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SEMINAIRE SUR

LES EQUATIONS NON-LINEAIRES

- I -

MEROMORPHIC EIGENVALUE PROBLEM, POLE INTERPRETATION AND MANY-PARTICLE PROBLEM FOR NON-LINEAR EQUATIONS.

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It is well known that infinite dimensional, completely integrable systems are connected with the linear eigenvalue problem. In general, the three dimensional (x, y, t) case, completely integrable system, appears as a condition of commutativity of two linear operators [1], [2]:

$$\frac{\partial L_n}{\partial t} - \frac{\partial L_m}{\partial y} = [L_m, L_n]$$

or

$$[L_m - \frac{\partial}{\partial t}, \quad L_n - \frac{\partial}{\partial y}] = 0$$

for

$$L_n = \sum_{i=0}^{n} u_i \frac{d^i}{dx^i}, \quad L_m = \sum_{i=0}^{m} v_i \frac{d^i}{dx^i}, \quad v_m = u_n = 1,$$

$$w_{m-1} = u_{n-1} = 0 .$$

This system, in u_i, v_j, is equivalent to the existence of many common eigenfunctions for L_n, L_m:

$$L_n\Psi = \Psi_y, \qquad L_m\Psi = \Psi_t$$

for $\Psi = \psi(x, y, t, k)$ with asymptotics for $k \to \infty$

$$\Psi = e^{kx + k^n y + k^m t}(1 + 0(k^{-1})) .$$

Analogically, we can consider y or t the independent case. In the last case, corresponding two-dimensional systems are called Lax systems [3], [4], while one-dimensional are called Zaharov-Shabat [1]. This way, when one of the operators is Schrodinger ($n = 2$ or $m = 2$) there arises the Korteweg-de Vries (KdV) equation $u_t = 6uu_x + u_{xxx}$ and higher KdV equations corresponding to the case $n = 2$, $m = 2k + 1$ in the y-independent case [3], [4]; Boussinesque equation $u_{yy} = \frac{\partial}{\partial x}(6uu_x + u_{xxx})$ (for $n = 2$, $m = 3$ in the t-independent case), Kadomzev-Petviaskvili equation [5] ($n = 2$, $m = 3$) , etc.

All other completely integrable systems (non-linear Schrödinger, sin -Gordon ...) corresponding to different n and m are scalar and matrix cases. Meromorphic solutions (among them, in particular, multisoliton, rational and elliptic) play an important role and are, in general, the only ones that can be analytically investigated.

In the papers [6], [7], [8], it was shown that for meromorphic in x solutions $u(x, y, t)$ the evolution of poles $a_i = a_i(y, t)$ corresponds to the many particle Hamiltonian with potential x^{-2} [6], [9]:

$$H = \frac{1}{2} \Sigma_{i \in I} b_i^2 + G \Sigma_{i \neq j} (a_i - a_j)^{-2}$$

or to Hamiltonian flows commuting with H [6], [8] (there are infinitely many such flows and H is completely integrable).

Here, we investigate the pole interpretation (the motion of poles) for "quasi-potentials" u_i, v_j by considering the eigenvalue problem for operators $L_n - \frac{\partial}{\partial y}$ for a meromorphic eigenfunction. We consider the case $n = 2$, when the motion of the poles corresponds to potential x^{-2}. In general for $n > 2$, there arises another completely integrable, many-particle system.

Our results give, e.g., exact formulae for rational and elliptic solutions of Zakharov-Shabat and Lax systems for $\min\{n, m\} = 2$.

We solve, for meromorphic potentials $u(x, t)$, the eigenvalue problem for the time-dependent Schrödinger equation:

$$(1) \qquad \frac{d^2}{dx^2} + u(x, t)\,\psi = \frac{d}{dt}\psi$$

for $\psi = \psi(x, t, k)$ and spectral parameter k, provided ψ nontrivially depends on k and is meromorphic as a function on x. Thus, we restrict ourselves to the function $u(x, t)$ which, as a function in x, is meromorphic with poles of order 2 with residues -2. Thus,

$$(2) \qquad u(x, t) = -2 \sum_{i\in I} (x - a_i)^{-2} ,$$

where $a_i = a_i(t) : i \in I$. Considering asymptotics of $\psi(x, t, k)$ for $k \to \infty$ as $e^{kx+k^2t}(1 + 0(\frac{1}{k}))$, thus giving proper normalization of ψ, we can write $\psi(x, t, k)$:

(3) $$\psi(x, t, k) = e^{kx+k^2t} \times \sum_{n=0}^{\infty} \psi_n k^{-n}$$

in the neighborhood of $k = \infty$. Assuming meromorphicity of ψ on x and the form (2) of $u(x, t)$ we obtain that $\psi_n = \psi(x, t)$ is a meromorphic function in x with first-order poles. According to (2), we can write ψ_n in the following form:

(4) $$\psi_0 = 1 \quad \text{and}$$

(5) $$\psi_n = \sum_{i\in I} w_i^n (x - a_i)^{-1} \quad : \quad n \geq 1 \ .$$

Here $w_i^n = w_i^n(t)$. Substituting (3), (4), (5) into (1), we obtain the following system:

(6) $$\psi_{n+1,\, x} = \frac{1}{2}\psi_{n,t} - \frac{1}{2}\psi_{uxx} - \frac{1}{2}u\psi_n : \quad n \geq -1$$

and thus

(7) $$\psi_{1,x} = -\frac{1}{2}u \ .$$

In terms of w_i^n we obtain the following system of equations of w_i^n :

(8) $$w_i^1 = -1 : \quad i \in I \ ;$$

(9) $$-w_i^{k+1} = \frac{1}{2}a_{it}w_i^k + \sum_{j\neq i} w_j^k (a_i - a_j)^{-1} : \quad i \in I$$

(10) $$\frac{1}{2}w_{it}^k + \sum_{j\neq i} (w_i^k - w_j^k)(a_i - a_j)^{-2} = 0 : \quad i \in I \ .$$

Now we can define two matrices A and L by putting [6], [8], [9]:

(11) $$L_{ij} = \delta_{ij} b_{it} + (1 - \delta_{ij}) \sqrt{-G}\,(a_i - a_j)^{-1}$$

and

(12) $$A_{ij} = \delta_{ij}\Big(-\sqrt{-G} \sum_{k \neq i} (a_i - a_k)^{-2}\Big) + (1 - \delta_{ij}) \times \sqrt{-G}\,(a_i - a_j)^{-2}$$

for $i, j \in I$.

Then, as it is well known, [6], [8], [9], the Lax system

(13) $$\frac{dL}{dt} = [A, L]$$

is equivalent to the many-particle system with Hamiltonian H :

(14) $$a_{itt} = 2G \sum_{j \neq i} (a_i - a_j)^{-3} : \quad i \in I \ .$$

However, if we take $G = -4$, then the system (9), (10) is equivalent to the following equation:

(15) $$\vec{w}^{k+1} = -\frac{1}{2} L \vec{w}^k$$

and

(16) $$\vec{w}^k_t = A\vec{w}^k \ ,$$

respectively. Here $\vec{w}^k = (w^k_i)_{i \in I}$ -vector corresponding to residues of Ψ_k . Consistency of the system (15), (16) is obviously equivalent to the system

(17) $$\Big(-\frac{1}{2}L\Big)_t = \Big[A, -\frac{1}{2}L\Big]$$

that is equivalent to (13) for $G = -4$. Thus, the poles a_i satisfy (14) with $G = -4$.

Conversely, for any system $a_i = a_i(t)$ of particles satisfying (18), we can effectively construct the potential $u(x, t)$ (2) and $\psi(x, t, k)$ (3) satisfying (1). In particular, we can easily solve the eigenvalue problem (1) for the rational function u with poles of order 2.

Results (14) or (17), together with the construction of higher KdV equations through the Sturm-Liouville problem [3], [4], give us a complete description of the pole motion of the k-th order KdV equation, answering the question [8]. If we consider the k-th KdV equation $u_t = \frac{\partial}{\partial x} \cdot \frac{\delta}{\delta u}\int R_k[u, u_x, \ldots]dx$ corresponding to $n = 2$, $m = 2k + 1$ in the y-independent case [3], then the motion of poles corresponds to Hamiltonian J_{2k+1} [6], [7], [8]:

$$J_{2k+1} = \frac{1}{2k+1} \operatorname{tr}(L^{2k+1})$$

with the restriction $\operatorname{grad} H \equiv \operatorname{grad} J_2 = 0$ for $G = -4$. e.g., all rational solutions of higher KdV are easily examined.

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