. Sur la structure des groupes infinis de transformations. Oeuvres Complètes, partie II, vol. 2, p. 609.

[2] E. CARTAN. La structure des groupes infinis. (Sem. Math.), ibid, p. 1347.

[3] M. KURANISHI. On the local theory of continuous infinite pseudogroups II. Nagoya Math. J., 1961.

[4] M. KURANISHI. The classification of complex simple continuous transitive infinite pseudo-groups according to E. Cartan. Mimeographed notes, Princeton University, 1961.

[5] Y. MATSUSHIMA. On a theorem concerning the prolongation of a differential system. Nagoya Math. J., vol. 6, Oct. 1953.

[6] Y. MATSUSHIMA. Sur les algèbres de Lie linéaires semi-involutives. Coll. de Topologie de Strasbourg, 1954.

[7] A.M. RODRIGUES. On Cartan pseudogroups. To appear, Nagoya Math. J.