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# ON FIBRATION OF SPHERES BY SPHERES AND HOPF INVARIANT

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1. Let $f: S_{2n-1} \to S_n$ be a map which we may assume to be simplicial. If $x_1, x_2$ are interior points of $n$-simplexes of $S_n$, then $z_i^{n-1} = f^{-1}(x_i)$, $i = 1, 2$, are $(n-1)$ cycles in $S_n$. Since $H_{n-1}(S_{2n-1}) = 0$, the cycles are boundaries, say $z_i^{n-1} = \partial c_i$. The linking coefficient of $z_1^{n-1}$ and $z_2^{n-2}$ is called the *Hopf number* $H(f)$ of $f$.

REMARK. $H(f)$ is clearly the degree of the map $f | c_i^n$. We have the following properties of $H(f)$ [5] :

1) $H(f)$ depends only on the homotopy class of $f : S_{2n-1} \to S$ . It follows that $H$ is a homomorphism :

$$H : \Pi_{2n-1}(S_n) \to Z.$$

2) If $n$ is odd, $H(f)$ is zero.

H. Hopf has shown [5] that, for $n = 2$, 4 and 8, there exist maps of Hopf invariant 1. He raised for what other values of $n$ do maps $f : S_{2n-1} \to S_n$ with $H(f) = 1$ exist ? Adams [1] has shown recently that these are the only values of $n$ for which maps with Hopf invariant one exist.

This problem is intimately connected with the existence of fibrations of spheres by spheres over spheres. Suppose $S_{2n-1}$ is a fibre space over $S_n$ with fibre $S_{n-1}$. We have :

PROPOSITION 1. *If $f : S_{2n-1} \to S_n$ is a fibre mapping, then $H(f) = 1$.*

More generally, we consider the following situation [10] : suppose that $E$ is a topological space aspherical in dimensions $\leqslant n$. If $E$ is fibred by $k$-spheres $S_k$ over $S_n$, then, form the homotopy-exact sequence, it follows that $k = n-1$. One has the following (cf, [3] also) :

PROPOSITION 1'. *Let $E$ be a topological space aspherical in dimensions $\leqslant n$. If $p : E \to S_n$ is a fibre space with fibre $S_{n-1}$, then $H(p) = 1$.*

**2.** Let $E$ be aspherical in dimensions $\leqslant n$ ; suppose that $E$ is a $(n-1)$ sphere bundle over $S_n$ . We have the following :

PROPOSITION 2. *The fibre $S_{n-1}$ is contractible to a point in E.*

PROPOSITION 3. *Let E be as above : if E is a (n-1) sphere bundle over $S_n$, then n is a multiple of 4.*

PROOF. If $(n-1)$ is even, then the inclusion $S_{n-1} \to E$ maps $\Pi_{n-1}(S_{n-1})$ isomorphically into $\Pi_{n-1}(E) = 0$ ([9] cf § 28. 3) a contradiction. If $n > 2$ and $n \equiv 1$ mod. 4, then the image of $\Pi_{n-1}(S_{n-1})$ in $\Pi_{n-1}(E)$ is either infinite cyclic or finite cyclic of even order ([9] cf § 28. 3). Thus $n = 1$, 2 or $n = 4$m.

REMARK. In fact, one can prove, by an argument similar to that used by J. Adam (cf [2] cor. 2.3), that if such a fibration exists, then $n$ is a power of 2.

Examples of such fibrations are furnished by the classical fibrations of Hopf [5].

PROPOSITION 4. *Let E, aspherical in dimensions $\leqslant n$, be a fibre space over $S_n$ with fibre $S_{n-1}$ and any structure group; then $n = 1$ or even.*

PROOF. Cf [9] §28. 7 theorem.

REMARK. Proposition 3 restricts that the structure group is $O(n)$ while proposition 4 shows shows that $n$ is only even for any structure group. If $E$ is a fibre space (in the sense of Serre) over $S_n$ with fibre $S_{n-1}$ one can obtain the same results as here Cf [3].

**3.** In this section we determine completely the $O(n)$ fibrations of $E$ by $S_{n-1}$ over $S_n$ using some results of Milnor [7]. For the sake of completeness we include them here.

THEOREM.1. *There exists an $O(m)$ - bundle $\xi$ over $S_n$ with the n dimensional Stiefel-Whitney class $w^n \neq 0$ only if $n = 1, 2, 4$ or 8.* We prove this using the following result of Wu.

THEOREM 2. (Wu). *For any $0(m)$-bundle $\xi$ over a complex K, the Pontrjagin class $p_k$ is related to the Stiefel-Whitney classes $w^1, \ldots w^{4k-1}$ by*

$$(p_k)_4 = P(w^{2k}) + i_* f_k(w^1, \ldots w^{4k})$$

*where $(p_k)_4$ is the image of $p_k$ by the homomorphism $H^*(K ; Z) \to H^*(K ; Z_4)$ induced by $Z \to Z_4$, i denotes the inclusion $Z_2 \to Z_4$, P the Pontrjagin operation : $H^{2k}(K ; Z_2) \to H^{4k}(K ; Z_4)$ and $f_k$ a polynomial with coefficients in $Z_2$.*

Following Hinzebruch we define the $4k$ - dimensional Pontrjagin class $p_k$ of an $O(m)$ - bundle as $(-1)^k$ times the Chern class $c_{2k}$ of the unitary extension induced by the inclusion $O(m) \to U(m)$.

PROOF. It is sufficient to consider the case of the universal bundle over the Grassmann space $G_m(k)$, with $m$ sufficiently large. Consider the exact coefficient sequence $0 \to Z_2 \xrightarrow{i} Z_4 \xrightarrow{j} Z_2 \to 0$ : we have :

$$i_* P(w) = w \cup w \qquad \text{for any cohomology class } w.$$

Also $$i_* ((p_k)_4) = (p_k)_2 = w^{2k} \cup w^{2k};$$

it follows that $$(p_k)_4 - P(w^{2k}) \in ker\ j_* = i_* H^{4k}(G_m(k) ; Z_2).$$

Since the cohomology ring $H^*(G_m(k) ; Z_2)$ is generated by the Stiefel-Whitney classes, the theorem follows.

COROLLARY. *If the Stiefel-Whitney classes* $w^1, \dots w^{4k-1}$ *are zero, then* $(p_k)_4 = i_* w^{4k}$.

PROOF. We will show that the coefficient of $w^{4k}$ in $f_k$ is non-zero. Let $\Gamma$ denote the universal $U(m)$-bundle over $G_m(C)$, the complex Grassmann space. It is known that the cohomology ring $H^* (G_m(C) ; Z)$ is a polynomial ring generated by the Chern classes of $\Gamma$. The inclusion $U(m) \to O(2m)$ induces an $O(2m)$ bundle $\Gamma_R$ over $G_m(C)$. Applying theorem 2 to $\Gamma_R$, the relations

$$w^{2r+1}(\Gamma_R) = 0, \qquad w^{2r}(\Gamma_R) = (C^r(\Gamma))_2$$

and $$p_k(\Gamma_R) = C_k(\Gamma)^2 - 2C_{k-1}(\Gamma)\, C_{k+1}(\Gamma) + \dots \pm 2C_0(\Gamma)\, C_{2k}(\Gamma)$$

show that the polynomial $f_k$ must satisfy

$$f\ (0.\ w^2, 0, \dots, w^{4k}) = w^{2k-2} w^{2k+2} + \dots + w^0 w^{4k}$$

In particular, we have $f_k(0,0,\dots,0, w^{4k}) = w^{4k}$.

We also make use of the following result of Bott :

THEOREM. 3.(Bott). *For any* $O(m)$*-bundle* $\xi$ *over* $S_{4k}$, *the Pontrjagin class* $p_k \in H^{4k}(S_{4k}; Z)$ *is always divisible by* $(2k-1)!$. For a proof refer [6].

PROOF OF THEOREM 1. According to Wu [12] such a bundle exists only if $n = 2^m$. Hence we can assume that $n = 4k$, $k > 2$. The identiy :

$$(p_k)_4 = i_* w^{2k} \in H^{4k}(S_{4k} ; Z_4) = Z_4$$

holds, since the lower dimensional Stiefel-Whitney classes are zero. Hence $w^{4k} = 0$ if and only if $p_k$ is divisible by 4. But by Bott's theorem, $p_k$ is divisible by $(2k-1)!$ Thus $w^{4k} = 0$ for $k > 2$. As a corollary we have the following :

THEOREM. *Let E be aspherical in dimensions* $\leqslant n$. *E is a* $(n-1)$*-sphere bundle over* $S_n$ *only if* $n = 1, 2, 4$ *or* $8$.

PROOF: $w^n = 0$ *if* and only if $E$ admits a cross-section.

Taking $E = S_{2n-1}$, we have :

COROLLARY. *There do not exist fibrations of* $S_{2n-1}$ *by* $S_{n-1}$ *over* $S_n$ *with structure group* $O(n)$ *for* $n > 8$.

## 4. Classification Theorem.

It is well-known that the equivalence classes of fibre spaces over $S_n$ are in 1-1 correspondance with equivalence classes of $\Pi_{n-1}(O(n))$ by the operations of $\Pi_0(O(n))$. Since $\Pi_1(SO(2)) = Z$, there exist an infinity of inequivalent fibre spaces. From the homotopy sequence of the fibration $S_1 \to S_3 \to S_2$ we see that $\Pi_2(S_2) \approx \Pi_1(S_1)$ or the fibre space corresponds to 1. Now let $E$ be a 1-sphere bundle over $S_2$ with $\Pi_i(E) = 0$, $i \leqslant 2$ ; by the same argument it follows that $E$ also corresponds to 1. Thus the two bundles are equivalent and $E$ is topologically homeomorphic to $S_3$.

In the case $n = 4$, since $\Pi_3(SO(4)) = Z + Z$, we have a fibre space $B_{m,n}$ corresponding to each pair $(m, n)$. It is easy to see that the primary obstruction for the existence of a cross-section is $n\beta_3$ where $\beta_3$ is a generator of $\Pi_3(SO(4))$ defined by $\sigma : S_3 \to SO(4)$ ([9], § 22. 6). Hence the fibre spaces $B_{m,o}$ have cross-sections. We will show that $B_{m,\pm 1}$ is equivalent to $S_3 \to S_1 \to S_4$. This follows from the fact that the characteristic map represents a generator of $\Pi_3(S_3) \approx \Pi_4(S_4)$. Moreover, the fibration $S_3 \to E \to S_4$ with $\Pi_i(E) = 0$ for $i < 4$, is equivalent to $B_{m,\pm 1}$. Therefore $E$ is topologically homeomorphic to $S_7$. For $n = 8$, there exist an infinity of inequivalent bundles $B_{m,n}$ corresponding to each pair $(m, n)$ since $\Pi_7(SO(8)) = Z + Z$. It can be seen as above that the primary obstruction for the existence of a cross-section is $n\beta_8$, where $\beta_8$ is defined by $\beta_8(x)y = x \cdot y$, $x, y$ being Cayley numbers. We will show that the classical fibration $S_7 \to S_{15} \to S_8$ corresponds to $m \pm 1$, Suppose it corresponds to $(m, n)$. Let $\tilde{B}_{m,n}$ be its associated principal bundle : if $y_o \in S_7$, denote by $y_o^* ; \tilde{B}_{m,n} \to S_{15}$ the principal map, we have the following

$$\begin{array}{ccc}
\Pi_8(S_8) & \xrightarrow{\ \partial\ } & \Pi_7(SO(8)) \\
\downarrow & & \downarrow y_o^* \\
\Pi_8(S_8) & \longrightarrow & \Pi_7(S_7)
\end{array}$$

By the defintion of the index $(m, n)$ $\partial$ maps a generator of $\Pi_8(S_8)$ into $n$ times a generator of $\Pi_7(SO(8))$ ; but $n$ times a generator of $\Pi_7(SO(8))$ is carried by $y_o^*$ to $n$ times a generator of $\Pi_7(S_7)$, a contradiction since $\Pi_8(S_8) \approx \Pi_7(S_7)$. By the same argument we can prove that the fibration $S_7 \to E \to S_8$ with $\Pi_i(E) = 0$, $i < 8$, is equivalent to $B_{m,\pm 1}$. Therefore we have again $E$ is topologically homeomorphic to $S_{15}$. Thus we have :

THEOREM. *Suppose that $S_{n-1} \to E \to S_n$, with $\Pi_i(E) = 0$, $i < n$, is a sphere-bundle. Then $E$ is topologically homeomorphic to $S_3$, $S_7$ or $S_{15}$.*

REMARK. In fact it follows, from a remark of J. Milnor ([ 8 ] p 403), that $E$ is differentiably homeomorphic to $S_3$, $S_7$ or $S_{15}$ if $m = 0$.

**Appendix.**

**1.** An equally interesting problem is the following : is $S_n$ a $k$ - sphere bundle over some space $B$ ?

Let $\tilde{B}$ be the principal bundle of $S_k \to S_m \to B$ and let $y : \tilde{B} \to S_m$ be the map defined by a point $y \in S_k$. Then the kernel of $i_* : \Pi_i(S_k) \to \Pi_i(S_m)$, induced by the inclusion of $S_k$ in $S_m$, is contained in the image of $y'_* : \Pi_i(SO(k+1)) \to \Pi_i(S_k)$ where $y' = y \,|SO(k+1)$ [9, 17. 13]. This means that $y'_* : \Pi_k(SO(k+1)) \to \Pi_k(S_k) \approx Z$ is onto. Let $f : S_k \to SO(k+1)$ be a mapping which corresponds to a generator of $y'_* \; \Pi_k(SO(k+1))$ ; that means $y' : SO(k+1) \to S_k$ is such that $y'f$ = identity. Since $y'$ is topologically equivalent to the fibre mapping $SO(k+1) \to S_k$, this implies that $S_k$ is parallelisable. Hence $k = 1, 3$ or $7$[ 7]. Thus we have the following :

PROPOSITION 1. *The sphere $S_n$ is the bundle space of a $k$ - sphere over some space $B$, only if $k = 1, 3$ or $7$.*

REMARK. Similarly let $B$ be a differentiable manifold ; if $E$ (a compact differentiable manifold) is a differentiable fibre space over $B$ (locally trivial) with differentiably contractible fibre $F$, it can be shown that $F$ is parallelisable.

We observe the following :

PROPOSITION 2. *If $S_m$ is a $k$ - sphere bundle over $B$, then $S_k$ is a $H$ - space, that is, $S_k$ admits a continuous multiplication with a two-sided identity.*

PROOF. Consider the homotopy sequence ..... $\to \Pi_k(SO(k+1)) \xrightarrow{p_*} \Pi_k(S_k) \xrightarrow{\partial} \Pi_{k-1}(SO(k)) \to \ldots$ of the fibration $SO(k) \to SO(k+1) \xrightarrow{p} S_k$. Since $S_k$ is parallelisable (1), we have $\partial(i) = 0$ ($i$= identity of $\Pi_k(S_k)$).

Let $J : \Pi_{k-1}(SO(k)) \to \Pi_{2k-1}(S_k)$ be the G. W. Whitehead homomorphism ; it is known that $J\partial = -[\ ,]$ where $[\ ,]$ is the Whitehead product by $i$ ; thus $[i, i] = 0$, or equivalently $S_k$ is a $H$ - space.

REMARK. After Adams [1], $S_k$ is a $H$ - space only if $k = 1, 3,$ or $7$.

**2.** In what follows we assume that $B$ is a locally finite polyhedron. Since the fibre is contractible in $S_m$, we have :

$$\Pi_i(B) \approx \Pi_i(S_m) + \Pi_{i-1}(S_k).$$

Thus we have $\Pi_i(B) = 0$ for $i \leqslant k$. We assert that the dimension of the polyhedron $B$ is $\geqslant k + 1$ ; if the dim $B \leqslant k$, then it can be seen easily that $B$ is contractible, consequently $\Pi_i(B) = 0$ for all $i$, a contradiction. Hence dim $B \geqslant k + 1$ so that $m \geqslant 2k + 1$.

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(1) Dr. A. DOLD has informed me that if $S_n$, endowed of an arbitrary differentiable structure is parallelisable, then it is an $H$ - space. However, it is not clear whether the tangent structure of $S_n$ is independent of its differentiable structure.

It is known that $m$ is odd (1). Thus we have the following :

THEOREM. *If $S_m$ is a $k$-sphere bundle over a locally finite polyhedron $B$, then $m$ is an odd number $\geqslant 2k + 1$ and $k = 1$, $3$ or $7$.*

The following are the only known examples of such fibrations :

$S_{2k+1}$ is a circle bundle over $P_k(C)$ the complex projective space.

$S_{4k+3}$ is a 3-sphere bundle over the quaternion projective space $P_k(Q)$.

**3. Classification theorem.** Since the fibration $S_{2n+1} \to B$, $k > 1$, is a principal 3-universal bundle with group $SO(2)$, it follows from the classification theorem that the equivalence classes of bundles with group $SO(2)$ and base space a 2-complex $K_2$ are in (1-1) correspondance with the homotopy classes of maps $f : K_2 \to B$. Consider a mapping $f : K_2 \to B$ and let $m_f$ be the integer such that $f_*(\alpha) = m_f \beta$ where $\alpha$ is a generator of $H^2(B)$ and $\beta$ a generator of $H^2(K_2)$. Then there is a 1-1 correspondance between homotopy classes of maps $f : K_2 \to B$ and the integers $Z$. In fact, the bundle $S_{2n+1} \to B$ corresponds to $\pm 1 \in Z$. For if it corresponds to $p \in Z$, then the space $B_p = \frac{S_{2n+1}}{H_p}$ where $H_p$ is a cyclic subgroup of $SO(2)$ of order $p$, is a bundle over $B$ with group $SO(2)/_{H_p} \cong SO(2)$. Since $B$ is simply connected, this cannot happen unless $p = \pm 1$.

It follows from the homotopy exact sequence that $\Pi_1(B) = 0$, $\Pi_2(B) = \Pi_{2n+1}(B) = Z$, $\Pi_i(B) = 0$ for $2 < i < 2n+1$ and $\Pi_i(B) = \Pi_i(S_{2n+1})$ for $i > 2n+1$. Comparing this with the classical bundle $S_{2n+1}$ over the complex projective space $P_n(C)$, one can see that $B$ is of the same homotopy type as $P_n(C)$.

REMARK. We assume that $B$ is a compact, connected metric space ; suppose that $B$ admits a locally strongly transitive group of isometries [11]. Then it follows, from [11], that $B$ is homeomorphic to a sphere, or complex or quaternion projective space. Thus we have the following :

THEOREM. *If $S_{2n+1}$ is a $k$-sphere bundle over a compact, connected metric space $B$ which admits a locally strongly transitive group of isometries, then $k = 1$, $3$ or $7$ and $B$ is homeomorphic to a complex or quaternion projective space or to a sphere.*

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(1) The fact that $m$ is odd can be seen as follows : we obtain from Gysin's sequence [4] $\ldots \to H^i(B) \to H^i(S_m) \to H^{i-k}(B) \to H^{i+1}(B) \to \ldots$ that $H^{i-k}(B) \cong H^{i+1}(B)$, $0 < i < m-1$. Thus the Poincaré polynomial of $B$ is of the form $P(B;t) = 1 + t^{k+1} + \ldots + t^{m(k+1)}$ where $\dim B = m(k+1)$ : hence the Euler characteristic $\chi(B) = P(B,-1) = \dim B/k+2 > 0$. Thus, if $m$ is even, we obtain from $\chi(S_m) = \chi(S_k)\,\chi(B)$ that $k$ is even, a contradiction to Prop. 1.

**Bibliographie.**

[1] ADAMS J. F, Non-existence of elements of Hopf invariant 1, Bull. Amer. Math. Society 64 (1958), 279-282.

[2] ADEM J, The iteration of the Steenrod squares in algebraic topology, Proc. Nat. Acad. Sci. U.S.A., 38 (1953), 729-726.

[3] CARTAN H, Séminaire E.N.S., 1958-59, exposés 5 et 6.

[4] GYSIN W, Zur homologietheorie der abbildungen und faberungen von manigfaltigkeiten, Comm. Math. Helv., 14 (1941), 61-122.

[5] HOPF H, Wher die abbildungen von sphären auf sphären neidrigerer dimension, Fund. Math., 25 (1935), 427-440.

[6] KERVAIRE M, On the Pontrjagin classes of certain SO(n)-bundles over manifolds, Amer. Jour. Math. L (1958), 42,632-638.

[7] MILNOR J, Some consequences of a theorem of Bott, Ann. of Math. 68 (1958), 444-449.

[8] MILNOR J, On manifolds homeomorphic to the 7-sphere, Ann. of Math. 64 (1956), 399-405.

[9] STEENROD N, The Topology of fibre bundles, Princeton Univ. Press, 1951.

[10] WADA H, On the structure of a sphere bundle, Tohoku Math. J. 2nd series, 3 (1951), 136-139.

[11] WANG H.C, Two-point homogeneous spaces, Ann. of Math. 55 (1952), 177-191.

[12] WU W.T, Classes caractéristiques et i-carrés d'une variété, C.R. Acad. Sci. Paris 230 (1950), 918-920.