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SEMINAIRE SUR

LES EQUATIONS NON-LINEAIRES

- I -

ONE CLASS OF MEROMORPHIC SOLUTIONS OF GENERAL TWO-DIMENSIONAL NON-LINEAR EQUATIONS, CONNECTED WITH THE ALGEBRAIC INVERSE SCATTERING METHOD.

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0. It is well known that the most general form of "exactly solvable" (or "completely integrable") systems of non-linear two-space dimensional (in x and y) partial differential equations can be written as a condition of commuting of two differential operators [1], [2]. Non-trivial two-dimensional systems in variables x, y, t can be obtained if each of these operators have differentiation, say in x, and differentiation of order only one in y and t [5]. Such systems were at first introduced by Zakharov and Shabat [1] as a natural generalization of Lax pair [3]. These systems are the culmination of investigations started by Burchnall and Chaundy [4] and we write them in the form

$$\frac{dL_m}{dt} - \frac{dL_n}{dy} = [L_n, L_m] \quad ,$$

where L_n and L_m are differential operators $L_n = \sum_{i=0}^{n} u_i \frac{d^i}{dx^i}$, $L_m = \sum_{i=0}^{m} v_i \frac{d^i}{dx^i}$, of orders n and m respectively, where $u_n = v_m = 1$, $u_{n-1} = v_{m-1} = 0$.

We present here a special class of solutions of this equation based on the following two observations. First of all, it was shown in our previous papers [6], [7], that for meromorphi in x solutions $u(x, y, t)$ the evolution of poles $a_i(y, t)$ is described as an evolution in y and t directions according to two Hamiltonian flows commuting simultaneously with the flow describing the interaction of particles with inverse square potential. On the other side, it was shown in [6], [7] that there are linearized equations, so called higher Burgers-Hopf (BH) equations, the poles of meromorphic solutions of which evolve also according to these Hamiltonian flows. These two observations lead to the conclusion that it is possible to construct directly the solutions of two-dimensional systems starting from linear partial differential equations. In this paper, we prove this and present a method for the construction of this class of solutions. It is possible to construct them starting from Gelfand-Levitan-Marchenko equation. Instead of doing this, we derive meromorphic solutions directly using commutativity conditions. We also examine the behavior of their poles.

1. It is known that the evolution of poles of most non-linear, completely integrable systems, is connected with the Hamiltonian for a system of particles interacting via potential $G\wp(x)$ for Weierstrass elliptic function $\wp(x)$.

Thus, we consider the Hamiltonian [9]

(1) $$H_{\wp} = \frac{1}{2}\sum_{i\in I} b_i^2 + G\sum_{i\neq j}\wp(a_i - a_j)$$

for an arbitrary number of particles a_i: $i \in I$. This Hamiltonian has infinitely many first integrals. These integrals come from the Lax representation

(2) $$\frac{dL}{dt} = [A, L]$$

for system (1), where the matrix $L = (L_{ij})_{i,\, j \in I}$, in (2) has the form

(3) $$L_{ij} = (1 - \delta_{ij}) \sqrt{-G}\ \alpha(a_i - a_j) + \delta_{ij} b_i ,$$

with $\alpha^2(x) = \wp(x)$. Then the functionals $J_n = \frac{1}{n} \mathrm{tr}(L^n)$, $n \geq 1$ are the first integrals of $H_\wp$. Moreover, J_n are in involution and they are sums of polynomials in $\dot{a}_i$, $\wp(a_i - a_j)$, G with rational coefficients. The form of the first terms of J_n is the following

(4) $$J_n = \frac{1}{n} \sum_{i \in I} b_i^n + G \sum_{i \neq j} (b_i^{n-2} + b_i^{n-3} b_j + \ldots + b_j^{n-2}) \wp(a_i - a_j) + \ldots$$

For the degenerate case $\wp(x) = x^{-2}$ and finite I there are very simple formulae [6], [9] for the solution of the Cauchy problem for any J_n.

If we consider such trajectories $x_i(t)$ of Hamiltonian J_n that all the integrals J_m vanish, $J_m = 0$, $m = 1, 2, \ldots,$ then we come to the poles of higher Burgers-Hopf equation.

As is well known [8], the usual BH equation has the form $u_t = u_{xx} - 2uu_x$. The higher BH equations like the ordinary one are linearized using Hopf-Cole substitution $u = -(\log \varphi)x$. Thus, n th higher BH equation has the form

$$u_t = -\frac{\partial}{\partial x}\left\{\frac{\partial^n}{\partial x^n}(\exp[-\textstyle\int u dx])\exp[\textstyle\int u dx]\right\} = BH_n[u]$$

, or $BH_n[u]$ can be defined as a polynomial in $u, u_x, u_{xx},$ by induction

$$BH_n[u] = \frac{d}{dx} C_n[u]$$

and $C_{n+1}[u] = \frac{d}{dx} C_n[u] - uC_n[u]$, $C_0[u] = -1$.

We must emphasize that the notations here and in [6] differ only in the sign of u . Then, the basic information on higher BH is contained in the following

Theorem 1: The equation $u_t = BH_n[u]$ reduces by the transformation $u = -(\log \varphi)_x$ to the equation $\varphi_t = \varphi_{x\ldots x} = \varphi_{nx}$. Meromorphic solutions $u(x, t)$ of $u_t = BH_n[u]$ have the form $u(x, t) = -\sum_{i\in I} (x - a_i)^{-1}$ for $a_i = a_i(t)$: $i \in I$ and meromorphic $u(x, t) = -\sum_{i\in I} (x - a_i)^{-1}$ satisfy $u_t = BH_n[u]$, iff

$$(5) \quad -\dot{a}_i = n!\sum_{i\notin\{j_1, \ldots, j_{n-1}\}} (a_i - a_{j_1})^{-1}\ldots(a_i - a_{j_{n-1}})^{-1}, \; i \in I .$$

Also, the system (5) has a very simple description:

Proposition 2: The system (5) is embedded into a system with Hamiltonian

$$J_n \quad \text{for } \wp(x) = x^{-2} \text{ and } G = -1 .$$

Moreover, the system (5) corresponds to the following submanifold

$$b_i = \sum_{j\neq i} (a_i - a_j)^{-1}: \quad i \in I .$$

For finite I the trajectories $a_i(t)$: $i \in I$ of J_n with $G = -1$ on which all the integrals J_m vanish, $J_m = 0$ are precisely the solutions of the system (5).

2. For any integer $n \geq 2$, $m \geq 2$ we consider the following system of equation in partial derivatives for

functions $u_i(x, y, t)$ and $v_j(x, y, t)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

$$\frac{dL_m}{dt} - \frac{dL_n}{dy} = [L_n, L_m] , \tag{6}$$

where

$$L_n = \sum_{i=0}^{n} u_i \frac{d^i}{dx^i} , \quad L_m = \sum_{j=0}^{m} v_j \frac{d^j}{dx^j} , \quad u_n = v_m = 1 , \tag{7}$$

$$u_{n-1} = v_{m-1} = 0 .$$

The main result of this paper is the following:

Theorem 3: For the system (6) and any initial condition $u(x)$ there exists such a solution u_i, v_j of (6) and that $u_{n-2}(x, 0, 0) = u(x)$, where $u_{n-2} = -nw_x$, $v_{m-2} = -mw_x$ and

$$w_t = BH_n[w], \quad w_y = BH_m[w] \tag{8}$$

and other u_i, v_j : $i = 1, \ldots, n$, $j = 1, \ldots, m$ are defined by induction

$$u_k = -\sum_{s=k+1}^{n-2} C_s^{k+1} w^{(s-k-1)} u_s - C_n^{k+1} w^{(n-k-1)} : \tag{9}$$

$$k = n - 2, \ldots, 0 , \text{ and}$$

$$v_k = -\sum_{s=k+1}^{m-2} C_s^{k+1} w^{(s-k-1)} v_s - C_m^{k+1} w^{(m-k-1)} : \tag{10}$$

$$k = m - 2, \ldots, 0 .$$

Here we denote by $w^{(j)}$ the j th derivative of w: $w^{(j)} = w_{jx}$ and $C_i^k = \binom{i}{k}$ are binomial coefficients.

Proof: As usual [10], we consider the common eigenfunction $\psi(x, y, t, k)$ for two operators $L_n - \frac{\partial}{\partial t}$ and $L_m - \frac{\partial}{\partial y}$ having the form

(11) $\psi(x, y, t, k) = (1 + wk^{-1})\exp(kx + k^n t + k^m y)$,

where k is a spectral parameter. Then, it is clear that the function ψ from (11) is the eigenfunction for $L_n - \frac{\partial}{\partial t}$ and $L_m - \frac{\partial}{\partial y}$:

(12) $L_n\psi = \psi_t$

$L_m\psi = \psi_y$.

Iff the following system of equations in w, u_i, v_j is satisfied:

(13) $u_m = -\sum_{s=k+1}^{n-2} C_s^{k+1} w^{(s-k-1)} u_s - C_n^{k+1} w^{(n-k-1)}$

$m = n - 2, \ldots, 0$ and

(14) $w_t = \sum_{s=o}^{n-2} w^{(s)} u_s + w^{(n)}$ and an analogous system for v_j changing u_i to v_i and n to m . Then ψ depends on an arbitrary parameter k and $\psi(k)$ is a null function for the commutator $[L_n - \frac{\partial}{\partial t}, L_m - \frac{\partial}{\partial y}]$. This operator is the operator on $\frac{\partial}{\partial x}$ and so cannot have infinite dimensional null subspace unless it is zero. Thus, (6) is satisfied when (13)-(14) for w, u_i, v_j is true. By induction, it is easy to show that assuming (13) the right side in (14) is exactly $BH_n[w]$. As (9) is equivalent to (13) and (8) to (14), the theorem is proved.

<u>Corollary 4</u>: For any $h(x)$ and any solution $\varphi(x, y, t)$ of the linear problem

(15) $\varphi_t = \varphi_{\underbrace{x\ldots x}_{n}}, \quad \varphi_y = \varphi_{\underbrace{x\ldots x}_{m}}$

with

$$\frac{d^2}{dx^2} \log \varphi(x;\ 0,\ 0) = h(x)$$

there exists a solution u_i, v_j of (6) defined by (9) and (10), where

(16) $w(x,\ y,\ t) = \frac{-d}{dx} \log \varphi(x,\ y,\ t)$.

Exact formulae for solutions (9), (10), (16) can be obtained using (15).

On the other side, formulae for rational solutions of (6) having the form (8) - (10) can be written, using theorem 1 and proposition 2 and result from [11], [12].

In fact, if initial conditions $a_1^0, \ldots, a_k^0, a_{1t}^0, \ldots, a_{kt}^0, a_{1y}^0, \ldots, a_{ky}^0$ satisfy

$$-a_{it}^0 = n!\Sigma\{j_1, \ldots, j_{n-1}\} \not\ni i(a_i^0 - a_{j_1}^0)^{-1} \ldots (a_i^0 - a_{j_{n-1}}^0)^{-1};$$

$$-a_{iy}^0 = m!\Sigma\{j_1, \ldots, j_{m-1}\} \not\ni i(a_i^0 - a_{j_1}^0)^{-1} \ldots (a_i^0 - a_{j_{m-1}}^0)^{-1}:$$

$$i = 1, \ldots, k \ .$$

Then the function $w(x,\ y,\ t) = -\sum_{i=1}^{k} (x - a_i)^{-1}$ being the solution of (8) can be written as

(18) $w(x,\ y,\ t) = -\frac{d}{dx} \log \chi(x,\ y,\ t)$.

Here $\chi(x,\ y,\ t) = \chi_{y,t}(x)$ is a characteristic polynomial on x of matrix $M(y,\ t)$: $M(y,\ t) = \text{diag}\,(a_1^0, \ldots, a_k^0) + (y - y_0)\, L^m(a_i^0, a_{iy}^0) + (t - t_0)\, L^n(a_i^0, a_{it}^0)$, where

$L(a_i, b_i)$ is the matrix for $\wp(x) = x^{-2}$, $G = -1$.

The solution (18) $w(x, y, t) = -\sum_{i=1}^{k} (x - a_i)^{-1}$

satisfy (8) provided

(20) $a_i(t_0, y_0) = a_i^0$, $a_{it}(t_0, y_0) = a_{it}^0$, $a_{iy}(t_0, y_0) = a_{iy}^0$.

Results analogous to these can also be obtained for matrix differential operators, and will be published in another paper.

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