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1. Introduction.

It is part of the classical material of homotopy theory that if a space X admits a continuous multiplication (that is if X is furnished with an H-space structure) then that multiplication is transferred to the terms of the Postnikov decomposition of X (see e.g., [9, 13]). We may express this fact precisely as follows. Let $X_{(n)}$ be the n^{th} term of the Postnikov decomposition of X and let $p_n : X \to X_{(n)}$ be the fibre projection. Then if X is furnished with an H-structure map $\Phi : X \times X \to X$, we can define an H-structure map $\Phi_n : X_{(n)} \times X_{(n)} \to X_{(n)}$ such that p_n is an H-map ($\Phi_n(p_n \times p_n) \simeq p_n \Phi$); moreover Φ_n is then uniquely determined up to homotopy.

The dual problem has recently been considered by Curjel [5] : given a comultiplication (H'-structure) on the simply-connected space X, then it is possible to define a comultiplication on each homology section of X in such a way that the inclusion map $i_n : X_n \to X$ is an H'-map.

The present paper consists in a very broad sense of a generalized treatment of of this type of problem. There are many other structures that have appeared in homotopy theory in recent years and Peterson has proposed in [15] a systematic treatment of such structures. Moreover various notions not evidently falling within the scope of such a unified theory in fact turn out to be susceptible of treatment (for example, the Borsuk-Svarc notion of genus [16], and the associativity problem for H or H'-structures, which may be translated into a uniqueness problem for related structures). Thus the first step in the generalization is to define the notion of a *structure system* and of a *structure* on an object X of a given category $\mathcal{C}$ relative to the given structure system. These systems divide themselves naturally into two mutually dual classes, which we call *left* and *right*

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structure systems, whose associated structures generalize H - structures and H'-structures respectively. Similary we introduce a general notion of structure-preserving maps, or *homomorphisms* of structured objects, generalizing the notion of H - maps and H' - maps. The generality of our approach here enables us to consider maps which are homomorphims with respect to a given transformation of structure system; thus we may study questions, of fairly frequent occurrence in homotopy theory, when one wishes to know whether an object X admitting a given structure in fact admits an even sharper structure. For example, the Sugawara theorem giving sufficient conditions for a space with commutative loop-space to be an H - space may be formulated (and proved !) within the framework of the general theory here elaborated.

Now let $\mathfrak{R}_1, \mathfrak{R}_2$ be two (right) structure systems over a given category $\mathcal{C}$ and let $\mathfrak{F} : \mathfrak{R}_1 \to \mathfrak{R}_2$ be a transformation of systems. We thus have a notion of an $\mathfrak{F}$ - homomorphism f of structured objects X and Y in $\mathcal{C}$ (Definition 2.7r). Suppose indeed that $f : X \to Y$ is a map in $\mathcal{C}$. Then we may ask the question : if Y is $\mathfrak{R}_2$ - structured may we $\mathfrak{R}_1$- structure X so that f is an $\mathfrak{F}$ - homomorphism, and if so, is the $\mathfrak{R}_1$- structure on X unique ? Similary we could suppose X $\mathfrak{R}_1$ - structured and raise the similar question about Y. Thus, if we also study left structure systems we get in all four types of questions; and all problems of transfer of structure hitherto considered in the literature may be subsumed under one or other of these four heads.

Fortunately it turns out that this generalization not only provides a common formulation of apparently little related problems; it also suggests a general procedure for their solution. All four types of question prove to be susceptible of study by the standard techniques of obstruction theory. The obstruction arguments may appear in the context of extension, cross-section or compression problems. However, this does not affect the essential unity of the treatment[1], and we are led to a study of cohomology groups (of certain maps depending on the structure systems involved) with coefficients in homotopy groups (of certain maps depending on the structure systems involved). Insofar as this paper is concerned we are content to describe conditions under which the groups, and therefore the obstructions, vanish. This turns out to be adequate to deal with questions of the sort referred to in the first two paragraphs, and to provide substantial generalizations of them; but we hope to take up in a later paper the problem of computing at least the first obstruction in concrete cases.

Each of the four types of problem involves an existence and a uniqueness question. The technique we adopt of replacing the problem by a standard obstruction problem leads us to a fairly comprehensive answer to the existence question; but it is considerebly less satisfactory for tackling the uniqueness question. For the technique requires us to replace a structure by what we call a *strong* structure. Every structure may be associated with a strong structure but the correspondence fails in general to be one-one. Thus while a solution of the existence problem for compatible strong structures implies a solution of the problem for structures, a solution of the uniqueness problem (which we provide) does not in general imply a solution of the uniqueness problem for arbitrary structures. Fortunately, we are able to pick out a subclass of the class of structures which is large enough to include H- structures and H'- structures and for which the correspondence referred to above is in fact one-one.

The plan of the paper is as follows. In section 2 we give the basic definitions and discuss several examples in some detail. We concentrate attention on *right* structure systems partly for the sake of definiteness and partly because the new results which we obtain in very explicit applications do refer to right systems. We lay particular stress on the notion of the *fibre* of a structure system $\mathfrak{R}$; our results stem from a careful study of the homotopy structure of this fibre (which is a covariant functor N from the category $\mathcal{C}$ to the category $\mathfrak{T}$ of reasonable (based) spaces, together with a natural transformation of N into a functor T appearing in the description of the structure system $\mathfrak{R}$). We are particularly interested in the structure system $\mathcal{G}_n$, the *n- genus structure system*. Here the category $\mathcal{C}$ is the category $\mathfrak{T}^2$ of 'pairs' drawn from the category $\mathfrak{T}$; this is the category $\mathfrak{P}(\mathfrak{T})$ of [7]. By specialization we obtain from $\mathcal{G}_n$ the structure system $\mathcal{K}_n$ over $\mathfrak{T}$, *the n- cat structure system over $\mathfrak{T}$*, and $\mathcal{K}_n^2$ over $\mathfrak{T}^2$, the *n- cat structure system over $\mathfrak{T}^2$*.

In section 3 we set out the general theory and formulate the solutions to the four types of problem in terms of the vanishing of cohomology groups containing potential obstruction elements. In section 4 we concentrate attention on the systems $\mathcal{G}_n$ (and the related systems $\mathcal{K}_n, \mathcal{K}_n^2$). We compute the connectivity of $N(\Phi)$ in terms of the connectivities of Φ, f_1, f_2 for a given map $\Phi : f_1 \to f_2$ in $\mathfrak{T}^2$, where N is the fibre of $\mathcal{G}_n$ (Theorem 4.3) and are thus able to generalize Curjel's result and to produce other concrete results in this direction. For example we prove that if $X^{(m)}$ is the m-skeleton of X the n-cat $X^{(m)} \leq$ cat X provided X is not contractible[2].

In section 5 we take up the uniqueness question under the hypothesis which allows our technique to be applicable. We show in particular how our general theorems permit an attack on the problem of the commutativity and associativity of transferred H and H'- structures. As a particular result we show that if X is 1- connected[3] then $X^{(m)}$ admits a unique (up to homotopy) H'- structure compatible with a given H'- structure on X and that this structure is commutative (associative) provided the structure on X is commutative (associative).

For clarity we explain here than an H- structure (H'- structure) on X is a continuous multiplication (comultiplication) on X with two-sided homotopy unit. Also we have, as elsewhere, renormalized cat, reducing its value by one unit compared with the classical convention adopted, for example, in [8], so that here cat $X = 0$ means that X is contractible.

2. Definition and examples.

Let $\mathcal{C}$ be an arbitrary category and let $\mathcal{T}$ be the category of based topological spaces (of the based homotopy type of countable CW-complexe) and based maps.

DEFINITION 2.1 r. A right structure system $\mathcal{R}$ *over* $\mathcal{C}$ *is a triple of covariant functors* $R, P, T : \mathcal{C} \to \mathcal{T}$ *together with a pair of natural transformations* $d : R \to P$, $j : T \to P$. *An object* $X \in \mathcal{C}$ *is said to be* $\mathcal{R}$-structured *if it is furnished with a map* $\Phi : RX \to TX$ *such that*

$$j(X) \circ \Phi \simeq d(X) : RX \to PX.$$

The map Φ *is called an* $\mathcal{R}$-structure map *for* X *and its homotopy class* $\{\Phi\}$ *an* $\mathcal{R}$-structure *for* X.

Let $\mathcal{R} = (R, P, T; d, j)$ be a right structure system. Then j determines in the standard way a functor $T' : \mathcal{C} \to \mathcal{T}$ and natural transformations

$$T \underset{s}{\overset{r}{\rightleftarrows}} T' \xrightarrow{j'} P$$

such that $j'r = j$, $sr = 1$, $rs \simeq 1$, and j' is a natural fibration. Let $\mathcal{R}' = (R, P, T'; d, j')$. Then if Φ is an $\mathcal{R}$-structure map for X, $r(X) \circ \Phi$ is an $\mathcal{R}'$-structure map for X and we set up in this way a (1-1) correspondence between $\mathcal{R}$-structures on X and $\mathcal{R}'$-structures on X. Thus there is no real loss of generality, in developing the general theory, in supposing that j is a natural fibration [4].

If $j : T \to P$ is a natural fibration let N be the fibre of j. We call the functor N, together with the natural inclusion $\chi : N \to T$ the *fibre* of $\mathcal{R}$; by abuse we may refer to N itself as the fibre of $\mathcal{R}$; and if j is not a natural fibration then we define the fibre of $\mathcal{R}$ to be the fibre of the associated system $\mathcal{R}'$.

Now let j be a natural fibration. Then, pulling j back by means of the transformation d, we obtain the commutative diagram

(2.2 r)
$$\begin{tikzcd} N \arrow[r, "i"] \arrow[d, "1"'] & M \arrow[r, "p"] \arrow[d, "t"] & R \arrow[d, "d"] \\ N \arrow[r, "\chi"'] & T \arrow[r, "j"'] & P \end{tikzcd}$$

where p is a natural fibration. We call the structure system $\overline{\mathcal{R}} = (R, R, M; 1, p)$ the *strong* system associated with $\mathcal{R}$. An $\overline{\mathcal{R}}$-structure (map) for X will be called a *strong* $\mathcal{R}$-structure (map).

PROPOSITION 2.3 r. *The function* $\{\sigma\} \to \{t(X) \circ \sigma\}$ *is a surjection of the strong* $\mathfrak{R}$-*structures for* X *onto the* $\mathfrak{R}$-*structures for* X.

PROOF. A strong $\mathfrak{R}$-structure map for X is a map $\sigma : RX \to MX$ such that $p(X) \circ \sigma \simeq 1$. Then $j(X) \circ t(X) \circ \sigma = d(X) \circ p(X) \circ \sigma \simeq d(X)$, so that $t(X) \circ \sigma$ is an $\mathfrak{R}$-structure map.

Now let $\Phi : RX \to TX$ be an $\mathfrak{R}$-structure map. Since $j(X)$ is a fibration we may assume that in fact $j(X) \circ \Phi = d(X)$. Then the map $\sigma : RX \to MX$, given by $\sigma(x) = (x, \Phi x)$ is a strong $\mathfrak{R}$-structure map such that $t(X) \circ \sigma = \Phi$.

Notice that a strong $\mathfrak{R}$-structure for X is just the homotopy class of a cross-section for $p(X)$. We call $\{t(X) \circ \sigma\}$ the $\mathfrak{R}$-structure for X associated with the strong $\mathfrak{R}$-structure $\{\sigma\}$.

DEFINITION 2.4 r. *Let* $\mathfrak{R}_i$ *be two right structure systems for* $\mathcal{C}$. *A* map $\mathcal{F} : \mathfrak{R}_1 \to \mathfrak{R}_2$ *is a triple of natural transformations* $\rho : R_1 \to R_2$, $\pi : P_1 \to P_2$, $\tau : T_1 \to T_2$ *such that* $d_2 \rho = \pi d_1$, $j_2 \tau = \pi j_1$.

REMARKS 2.5 r. (i) If $\rho = 1$ and Φ is an $\mathfrak{R}_1$-structure map for X, then $\tau(X) \circ \Phi$ is an $\mathfrak{R}_2$-structure map for X. We write $\{\tau(X) \circ \Phi\} = \mathcal{F}_*\{\Phi\}$ and call this structure the $\mathcal{F}$-image of $\{\Phi\}$.

(ii) Let $\mu : M_1 \to M_2$ be obtained by restricting $\rho \times \tau$ to M_1. Then $\overline{\mathcal{F}} = (\rho, \rho, \mu)$ is a map $\overline{\mathfrak{R}}_1 \to \overline{\mathfrak{R}}_2$. If $\{\sigma\}$ is a strong $\mathfrak{R}_1$-structure and $\rho = 1$, then

$$\mathcal{F}_*\{t_1(X) \circ \sigma\} = \{t_2(X) \bullet \overline{\mathcal{F}}_*(\sigma)\}. \tag{2.6 r}$$

(iii) We describe $\mathcal{S} = (1, d, t) : \overline{\mathfrak{R}} \to \mathfrak{R}$ as the *canonical* map from $\overline{\mathfrak{R}}$ to $\mathfrak{R}$. Then $\mathcal{F}\mathcal{S} = \mathcal{S}\overline{\mathcal{F}}$ for any $\mathcal{F} : \mathfrak{R}_1 \to \mathfrak{R}_2$. The $\mathcal{S}$-image of a strong $\mathfrak{R}$-structure $\{\sigma\}$ for X is just the $\mathfrak{R}$-structure associated with $\{\sigma\}$. (Compare (2.6 r)).

(iv) A map $\mathcal{F} : \mathfrak{R}_1 \to \mathfrak{R}_2$ induces a natural transformation of fibres $\nu : N_1 \to N_2$.

DEFINITION 2,7 r. *Let* X_i *be* $\mathfrak{R}_i$-*structured by* $\Phi_i : R_i(X_i) \to T_i(X_i)$, $i = 1, 2$, *and let* $\mathcal{F} : \mathfrak{R}_1 \to \mathfrak{R}_2$ *be a map. A map* $f : X_1 \to X_2$ *in* $\mathcal{C}$ *is called an* $\mathcal{F}$-homomorphism *if the diagram*

$$\begin{array}{ccccc}
R_1(X_1) & \xrightarrow{R_1(f)} & R_1(X_2) & \xrightarrow{\rho(X_2)} & R_2(X_2) \\
\downarrow \Phi_1 & & & & \downarrow \Phi_2 \\
T_1(X_1) & \xrightarrow{T_1(f)} & T_1(X_2) & \xrightarrow{\tau(X_2)} & T_2(X_2)
\end{array}$$

is homotopy-commutative.

REMARKS 2.8 r. (i) The most important cases of this notion seem to be (a) when $\mathcal{F} = 1$, and (b) when $f = 1$.

(ii) The horizontal maps of the diagram may just as well be written $R_1(X_1) \xrightarrow{\rho(X_1)} R_2(X_1) \xrightarrow{R_2(f)} R_2(X_2)$ and $T_1(X_1) \xrightarrow{\tau(X_1)} T_2(X_1) \xrightarrow{T_2(f)} T_2(X_2)$ respectively.

Let X_i be strongly $\mathfrak{R}_i$- structured by $\sigma_i : R_i(X_i) \to M_i(X_i)$ and let $\mathfrak{F} : \mathfrak{R}_1 \to \mathfrak{R}_2$ be a map. Then we omit the proof of the following proposition.

PROPOSITION 2.9 r. *If $f : X_1 \to X_2$ is an $\overline{\mathfrak{F}}$-homomorphism with respect to the strong structures $\{\sigma_1\}$, $\{\sigma_2\}$ it is an $\mathfrak{F}$-homomorphism with respect to the associated structures.*

Given any category $\mathcal{C}$, let $\mathcal{C}^2$ designate the *category of pairs* derived from $\mathcal{C}$; thus, an object of $\mathcal{C}^2$ is a map f of $\mathcal{C}$ and a map in $\mathcal{C}^2$ from f to f' is a pair of maps (u, v) in $\mathcal{C}$ such that $f'u = vf$. We describe now a procedure for inducing a right structure system $\mathfrak{R}^2$ over $\mathcal{C}^2$ from a right structure system $\mathfrak{R}$ over $\mathcal{C}$. Precisely, given $\mathfrak{R} = (R, P, T; d, j)$ we define $\mathfrak{R}^2 = (R^2, P^2, T^2; d^2, j^2)$ as follows. For any map $f : X \to Y$ in $\mathcal{C}$ and any map $(u, v) : f \to f'$ in $\mathcal{C}^2$,

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{v} & Y' \end{array}$$

we set $R^2(f) = R(X)$, $R^2(u, v) = R(u)$, $P^2(f) = P(Y)$, $P^2(u, v) = P(v)$, $T^2(f) = T(Y)$, $T^2(u,v) = T(v)$, $d^2(f) = P(f) \circ d(X) = d(Y) \circ R(f)$, $j^2(f) = j(Y)$. We call $\mathfrak{R}^2$ the structure system over $\mathcal{C}^2$ induced by $\mathfrak{R}$. It is plain how a map $\mathfrak{F} : \mathfrak{R}_1 \to \mathfrak{R}_2$ induces a map $\mathfrak{F}^2 : \mathfrak{R}_1^2 \to \mathfrak{R}_2^2$; it is also clear that if N^2 is the fibre of $\mathfrak{R}^2$, then

$$N^2(f) = N(Y). \tag{2.10 r}$$

On the other hand $\overline{\mathfrak{R}^2} \neq \overline{\mathfrak{R}}^2$; we will, where no ambiguity is to be feared, say that f is $\mathfrak{R}$-*structured* if it is $\mathfrak{R}^2$- structured.

PROPOSITION 2.11 r. *If either X or Y is $\mathfrak{R}$-structured then $f : X \to Y$ may be $\mathfrak{R}$-structured.*

PROOF. If X is structured by $\Phi : RX \to TX$ we may structure f by $Tf \circ \Phi$. If Y is structured by $\psi : RY \to TY$ we may structure f by $\psi \circ Rf$.

It is plain how a covariant functor $F : \mathfrak{D} \to \mathcal{C}$ induces a transformation F^* of right structure systems over $\mathcal{C}$ into right structure systems over $\mathfrak{D}$. In particular let $F_1 : \mathcal{C} \to \mathcal{C}^2$ be the functor $F_1 X = 1_X$. For this functor F we have evidently

$$F_1^* \mathfrak{R}^2 = \mathfrak{R}. \tag{2.12 r}$$

It is also plain that a functor $D : \mathfrak{T} \to \mathfrak{T}$ induces a transformation D_* of right structure systems (over a given category $\mathcal{C}$).

We now turn briefly to *left* structure systems. We will be content to give the definition and leave to the reader the propositions and auxiliary definitions corresponding to (2.2r)-(2.12r).

DEFINITION 2.11. *A* left structure system $\mathfrak{L}$ *over* $\mathcal{C}$ *is a triple of covariant functors* $L, W, S : \mathcal{C} \to \mathcal{T}$ *together with a pair of natural transformations* $d : W \to L$, $j : W \to S$. *An object* $X \in \mathcal{C}$ *is said to be* $\mathfrak{L}$-structured *if it is furnished with a map* $\Phi : SX \to LX$ *such that*

$$\Phi \circ j(X) \simeq d(X) : WX \to LX.$$

The map Φ *is called an* $\mathfrak{L}$- structure map *for* X *and its homotopy class* $\{\Phi\}$ *an* $\mathfrak{L}$-structure *for* X.

We now give a number of examples of the concepts thus far introduced.

EXAMPLE 2.13. Let $f : Y \to X$ be a map in $\mathcal{T}$, and let C_f be the (based) mapping cylinder of f. Then there are maps

$$Y \xrightarrow{i} C_f \underset{\lambda}{\overset{k}{\rightleftarrows}} X, \qquad ki = f,$$

where i is an inclusion cofibration and k is a homotopy equivalence with inverse λ. We may use i and λ to embed Y and X in C_f. Set $R(f) = X$, $P(f) = X^n$, $d(f) =$ diagonal map; let $T(f)$ be the subspace of C_f^n consisting of n-triples $(z_1, \dots, z_n)$ where at least one $z_i \in Y$, and let $j(f) : T(f) \to X^n$ be the restriction of k^n to $T(f)$. The right structure system $\mathcal{G}_n = (R, P, T; f, j)$ over $\mathcal{T}^2$ is called the *n-genus* structure system. A $\mathcal{G}_n$-structure for f is then an n-genus structure for f and we say that genus $f < n$ iff admits an n-genus structure.

The notion of genus (originally due to Borsuk who applied it only to covering maps[5]) was defined for arbitrary fibre maps by A.S. Svarc [16] and later, but independently, for principal fibre bundles by M. Ginsberg [11]. It is not hard to show that our definition coincides with theirs if f is a fibre map[6], except that we lower the value of the genus by 1, so that, with our convention, genus $f = 0$ if and only if f is a domination.

Let $F_o : \mathcal{T} \to \mathcal{T}^2$ be the functor which associates with X the map $* \to X$. We set $\mathcal{K}_n = F_o^*(\mathcal{G}_n)$ and call $\mathcal{K}_n$ the *n-cat* structure system over $\mathcal{T}$ and write *cat* $X < n$ if X admits an n-cat structure. This notion coincides (except for the shift of 1) with that of Lusternik-Schnirelmann category.

We may now introduce the $\mathcal{K}_n^2$- structure system over $\mathcal{T}^2$ which we call the n-cat structure system over $\mathcal{T}^2$ and write *cat* $f < n$ if f admits an n-cat structure. Notice that $\mathcal{G}_n \neq \mathcal{K}_n^2$; in fact $\mathcal{K}_n = F_o^*(\mathcal{G}_n) = F_1^*(\mathcal{K}_n^2)$.

To justify the notation genus $f < n$ we should show that if f admits an n-genus structure it admits an $(n+1)$-genus structure. This is best done by exhibiting a map

$\mathcal{F} = (\rho, \pi, \tau) : \mathcal{G}_n \to \mathcal{G}_{n+1}$ with $\rho = 1$. In fact, for $f : Y \to X$, we set

$$\pi(f)(x_1, \ldots, x_n) = (x_1, \ldots, x_n, x_n), \quad x_i \in X$$
$$\tau(f)(z_1, \ldots, z_n) = (z_1, \ldots, z_n, z_n), \quad z_i \in C_f.$$

We study the fibre of the structure system $\mathcal{G}_n$. Let K be the fibre of k and F the fibre of f. Thus we have a fibration

(2.14) $$(K, F) \to (C_f, Y) \xrightarrow{k} (X, X),$$

since $k | Y = f$. If k is replaced by the equivalent fibre map $\bar{k} : \bar{C}_f \to X$ and f by the equivalent fibre map $\bar{f} : \bar{Y} \to X$ then $\bar{Y} \subseteq \bar{C}_f$ and $\bar{k} | \bar{Y} = \bar{f}$. Thus (K, F) is strictly the fibre of $\bar{k} : (\bar{C}_f, \bar{Y}) \to (X, X)$. Now $(C_f^n, T(f))$ is just the n^{th} Cartesian power of the pair[7] (C_f, Y),

$$(C_f^n, T(f)) = (C_f, Y)^n.$$

It is easy to verify, using superscript bars as above for equivalent fibre spaces and equivalent fibre maps, that $\overline{K^n} = (\bar{K})^n$ and $(\bar{C}_f, \bar{Y})^n = \overline{(C_f, Y)^n}$. Thus $\bar{k}^n | \overline{T(f)} = \overline{j(f)}$ and we may raise (2.14) to the n^{th} power to obtain the fibration

(2.15) $$(K, F)^n \to (C_f^n, T(f)) \xrightarrow{k^n} (X^n, X^n).$$

It follows that $N(f)$ is given by

$$(K, F)^n = (K^n, N(f)).$$

Notice that K is *contractible* since $k : C_f \to X$ is a homotopy equivalence.

EXAMPLE 2.17. Let $\mathcal{R}$ be an arbitrary right structure system over $\mathcal{C}$. Then, as described by Peterson in [15] we may obtain from $\mathcal{R}$ the so-called associated weak structure system $\mathcal{R}_w$; this is a natural generalization of the definition of *weak cat* in [4]. Precisely, let $\mathcal{R} = (R, P, T; d, j)$. Let $q : P \to Q$ be the cofibre of j and let $j_w : T_w \to P$ be the fibre of q. Then $\mathcal{R}_w = (R, P, T_w; d, j_w)$. Moreover there is evidently a natural transformation $\tau : T \to T_w$ such that $j_w \tau = j$. Thus $(1, 1, \tau)$ is a map $\mathcal{R} \to \mathcal{R}_w$ so that if X may be $\mathcal{R}$-structured it may be *weakly* $\mathcal{R}$-structured. The converse is, in general false (see [4]) but has been proved to hold under certain connectivity conditions if $\mathcal{R} = \mathcal{K}_n$ (see [3]). Notice that

(2.18) $$N(\mathcal{R}_w) = \Omega Q.$$

The suspension function $\Sigma : \mathcal{T} \to \mathcal{T}$ transforms every right homotopy system $\mathcal{R}$ into a right homotopy system $\Sigma_* \mathcal{R}$. It has been proved (see [3, 10]) that the *conilpotency* of X is less than n, *conil* $X < n$, if and only if X admits a $\Sigma_* \mathcal{K}_{nw}$-structure. It is not difficult to describe conilpotency in terms of structures so that this result expresses, in some sense, the equivalence of two structures.

EXAMPLE 2.19. We give here an example of a left structure system. Let $\mathcal{C} = \mathcal{T}$ and let $L(X) = X$, $W(X) = X \vee X$, $S(X) = X \times X$; $d(X) : X \vee X \to X$ is the folding map and

$j(X) : X \vee X \to X \times X$ is the inclusion. Then $\mathcal{H} = (L, W, S; d, j)$ is a left structure system and an $\mathcal{H}$-structure map for X is just an H-space multiplication on X. The cofibre of $\mathcal{H}$, U, is given by $U(X) = X \# X$. If X_i is furnished with the $\mathcal{H}$-structure $\{\Phi_i\}$, $i = 1, 2$, then a homomorphism $f : X_1 \to X_2$ (with $\mathcal{F} = 1 : \mathcal{H} \to \mathcal{H}$) is just an H-map. Now the fibre of j is the «flat» product $X \flat X$, embedded by $i : X \flat X \to X \vee X$ and X admits a weak $\mathcal{H}$-structure if and only if $di \simeq 0 : X \flat X \to X$. It is proved in [10] that X admits a weak $\mathcal{H}$-structure if and only if $nil\ X \leq 1$, i.e., if and only if ΩX is homotopy-commutative. It is not difficult to describe nilpotency in terms of (left) structures, so this result may also be viewed as expressing the equivalence of two structures [8].

3. The general theory.

Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two (right) structure systems over $\mathcal{C}$ and let $\mathcal{F} : \mathcal{R}_1 \to \mathcal{R}_2$ be a map. Let $f : X_1 \to X_2$ in $\mathcal{C}$. Then the two questions we here consider are

(i) given an $\mathcal{R}_2$-structure $\{\Phi_2\}$ on X_2, can we furnish X_1 with an $\mathcal{R}_1$-structure $\{\Phi_1\}$ such that f is an $\mathcal{F}$-homomorphism; and

(ii) given an $\mathcal{R}_1$-structure $\{\psi_1\}$ on X_1, can we furnish X_2 with an $\mathcal{R}_2$-structure $\{\psi_2\}$ such that f is an $\mathcal{F}$-homomorphism ?

We are also interested in the possible uniqueness of $\{\Phi_1\}$ and $\{\psi_2\}$ but we lay less stress on that aspect in this section.

We first state and prove a basic lemma.

LEMMA 3.1 r. *Let*

$$\begin{tikzcd} F_1 \arrow[r, "i_1"] \arrow[d, "f"] & E_1 \arrow[r, "p_1"] \arrow[d, "e"] & B_1 \arrow[d, "b"] \\ F_2 \arrow[r, "i_2"] & E_2 \arrow[r, "p_2"] & B_2 \end{tikzcd}$$

be a commutative diagram in which the horizontal rows are fibre sequences and let $t : B_1 \to E_2$ *be a map such that* $p_2 t = b$. *Then*

(i) if f is 0-connected the obstructions to constructing a cross-section $s_1 : B_1 \to E_1$ *with* $es_1 \simeq t$ *belong to the groups* [9] $H^k(B_1 : \pi_k(f))$, $k = 1, 2, \ldots$;

(ii) if F_2 *is 0-connected the obstructions to constructing a cross-section* $s_2 : B_2 \to E_2$ *with* $s_2 b \simeq t$ *belong to the groups* $H^{k+1}(b; \pi_k(F_2))$, $k = 1, 2, \ldots$.

PROOF. (i) Let $F_2 \xrightarrow{\rho} Z \xrightarrow{q} B_1$ be the fibration over B_1 induced by b from the fibration $F_2 \xrightarrow{i_2} E_2 \xrightarrow{p_2} B_2$. Then t determines a cross-section $u : B_1 \to Z$. Now consider the diagram

$$\begin{tikzcd} F_1 \arrow[r, "i_1"] \arrow[d, "f"] & E_1 \arrow[r, "p_1"] \arrow[d, "m"] & B_1 \arrow[dl, shift left, "u"] \\ F_2 \arrow[r, "\rho"] & Z \arrow[ur, shift left, "q"] & \end{tikzcd}$$

The problem then is to pull u back into E_1; that is, to construct $s : B_1 \to E_1$ with $ms \simeq u$. For then, if $n : Z \to E_2$ is the map induced by e, so that $nm = e$, $nu = t$, we have $es = nms \simeq nu = t$ and $p_1 s = qms \simeq qu = 1$, so that s is homotopic to a cross-section s_1 of p_1 such that $es_1 \simeq t$. Now the problem of pulling back, or compressing, u into E_1 is a standard obstruction problem[10] and the obstructions all lie in the groups $H^k(B_1; \pi_k(m))$. Now $(f, m)_* : \pi_k(i_1) \cong \pi_k(\rho)$ so that (see [7]) $(i_1, \rho)_* : \pi_k(f) \cong \pi_k(m)$, and (i) is proved. To prove (ii) we consider the simple diagram

$$\begin{tikzcd} & & B_1 \arrow[dl, "t"'] \arrow[d, "b"] \\ F_2 \arrow[r, "i_2"] & E_2 \arrow[r, "p_2"] & B_2 \end{tikzcd}$$

The problem now is that of extending a cross-section over B_1 to B_2, for $p_2 t = b$. This is a standard obstruction problem[11] and the obstructions all lie in the groups $H^{k+1}(b; \pi_k(F_2))$.

REMARK 3.2 r. It is convenient to have «relative» forms of the two parts of this lemma in which we suppose that $s_1(s_2)$ is already given on some subcomplex $A_1(A_2)$ of $B_1(B_2)$. In case (i), the case we shall be applying in the next section, the obstruction groups are simply relativized as $H^k(B_1, A_1; \pi_k(f))$. In case (ii) we have to replace b by the «union» of b and the inclusion map $A_2 \to B_2$; we will not make an explicit statement.

REMARK 3.3 r. We observe that precisely the same type of reasoning shows that if s_1, s'_1 are two cross-sections of p_1 such that $es_1 \simeq es'_1$, then the obstructions to a homotopy between s_1 and s'_1 lie in $H^{k-1}(B_1; \pi_k(f))$. A similar statement holds for case (ii).

Now let $\mathfrak{F} = (\rho, \pi, \tau) : \mathfrak{R}_1 \to \mathfrak{R}_2$ be a map from the (right) structure system $\mathfrak{R}_1$ to the structure system $\mathfrak{R}_2$. Then (2.5r(iv)) $\mathfrak{F}$ induces a natural transformation $\nu : N_1 \to N_2$ of fibres, and a map $\overline{\mathfrak{F}} = (\rho, \rho, \mu) : \overline{\mathfrak{R}}_1 \to \overline{\mathfrak{R}}_2$ of the associated strong structure systems. Let $f : X_1 \to X_2$ be a map in $\mathcal{C}$; consider the commutative diagram

$$\begin{tikzcd} N_1(X_1) \arrow[r, "i_1"] \arrow[d, "\nu \circ N_1(f)"] & M_1(X_1) \arrow[r, "p_1"] \arrow[d, "\mu \circ M_1(f)"] & R_1(X_1) \arrow[d, "\rho \circ R_1(f)"] \\ N_2(X_2) \arrow[r, "i_2"] & M_2(X_2) \arrow[r, "p_2"] & R_2(X_2) \end{tikzcd} \qquad \begin{cases} \nu = \nu(X_2), \\ \mu = \mu(X_2), \\ \rho = \rho(X_2) \end{cases}$$

Recall that a strong $\mathfrak{R}_i$-structure for X_i is a homotopy class $\{\sigma_i\} : R_i(X_i) \to M_i(X_i)$ containing a cross-section σ_i to p_i, $i = 1, 2$; and that if X_i is strongly $\mathfrak{R}_i$-structured by σ_i, then $f : X_1 \to X_2$ is an $\overline{\mathfrak{F}}$-homomorphism if

$$\sigma_2 \circ \rho \circ R_1(f) \simeq \mu \circ M_1(f) \circ \sigma_1 .$$

We thus infer from Lemma 3.1r(i), taking $t = \sigma_2 \circ \rho \circ R_1(f) : R_1(X_1) \to M_2(X_2)$

THEOREM 3.5 r. *Let $\nu \circ N_1(f)$ be 0-connected, and let X_2 be strongly $\mathfrak{R}_2$-structured. Then the obstructions to a strong $\mathfrak{R}_1$-structure for X_1 such that f is an $\overline{\mathfrak{F}}$-homomorphism lie in the groups $H^k(R_1(X_1); \pi_k(\nu \circ N_1(f)))$.*

COROLLARY 3.6 r. *Suppose in addition that $\nu \circ N_1(f)$ is $(Q-1)$-connected and that $\dim R_1(X_1) \leq Q-1$. Then if X_2 is strongly $\mathfrak{R}_2$-structured by σ_2 we may give X_1 a strong $\mathfrak{R}_1$-structure σ, such that f is an $\overline{\mathfrak{F}}$-homomorphism. Moreover if $\dim R_1(X_1) \leq Q-2$ then $\{\sigma_1\}$ is uniquely determined.*

The last statement here follows from Remark 3.3 r. From Corollary 3.6 r and Propositions 2.3 r, 2.9 r we infer

THEOREM 3.7 r. *Suppose that $\nu \circ N_1(f)$ is $(Q-1)$-connected, $Q \geq 1$ and that $\dim R_1(X_1) \leq Q-1$. Then if X_2 is $\mathfrak{R}_2$-structured by Φ_2 we may give X_1 an $\mathfrak{R}_1$-structure Φ_1 such that f is an $\mathfrak{F}$-homomorphism.*

REMARK 3.8 r. We may invoke Remark 3.2 r in the following way. Let $X_o \subseteq X_1$ and suppose X_o already $\mathfrak{R}_1$-structured in such a way that $f | X_o$ is an $\overline{\mathfrak{F}}$-homomorphism. Then if $\nu \circ N_1(f)$ is $(Q-1)$-connected and $\dim (R_1(X_1) - R_1(X_o)) \leq Q-1$ we may extend the strong $\mathfrak{R}_1$-structure to the whole of X_1 in such a way that f is an $\overline{\mathfrak{F}}$-homomorphism. There is a corresponding uniqueness statement if $\dim(R_1(X_1) - R_1(Y_1)) \leq Q-2$.

REMARK 3.9 r. In 3.6 r - 3.8 r we may replace the dimension condition by an appropriate cohomology dimension condition.

EXAMPLE 3.10 r. Take $f = 1 : X \to X$, $\mathfrak{R}_1 = \mathfrak{K}_n$, $\mathfrak{R}_2 = \mathfrak{K}_{nw}$. Then if $PX = X^n$, $TX \subseteq X^n$ is the «fat wedge»[12] and $QX = PX / TX$, we may identify $N(\mathfrak{K}_n)$ with the space $E(P; T, *)$ of paths on P originating in T and $N(\mathfrak{K}_{n\omega}) = \Omega Q$. Moreover $\nu : N(\mathfrak{K}_n) \to N(\mathfrak{K}_{nw})$ is the map induced by pinching T. We may identify $\nu_* : \pi_k(N(\mathfrak{K}_n)) \to \pi_k(N(\mathfrak{K}_{nw}))$ with the «excision» homomorphism $\pi_{k+1}(P, T) \to \pi_{k+1}(Q)$. It then follows from the Blakers-Massey theorem that if X is $(q-1)$-connected then ν is $((n+1)q-2)$-connected. Thus we infer the Berstein-Ganea theorem [3] that if X is $(q-1)$-connected and $\dim X \leq (n+1)q-2$ then $w \operatorname{cat} X < n \Rightarrow \operatorname{cat} X < n$.

Taking $t = \mu \circ M_1(f) \circ \sigma$, in (3.4 r) and invoking Lemma 3.1 r (ii) we infer

THEOREM 3.11 r. *Let $N_2(X_2)$ be 0-connected and let X_1 be strongly $\mathfrak{R}_1$-structured. Then the obstructions to a strong $\mathfrak{R}_2$-structure for X_2 such that f is an $\overline{\mathfrak{F}}$-homomorphism lie in the groups $H^{k+1}(\rho \circ R_1(f); \pi_k(N_2(X_2)))$.*

COROLLARY 3.12 r. *Suppose in addition that $N_2(X_2)$ is $(Q-1)$-connected and cohom. $\dim \rho \circ R_1(f) \leq Q$. Then if X_1 is strongly $\mathfrak{R}_1$-structured by σ_1 we may give X_2 a strong $\mathfrak{R}_2$-structure σ_2 such that f is an $\overline{\mathfrak{F}}$-homomorphism. Moreover if* *cohom. dim* $\rho \circ R_1(f) \leq Q-1$ *then* $\{\sigma_2\}$ *is uniquely determined.*

Of course cohom. dim g relates to the vanishing of certain cohomology groups of g. We now apply Propositions 2.3r, 2.9r to infer

THEOREM 3.13r. *Suppose that* $N_2(X_2)$ *is* $(Q-1)$*-connected,* $Q \geq 1$*, and that cohom. dim* $\rho \circ R_1(f) \leq Q$. *Then if* X_1 *is* $\mathcal{R}_1$*-structured by* Φ_1 *we may give* X_2 *an* $\mathcal{R}_2$*-structure* Φ_2 *such that* f *is an* $\mathcal{F}$*-homomorphism.*

We conclude this section by stating without proof the counterparts of Theorems 3.5r, 3.11r for left structure systems.

Let $\mathcal{L}_i = (L_i, W_i, S_i; d_i, j_i)$, $i = 1, 2$ be two left structure systems over $\mathcal{C}$ and let $\mathcal{F} = (\lambda, \omega, \chi) : \mathcal{L}_1 \to \mathcal{L}_2$ be a map. Let U_i be the cofibre of $\mathcal{L}_i$ so that $\mathcal{F}$ induces $u : U_1 \to U_2$. Let $f : X_1 \to X_2$ be a map in $\mathcal{C}$.

THEOREM 3.5 l. *Let* $L_2(f) \circ \lambda$ *be* 0*-connected, and let* X_2 *be strongly* $\mathcal{L}_2$*-structured. Then the obstructions to a strong* $\mathcal{L}_1$*-structure for* X_1 *such that* f *is an* $\overline{\mathcal{F}}$*-homomorphism lie in the groups* $H^k(U_1(X_1); \pi_k(L_2(f) \circ \lambda))$.

THEOREM 3.11 l. *Let* $L_1(X_1)$ *be* 0*-connected, and let* X_1 *be strongly* $\mathcal{L}_1$*-structured. Then the obstructions to a strong* $\mathcal{L}_2$*-structure for* X_2 *such that* f *is an* $\overline{\mathcal{F}}$*-homomorphism lie in the groups* $H^{k+1}(U_2(f) \circ u; \pi_k(L_2(X_2)))$.

EXAMPLE 3.14 l. Let $X \in \mathcal{T}$ and let $X(m, \dots, n)$ be obtained from X by killing its homotopy groups in dimensions $< m$ and $> n$. Then there is a fibration

$$X(n, \dots, \infty) \to X \to X(1, \dots, n-1).$$

Suppose that X is an H-space. Then we may apply Theorem 3.5 l to show that $X(n, \dots, \infty)$ is an H-space and Theorem 3.11 l to show that $X(1, \dots, n-1)$ is an H-space. It thus follows that $X(m, \dots, n)$ is an H-space for all $1 \leq m \leq n \leq \infty$. We return to this example in section 5.

There are of course statements on the obstructions to the uniqueness of strong $\mathcal{L}_1$-($\mathcal{L}_2$-) structures for $X_1(X_2)$ compatible with given strong $\mathcal{L}_2$-($\mathcal{L}_1$-) structures for $X_2(X_1)$, analogous to those for right structure systems. We leave the details to the reader[13].

4. The n-genus structure.

In this section we apply the main results of section 3 to the structure system $\mathcal{G}_n$. We have already computed the fibre N of $\mathcal{G}_n$ in (2.15, 16). We will take $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{G}_n$, $\mathcal{F} = 1$ throughout this section and will refer to $\mathcal{F}$-homomorphims, $\overline{\mathcal{F}}$-homomorphims as *homomorphims, strong homomorphims* respectively. Thus to apply Theorems 3.7r, 3.13r we must compute for a given map $f : X_1 \to X_2$ in $\mathcal{T}$, the connectivities of $N(f)$,

$N(X_2)$. The computation is rendered simple by the following lemma.

LEMMA 4.1. *Let* $\gamma = (Y, B, A)$ *be a triple*[14] *in* $\mathcal{T}$, $A \subseteq B \subseteq Y$. *Let* A *be* $(r-1)$*-connected,* (B, A) $(s-1)$*-connected and* (Y, B) $(t-1)$*-connected. Then* γ *is homotopically equivalent to a CW-triple* $\bar{\gamma} = (\bar{Y}, \bar{B}, \bar{A})$ *in which* $\bar{A} - *$, $\bar{B} - \bar{A}$, $\bar{Y} - \bar{B}$ *has no cells of dimension* $< k$, *where* $k = r, s, t$ *respectively.*

PROOF. Let $S(X)$ be the singular complex of X, for any $X \in \mathcal{T}$. Then $S(A) \subseteq S(B) \subseteq S(Y)$. Choose a minimal complex $M(A)$ for A in $S(A)$. Then choose a minimal extension $M(B)$ of $M(A)$ in $S(B)$ which is homotopically equivalent to $S(B)$. Finally choose a minimal extension $M(Y)$ of $M(B)$ in $S(Y)$ which is homotopically equivalent to $S(Y)$. Plainly the Milnor geometric realization of $(M(Y), M(B), M(A))$ yields the required triple $\bar{\gamma}$.

PROPOSITION 4.2. *(i) Let* (K, F) *be a* $(q-1)$*-connected pair in* $\mathcal{T}$. *Then* $(K, F)^n$ *is* $(nq-1)$*-connected. (ii) Let* $h : K_1, F_1 \to K_2, F_2$ *be a map of pairs. Suppose* K_i *contractible,* (K_i, F_i) $(q-1)$*-connected,* $i = 1, 2$, *and* h $(m-1)$*-connected, Then* $h^n : (K_1, F_1)^n \to (K_2, F_2)^n$ *is* $(m-1+(n-1)q)$*-connected.*

PROOF. (i) By lemma 4.1 we may suppose that $K - F$ has no cells of dimension $< q$. Then if $(K, F)^n = (K^n, N)$, $K^n - N$ has no cells of dimension $< nq$.

(ii) We may replace h by an inclusion map and we may then suppose that $K_1 = K_2 (= K)$. Thus we have a triple (K, F_2, F_1), where F_1 is $(q-2)$-connected, (F_2, F_1) is $(m-2)$-connected and (K, F_2) is $(q-1)$-connected. Moreover we may assume $m > q$ since if $m \leq q$ the conclusion is a trivial consequence of (i). We now apply Lemma 4.1 so that we may assume (K, F_2, F_1) minimal in the sense that $F_1 - *$ has no cells of dimension $\leq q-2$, $F_2 - F_1$ has no cells of dimension $\leq m-2$ and $K - F_2$ has no cells of dimension $\leq q-1$. Then $K - *$ has no cells of dimension $\leq q-2$ and all $(q-1)$-cells of K are in F_1. Thus if $(K, F_i)^n = (K^n, N_i)$, $N_2 - N_1$ has no cells of dimension $\leq m-2+(n-1)q$. It follows that $h^n | N_1 : N_1 \to N_2$ is $(m-2+(n-1)q)$-connected, whence the result follows (since K^n is contractible).

We can now state the main theorem of this section.

THEOREM 4.3. *Let* $f_i : Y_i \to X_i$ *be* $(q-1)$*-connected maps in* $\mathcal{T}$, $i = 1, 2$, *and let* $\Phi : f_1 \to f_2$ *be an* $(m-1)$*-connected map in* $\mathcal{T}^2$. *Let* N *be the fibre of the* n*-genus structure* $\mathcal{G}_n$. *Then* $N(f_i)$ *is* $(nq-2)$*-connected and* $N(\Phi)$ *is* $(m-2+(n-1)q)$*-connected. Thus, if also* f_2 *is* $\mathcal{G}_n$*-structured and* $\dim X_1 \leq m-2+(n-1)q$, f_1 *may be* $\mathcal{G}_n$*-structured in such way that* Φ *is a homomorphism.*

PROOF. If F_i is the fibre of f_i then F_i is $(q-2)$-connected. If Φ induces $g : F_1 \to F_2$

then g is $(m-2)$- connected. Recall from (2.15, 16) that $(K_i, F_i)^n = K_i^n, N(f_i)$, and Φ induces $h : K_1, F_1 \to K_2, F_2$ such that $h\,|F_1 = g$. Theorem 4.3 now follows from Proposition 4.2 and Theorem 3.7 r.

REMARK 4.4. Of course if f_2 is strongly $\mathcal{G}_n$- structured then f_1 may be strongly $\mathcal{G}_n$- structured in such a way that Φ is a strong homomorphism.

Taking the special case $Y_i = *$, we obtain

COROLLARY 4.5. *Let X_i be $(q-1)$-connected spaces, $i = 1, 2$ and let $f : X_1 \to X_2$ be an $(m-1)$- connected map. Then if X_2 is $\mathcal{K}_n$- structured and $\dim X_1 \leq (m-2)+(n-1)q$, X_1 may be $\mathcal{K}_n$- structured in such a way that f is a homomorphism.*

Passing to the $\mathcal{K}_n^2$- structure system on $\mathcal{T}^2$ and invoking Theorem 3.7 r, (2.10 r) and the proof of Theorem 4.3 we infer

THEOREM 4.6. *Let $f_i : X_i \to Y_i$ be a map of X_i into the $(q-1)$- connected space Y_i, $i = 1,2$, and let $\Phi = (\alpha, \beta) : f_1 \to f_2$ be a map such that β is $(m-1)$- connected. Then if f_2 is $\mathcal{K}_n^2$- structured and $\dim X_1 \leq (m-2)+(n-1)q$, f_1 may be $\mathcal{K}_n^2$- structured in such a way that Φ is a homomorphism.*

REMARK 4.7. (i) Corollary 4. 5 could, of course, in principle be deduced from Theorem 4.6. (ii) If, in Corollary 4. 5, we assume that $X_o \subseteq X_1$ is already $\mathcal{K}_n$- structured in such a way that $f|\,X_o$ is a homomorphism then we may extend the structure to X_1 in such a way that f is a homomorphism provided $\dim(X_1 - X_o) \leq (m-2)+(n-1)q$ (see Remark 3.8 r).

We draw some conclusions from Corollary 4.5 and Theorem 4.6.

THEOREM 4.8. *Let X be a countable connected CW-complex with $\operatorname{cat} X < n$, $n \geq 2$. Then for any section $X^{(k)}$ of X, $k \geq 1$, $\operatorname{cat} X^{(k)} < n$. Moreover if X is $\mathcal{K}_n$- structured we may structure all the sections simultaneously in such a way that the inclusions $X^{(k)} \subseteq X^{(k+1)}$, $X^{(k)} \subseteq X$ are homomorphism.*

PROOF. Define $X^{(o)} = *$ and obtain a $\mathcal{K}_n$- structure successively on each $X^{(k)}$. Thus let us suppose a structure already defined on $X^{(k)}$, $k \geq 0$, in such a way that $X^{(k)} \subseteq X$ is a homomorphism; we will extend this structure to $X^{(k+1)}$. Now the inclusion map $X^{(k+1)} \subseteq X$ is $(k+1)$- connected and X, $X^{(k+1)}$ are certainly 0- connected. Thus we may apply Remark 4. 7 (ii) with $X_o = X^{(k)}$, $X_1 = X^{(k+1)}$, $m = k+2$, $q = 1$. We have

$$\dim X_1 - X_o = k+1 \leq k+(n-1)\,.\,1\,,$$

provided $n \geq 2$. This proves the theorem. Of course the case $n = 1$ is properly excluded since the sections of a contractible space are not necessarily contractible.

REMARK 4.9. Theorem 4.8. is a special case of theorem on induced $\mathcal{G}_n$- structures (as well as of a theorem on induced $\mathcal{K}_n^2$ - structures) on the sections of a 0 - connected cellular map which the reader may readily enunciate.

Let X be a connected CW - complex. A pair consisting of a complex X_k and a map $i_k : X_k \to X$ is called a homology k - section of X if $k \geq 2$, $\dim X_k \leq k+1$, and

$$\begin{array}{llll} (4.10) & \text{(i)} & i_* : \pi_r(X_k) \cong \pi_r(X), & r < k \\ & & i_* \pi_k(X_k) = \pi_k(X), & \\ & \text{(ii)} & i_* : H_r(X_k) \cong H_r(X), & r \leq k \\ & & H_r(X_k) = 0 & \text{if } r > k. \end{array}$$

Notice that if X is 1 - connected then (4.10i) effectively just asserts that X_k is 1 - connected. Moreover it is known [6, 14, 17] that such X_k always exist and that we may take $X_2 \subseteq \ldots \subseteq X_k \subseteq X_{k+1} \subseteq \ldots$, $h : \cup X_k \simeq X$, with $h | X_k = i_k$ if X is 1-connected.

THEOREM 4.11. *Let X be a countable connected CW - complex with $cat\, X < n$. Then (i) if X is 1 - connected and $n \geq 2$, $cat\, X_k < n$ for any homology k - section. Moreover if X is $\mathcal{K}_n$ - structured we may structure all the X_k of a homology decomposition simultaneously so that $X_k \to X_{k+1}$ and $i_k : X_k \to X$ are homomorphisms. (ii) If $n \geq 3$ then $cat\, X_k < n$ for any homology k - section; and X_k may be structured so that i_k is a homomorphism.*

PROOF. (i) As for Theorem 4.8, we proceed inductively and have to satisfy an inequality $k+2 \leq k+(n-1)q$ with $q = 2$ to proceed from X_k to X_{k+1}. The non-relativized form of Corollary 4.5 deals with any homology k - section and thus with assertion (ii) since we then need $(n-1) \geq 2$.

REMARK 4.12. The case $n = 2$ of Theorem 4.11(i), asserting that $cat\, X_k < 2$ if $cat\, X < 2$ has already been proved by Curjel [5].

Let X be a CW - complex, let $Y = X(1, \ldots, m)$ and let $p : X \to Y$ be the projection (see Example 3.14 1). Then it was proved in [2] that

$$(4.13) \qquad cat_m X = cat\, p,$$

where cat_m is the *m - dimensional category* of Fox [8]. Of course $cat_m X \leq cat X$. We prove

THEOREM 4.14. *If X is a $(q-1)$ - connected countable CW - complex, $q \geq 1$, and if $cat_m X < n$, then $cat\, X < n$ provided that $\dim X \leq m+(n-1)q$. In particular, $cat_m X = cat\, X$ if $\dim X \leq m+(cat_m X)q$.*

PROOF. Consider the map $(1, p) : 1 \to p$,

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & Y \end{array}$$

and apply Theorem 4.6. Then X and Y are $(q-1)$- connected and p is $(m+1)$-connected. Thus, if $cat\, p < n$, $cat\, X < n$ provided $dim\, X \leq m+(n-1)q$. The second assertion follows by taking $n = cat_m X + 1$.

REMARK 4.15(i). Taking $q = 1$, $m = 1$, we recover the known result ([1]) that if X is connected and $cat_1 X < n$, $dim\, X \leq n$, then $cat\, X < n$. (ii) We may similary obtain results relating genus to m- dimensional genus.

We close this section by giving one application of Theorem 3.13r just by way of example. From that theorem, and part of Theorem 4.3 we infer

THEOREM 4.16. *Let* $f_i : Y_i \to X_i$ *be maps in* $\mathcal{T}$ *with* f_2 $(q-1)$*- connected. Let* $\Phi = (\beta, \alpha) : f_1 \to f_2$ *with cohom. dim* $\alpha \leq nq-1$. *Then if* f_1 *is* $\mathcal{G}_n$*- structured we may give* f_2 *a* $\mathcal{G}_n$*- structure such that* Φ *is a homomorphism.*

COROLLARY 4.17. *Let* X *be a countable connected CW- complex and let* $q : X \to Y$ *project* X *onto* X *modulo its* k*- section,* $Y = X / X^{(k)}$. *Then if* X *is* $\mathcal{K}_n$*-structured,* $n \geq 2$*, we may structure* Y *so that* q *is a homomorphism.*

COROLLARY 4,18. *Let* X *be a countable connected CW-complex and let* $q : X \to Z$ *project* X *onto* X *modulo a homology* k*- section,* $Z = X / i_k X_k$. *Then if* X *is* $\mathcal{K}_n$*- structured,* $n \geq 2$*, we may structure* Z *so that* q *is a homomorphism.*

REMARK 4.19. These corollaries, except for the case of 4.18 when $n = 2$ and X is not 1- connected, follow from Theorems 4.8 and 4.11 respectively by means of a different type of argument (see Theorem 3.2r of [15]).

5. H and H'-structures.

The results we have obtained in section 4 for special structure systems have made no mention of the uniqueness of the structures obtained. The reason for this is, as mentioned in the Introduction, that our uniqueness theorems in section 3 refer to *strong* structures and we have no means in general of passing from the uniqueness of strong structure to the uniqueness of structure. Howewer there is an important special case in which this passage may be effected and we now proceed to describe this. For simplicity we will confine attention to single structure systems so that $\mathcal{R}_1 = \mathcal{R}_2 (= \mathcal{R})$ and $\mathcal{F} = 1$.

Peterson has pointed in [15] to a very good feature of the structure systems that have so far been discussed in the literature, namely that Ωj is a *domination*[15]. Although this is really too strong an assumption for the general theory, we are content to make it here, and will describe $\mathcal{R}$ as *good* if it has this property. Now consider a fibre map $v : Y \to B$ and let $g : A \to B$ be a map. Then g induces a fibre map $u : X \to A$ and we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ u\downarrow & & \downarrow v \\ A & \xrightarrow{g} & B \end{array}$$

This diagram induces a *dual Mayer-Vietoris sequence* in homotopy (see [12]),

$$(5.1) \qquad \ldots \to \Pi_m(D, X) \xrightarrow{\alpha_m} \Pi_m(D, A) \oplus \Pi_m(D, Y) \xrightarrow{\beta_m} \Pi_m(D, B) \xrightarrow{\gamma_m} \Pi_{m-1}(D, X)$$
$$\ldots \xrightarrow{\alpha_1} \Pi_1(D, A) \oplus \Pi_1(D, Y) \xrightarrow{\beta_1} \Pi_1(D, B) \xrightarrow{\gamma_1} \Pi(D, X) \xrightarrow{\alpha}$$
$$\Pi(D, A) \times \Pi(D, Y).$$

Here $\alpha_m(x) = (u_*(x), h_*(x))$, $\beta_m(a, y) = g_*(a)v_*(y)^{-1}$, and we need not make γ_m explicit. Thus β_1 in general fails to be a homomorphism. Although $\beta: \Pi(D, A) \times \Pi(D, Y) \to \Pi(D, B)$ is not in general defined we have exactness at $\Pi(D, A) \times \Pi(D, Y)$ in the sense that $(a, y) \in \alpha\Pi(D, X)$ if and only if $g_*(a) = v_*(y)$.

If we consider in particular the exact sequence (5.1) associated with the diagram (2.2 r)

$$\begin{array}{ccc} MX & \xrightarrow{t} & TX \\ p\downarrow & & \downarrow j \\ RX & \xrightarrow{d} & PX \end{array}$$

we see than an $\mathfrak{R}$-structure is essentially an element $(1, \{\Phi\}) \in \Pi(RX, RX) \times \Pi(RX, TX)$ and this element belongs to $\alpha\Pi(RX, MX)$ because $d_*(1) = j_*\{\Phi\}$. Thus we choose an associated strong structure by picking $\{\sigma\} \in \Pi(RX, MX)$ such that $\alpha\{\sigma\} = (1, \{\Phi\})$. We would thus certainly be interested in the case when α is injective. Reverting to the general case (5.1) we prove

PROPOSITION 5.2. *Suppose Ωv has a cross-section and D is an H'-space. Then*
$$\alpha: \Pi(D, X) \to \Pi(D, A) \times \Pi(D, Y)$$
is injective.

PROOF. Since D is an H'-space it is a retract of $\Sigma\Omega D$. Thus $\Pi(D, E)$ is a subset of $\Pi(\Sigma\Omega D, E)$ and it thus suffices to prove the proposition when D is a suspension space. But then α is a group-homomorphism so it suffices to show that the kernel of α is the zero subgroup. This however follows at once by the exactness of (5.1) since β_1 is surely surjective (indeed $v_*: \Pi_1(D, Y) \to \Pi_1(D, B)$ is surjective for any D).

We are thus led to the following theorem.

THEOREM 5.3 r. *Let $\mathfrak{R}$ be a good (right) structure system over $\mathcal{C}$. (i) If RX is an H'-space then each $\mathfrak{R}$-structure for X is associated with a unique strong structure. (ii) If $f: X_1 \to X_2$ is a homomorphism with respect to $\mathfrak{R}$-structures Φ_1, Φ_2, and if RX_1 is an H'-space then f is a strong homomorphism with respect to strong $\mathfrak{R}$-structures σ_1,*

σ_2 *associated with* Φ_1, Φ_2.

PROOF. (i) is already proved. To prove (ii) we consider the diagrams

$$\begin{array}{ccc} RX_1 & \xrightarrow{Rf} & RX_2 \\ \downarrow \Phi_1 & & \downarrow \Phi_2 \\ TX_1 & \xrightarrow{Tf} & TX_2 \end{array} \qquad\qquad \begin{array}{ccc} RX_1 & \xrightarrow{Rf} & RX_2 \\ \downarrow \sigma_1 & & \downarrow \sigma_2 \\ MX_1 & \xrightarrow{Mf} & MX_2 \end{array}$$

where $t_i \sigma_i = \Phi_i$, $i = 1, 2$. We are given that the first diagram is homotopy-commutative and we want to prove the same property of the second. By Proposition 5.2. it suffices to prove that $p_2 \circ \sigma_2 \circ Rf \simeq p_2 \circ Mf \circ \sigma_1$ and $t_2 \circ \sigma_2\, Rf \simeq t_2 \circ Mf \circ \sigma_1$.

Now $\quad p_2 \circ \sigma_2 \circ Rf \simeq Rf \simeq Rf \circ p_1 \circ \sigma_1 = p_2 \circ Mf \circ \sigma_1$,

and $\quad t_2 \circ \sigma_2 \circ Rf \simeq \Phi_2 \circ Rf \simeq Tf \circ \Phi_1 \simeq Tf \circ t_1 \circ \sigma_1 = t_2 \circ Mf \circ \sigma_1$.

Thus the theorem is proved. Notice that f is a strong homomorphism with respect to *any* strong structure $\{\sigma_2\}$ associated with $\{\Phi_2\}$. Combining this with the uniqueness part of Corollary 3.6r we have

THEOREM 5.4r. *Let* $\mathfrak{R}$ *be a good (right) structure system over* $\mathcal{C}$ *with fibre* N *and let* $f : X_1 \to X_2$ *be a map in* $\mathcal{C}$. *Suppose that* $N(f)$ *is* $(Q-1)$*- connected and* $R(X_1)$ *is an* H'*- space such that* $\dim R(X_1) \leq Q-2$. *Then if* X_2 *is* $\mathfrak{R}$*- structured by* Φ_2 *there exists a unique* $\mathfrak{R}$*- structure* $\{\Phi_1\}$ *for* X_1 *such that* f *is a homomorphism.*

PROOF. Let $\{\sigma_2\}$ be any strong structure associated with $\{\Phi_2\}$. Then there exists a unique structure $\{\sigma_1\}$ for X_1 such that f is a strong homomorphism. If now $\{\Phi'_1\}$ is any $\mathfrak{R}$- structure with respect to which f is a homomorphism and if $\{\sigma'_1\}$ is the unique strong structure associated with $\{\Phi'_1\}$ then, by Theorem 5.3r(ii), f is a strong homomorphism with respect to σ'_1, σ_2 so that $\{\sigma'_1\} = \{\sigma_1\}$ and so $\{\Phi'_1\}$ is uniquely determined as $t_*\{\sigma_1\}$.

REMARK 5.5r. There is no difficulty in extending Theorems 5.3r, 5.4r to arbitrary $\mathfrak{F}$- homomorphims.

We may apply Theorem 5.4r to the case $\mathfrak{R} = \mathfrak{K}_2$. Certainly $\mathfrak{K}_n$ is a good structure system for all $n \geq 2$ so we infer from Theorems 4.3, 5.4r

THEOREM 5.5r. *Let* X_i *be* $(q-1)$*- connected spaces,* $i = 1,2$ *and let* $f : X_1 \to X_2$ *be an* $(m-1)$*- connected map. Then if* X_2 *is* $\mathfrak{K}_2$*- structured and* $\dim X_1 \leq m-3+q$, X_1 *may be uniquely* $\mathfrak{K}_2$*- structured in such a way that* f *is a homomorphism.*

PROOF. By Corollary 4.5, X_1 may be $\mathfrak{K}_2$- structured and so is an H'- space. Now by Theorem 4.3 $N(f)$ is $(m-2+q)$- connected so the result follows from Theorem 5.4r. Now a $\mathfrak{K}_2$- structure is just a comultiplication. We thus infer from Theorems 4.8 and 5.5r.

THEOREM 5.6. *Let X be a countable connected CW-complex. Then if X admits a comultiplication so do all its sections. Moreover if X is 1-connected then a given comultiplication on X induces unique*[16] *comultiplications on the sections of X.*

COROLLARY 5.7. *If X is 1-connected and the comultiplication on X is commutative so are the comultiplications on all the sections of X.*

REMARK 5.8. Theorem 5.6 and Corollary 5.7 would be false if we did not require X to be 1-connected. For S^1 admits many comultiplications, not all commutative, but V^2 admits only one comultiplication !

Similary we deduce from Theorems 4.11, 5.5r

THEOREM 5.9. *Let X be a countable 1-connected CW-complex. Then if X admits a comultiplication so do all its homology sections. Moreover if X is 2-connected then a given comultiplication on X induces unique*[17] *comultiplications on the homology sections of X.*

COROLLARY 5.10. *If X is 2-connected and the comultiplication on X is commutative so are the comultiplications on all the homology sections of X.*

We are also able to apply Theorem 5.4r to handle the question of the associativity of a comultiplication. Consider the right structure system $\mathfrak{A} = (R, P, T; d, j)$ over $\mathfrak{T}$ where $RX = X$, $PX = X \times X \times X$, $TX = X \vee X \vee X$, dX is the diagonal map and jX is the inclusion. Plainly X admits an $\mathfrak{A}$-structure if and only if it is an H'-space. Indeed if $\psi: X \to X \vee X \vee X$ is an $\mathfrak{A}$-structure map we obtain a comultiplication by composing ψ with any of the three projections $X \vee X \vee X \to X \vee X$. Conversely if $\Phi: X \to X \vee X$ is a comultiplication then $(\Phi \vee 1) \circ \Phi$ and $(1 \vee \Phi) \circ \Phi$ are $\mathfrak{A}$-structure maps. Let F be the fibre of $\mathfrak{A}$. We prove

LEMMA 5.11. *Let X_1, X_2 be $(q-1)$-connected and let $f: X_1 \to X_2$ be an $(m-1)$-connected map. Then (i) FX_1 is $(2q-2)$-connected, (ii) Ff is $((m-2)+q)$-connected.*

PROOF. $FX = E(X \times X \times X; X \vee X \vee X, *)$, the space of paths on $X \times X \times X$ beginning in $X \vee X \vee X$ and ending at the base point. Thus $\pi_r(FX) = \pi_{r+1}(X \times X \times X, X \vee X \vee X) = 0$ if X is $(q-1)$-connected and $r+1 < 2q$. This proves (i).

Now FX is the subspace $EX \times \Omega X \times \Omega X \cup \Omega X \times EX \times \Omega X \cup \Omega X \times \Omega X \times EX$ of $EX \times EX \times EX$. Set $Y_i = EX_i$, $A_i = \Omega X_i$, $i = 1, 2$; then f induces

$$h: Y_1, A_1 \to Y_2, A_2$$

and h is $(m-1)$-connected, while (Y_i, A_i) is $(q-1)$-connected. As in the proof of Proposition 4.2 we may suppose h an inclusion and $Y_1 = Y_2 (= Y)$ so that we have a triple (Y, A_2, A_1). Again we assume $m > q$, since (ii) follows trivially from (i) if $m \leq q$

and replace (Y, A_2, A_1) by a minimal triple in the sense of Lemma 4.1. Thus we take (Y, A_2, A_1) to be a CW- triple in which $A_1 - *$ has no cells of dimension $\leq q-2$, $A_2 - A_1$ has no cells of dimension $\leq m-2$, and $Y - A_2$ has no cells of dimension $\leq q-1$. Again as in Proposition 4.2 we deduce that $Y-*$ has no cells of dimension $\leq q-2$ and that all $(q-1)$- cells of Y are in A_1. It follows that $FX_2 - FX_1$ has no cells of dimension $\leq m-2+q$ and the lemma follows.

Now $\mathfrak{A}$ is plainly a good structure system. Thus from Theorem 5.4r and Lemma 5.11 we infer

PROPOSITION 5.12. *Let X_1, X_2 be $(q-1)$- connected and let $f: X_1 \to X_2$ be $(m-1)$- connected. Then if X_2 is $\mathfrak{A}$- structured and $\dim X_1 \leq m-3+q$, we may give X_1 a unique $\mathfrak{A}$- structure such that f is a homomorphism.*

COROLLARY 5.13. *Let X_1, X_2 be $(q-1)$- connected and let $f: X_1 \to X_2$ be $(m-1)$- connected. Then if X_2 is furnished with an associative comultiplication and $\dim X_1 \leq m-3+q$, the unique comultiplication on X_1 with respect to which f is a homomorphism is associative.*

PROOF. Let Φ_2 be the comultiplication on X_2 and Φ_1 the induced comultiplication on X_1 (see Theorem 5.5r). Set $\psi_i = (\Phi_i \vee 1) \circ \Phi_i$, $\overline{\psi}_i = (1 \vee \Phi_i) \circ \Phi_i$, $i = 1, 2$. Then $\psi_i, \overline{\psi}_i$ are $\mathfrak{A}$- structure maps for X_i and $\{\psi_2\} = \{\overline{\psi}_2\}$ since Φ_2 is associative. Moreover, f is a homomorphism both with respect to the structures $\{\psi_1\}, \{\psi_2\}$ and with respect to the structures $\{\overline{\psi}_1\}, \{\overline{\psi}_2\}$. Thus by Proposition 5.12 $\{\psi_1\} = \{\overline{\psi}_1\}$ so that Φ_1 is associative.

We thus deduce

THEOREM 5.14. *Let X be a countable 1- connected CW- complex furnished with an associative comultiplication. Then the induced comultiplications on the sections of X are also associative.*

THEOREM 5.15. *Let X be a countable 2- connected CW- complex furnished with an associative comultiplication. Then the induced comultiplications on the homology sections of X are also associative.*

REMARK 5.16. (i) Theorem 5.14 would be false without the requirement that X be 1- connected. That is, we could have an associative comultiplication on X compatible with a non-associative multiplication on its 1- section. (ii) We could refine Corollary 5.13 by replacing $X \times X \times X$ in the argument by the «compatible product» of the 3 copies of X, that is, the inverse limit of the system of projections of the 3 «coordinate planes» $X \times X$, of $X \times X \times X$, onto their factors. If we did this we would deduce that if X_i is furnished with a comultiplication Φ_i and if Φ_2 is associative and $\dim X_1 \leq m-4+2q$

then Φ_1 is associative. This would enable us to infer that the homology sections of a *1*-connected complex furnished with an associative comultiplication admit an associative comultiplication. Since the details of the necessary connectivity computation corresponding to Lemma 5.11 are involved, we have preferred to give the simpler argument.
(iii) Since a countable *1*-connected *CW*-complex with an associative comultiplication is a cogroup, we infer from Theorem 5.14 that a cogroup structure on such a space X induces a cogroup structure on all its sections; and from Theorem 5.15 that if X is *2*-connected then a cogroup structure on X induces a cogroup structure on its homology sections.

We now turn rather briefly to the dual story. The basic theory at the beginning of the section dualizes automatically. Thus $\mathfrak{L} = (L, W, S; d, j)$ is *good* if Σj is a domination and we have

THEOREM 5.31. *Let $\mathfrak{L}$ be a good (left) structure system over $\mathcal{C}$. (i) if LX is an H-space then each $\mathfrak{L}$-structure for X is associated with a unique strong structure. (ii) If $f: X_1 \to X_2$ is a homomorphism with respect to $\mathfrak{L}$-structures Φ_1, Φ_2 and if LX_2 is an H-space then f is a strong homomorphism with respect to strong structures σ_1, σ_2 associated with Φ_1, Φ_2.*

We then deduce the following two theorems; the first is analogous to Theorem 5.4r and the second goes in the opposite direction. Howewer we do not state these theorems in the same form as Theorem 5.4r since, in our applications, we will have nothing to do with dimensionality restrictions.

THEOREM 5.17. *Let $\mathfrak{L}$ be a (left) structure system over $\mathcal{C}$ with cofibre U and let $f: X_1 \to X_2$ be a map in $\mathcal{C}$. Then if X_2 is $\mathfrak{L}$-structured by Φ_2, there exists an $\mathfrak{L}$-structure $\{\Phi_1\}$ for X_1 such that f is a homomorphism provided all the groups $H^k(U(X_1); \pi_k(L(f)))$ vanish. If $\mathfrak{L}$ is good and LX_2 is an H-space, then $\{\Phi_1\}$ is uniquely determined by $\{\Phi_2\}$ provided all the groups $H^{k-1}(U(X_1); \pi_k(L(f)))$ vanish.*

THEOREM 5.18. *Let $\mathfrak{L}$ be a (left) structure system over $\mathcal{C}$ with cofibre U and let $f: X_1 \to X_2$ be a map in $\mathcal{C}$. Then if X_1 is $\mathfrak{L}$-structured by Φ_1, there exists an $\mathfrak{L}$-structure $\{\Phi_2\}$ for X_2 such that f is a homomorphism provided all the groups $H^{k+1}(U(f); \pi_k(LX_2))$ vanish. If $\mathfrak{L}$ is good and LX_2 is an H-space, then $\{\Phi_2\}$ is uniquely determined by $\{\Phi_1\}$ provided all the groups $H^k(U(f); \pi_k(LX_2))$ vanish.*

We apply these theorems to the structure system $\mathcal{H}$ of Example 2.19. $\mathcal{H}$ is certainly good; and, of course, X admits an $\mathcal{H}$-structure ($=H$-structure $=$ multiplication) if and only if X is an H-space. The cofibre of $\mathcal{H}$ is given by $U(X) = X \# X$. We consider the fibration of the connected space X,

$$X(n, \ldots, \infty) \xrightarrow{i} X \xrightarrow{p} X(1, \ldots, n-1),$$

of Example 3.14 1. Write this as $F \xrightarrow{i} X \xrightarrow{p} B$. Now $U(F)$ is $(2n-1)$- connected and $\pi_k(i) = 0$, $k \geq n$. Thus we may apply Theorem 5.17 to show that an H- structure on X induces a unique H- structure on each $X(n, \ldots, \infty)$. Again $\pi_k(B) = 0$, $k \geq n$, and a simple computation shows that $U(p)$ is $(n+q)$- connected if X is $(q-1)$- connected. Thus we may apply Theorem 5.18 to show that an H- structure on X induces a unique H- structure on each $X(1, \ldots, n-1)$. We may sum up in the following theorem[18].

THEOREM 5.19. *Let X be a countable connected CW- complex and consider the diagram*

$$\begin{tikzcd} X \arrow[r] & X(1, \ldots, n) \\ X(m, \ldots, \infty) \arrow[u] \arrow[r] & X(m, \ldots, n) \arrow[u] \end{tikzcd} \tag{5.20}$$

Then an H- structure on X induces a unique H- structure on each $X(m, \ldots, n)$, $1 \leq m \leq n \leq \infty$, such that the maps of (5.20) are homomorphisms. In particular if the H- structure on X is commutative so is the H- structure on $X(m, \ldots, n)$.

We may handle the associativity question essentially just for H'- structures. Suppressing the details, we are content to record

ADDENDUM TO THEOREM 5.19. If the H- structure on X is associative so are those on each $X(m, \ldots, n)$.

Footnotes.

1. A unified obstruction theory will appear in a forthcoming paper by Eckmann and Hilton.

2. I. e., provided $cat\ X > 0$. Of course the inequality fails in general if $cat\ X = 0$.

3. If X is not 1-connected the conclusion is false.

4. Of course, this hypothesis is very rarely verified in practice !

5. A systematic study of the genus of covering maps was carried out by Ganea in [18] .

6. Such an argument is implicit in [16] .

7. $(A, A_o) \times (B, B_o) = (A \times B, A \times B_o \cup A_o \times B)$.

8. Ganea gave in [19] a definition of *cocategory* which generalizes the notion of H-space in the way in which *cat* generalizes the notion of H'- space. He has recently succeeded in finding a structure for cocategory, in the present sense, so that it is brought within the scope of the theory developed in this paper. See A. M. S. Notices, Vol. 10, N° 3, 1963, abstract 600-2.

9. If $k \leq 2$, we have only cohomology sets.

10. It is of no importance that the map m is not necessarily an inclusion map.

11. It is of no importance that the map b is not necessarily an inclusion map.

12. I. e., $(x_1, \ldots, x_n) \in TX$ if and only if $x_i = *$ for at least one i, $1 \leq i \leq n$.

13. Similar results were obtained by Ganea for cocategory in [19]. These can presumably now be obtained by the methods described here, using his characterization of cocategory as a structure.

14. We could state the lemma for n- tuples but are content with the explicit case $n = 3$.

15. Thus, if we take j to be a fibration, Ωj has a cross-section.

16. Up to homotopy.

17. Up to homotopy.

18. This result is well-known (see, e. g., [9, 13]); we are here concerned with its relation to the general theory.

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