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TWO BLOW-UP REGIMES FOR L^2 SUPERCRITICAL NONLINEAR SCHRÖDINGER EQUATIONS

FRANK MERLE, PIERRE RAPHAËL, AND JÉRÉMIE SZEFTEL

ABSTRACT. We consider the focusing nonlinear Schrödinger equations $i\partial_t u + \Delta u + u|u|^{p-1} = 0$. We prove the existence of two finite time blow up dynamics in the supercritical case and provide for each a qualitative description of the singularity formation near the blow up time.

1. Introduction

1.1. The focusing nonlinear Schrödinger equation.

We consider in this paper the nonlinear Schrödinger equation

$$(NLS) \quad \begin{cases} iu_t = -\Delta u - |u|^{p-1}u, & (t,x) \in [0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & u_0 : \mathbb{R}^N \to \mathbb{C}, \end{cases} \quad (1)$$

in dimension $1 \le N \le 5$ with

$$1 < p < +\infty \text{ for } N = 1, 2 \text{ and } 1 < p < \frac{N+2}{N-2} \text{ for } N \ge 3.$$

From a result of Ginibre and Velo [4], (1) is locally well-posed in $H^1 = H^1(\mathbb{R}^N)$ and thus, for $u_0 \in H^1$, there exists $0 < T \le +\infty$ and a unique solution $u(t) \in \mathcal{C}([0,T), H^1)$ to (1) and either $T = +\infty$, we say the solution is global, or $T < +\infty$ and then $\lim_{t \uparrow T} |\nabla u(t)|_{L^2} = +\infty$, we say the solution blows up in finite time.

Recall that (1) admits the following conservation laws in the energy space H^1:

$L^2 - \text{norm}: \quad \int |u(t,x)|^2 dx = \int |u_0(x)|^2 dx,$

$\text{Energy}: \quad E(u(t,x)) = \frac{1}{2}\int|\nabla u(t,x)|^2 dx - \frac{1}{p+1}\int|u(t,x)|^{p+1}dx = E(u_0),$

$\text{Momentum}: \quad Im(\int \nabla u(t,x)\overline{u(t,x)}dx) = Im(\int \nabla u_0(x)\overline{u_0(x)}dx).$

1.2. The scaling symmetry and the virial law.

The scaling symmetry $\lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x)$ leaves the homogeneous Sobolev space $\dot{H}^{\sigma_c}$ invariant with

$$\sigma_c = \frac{N}{2} - \frac{2}{p-1}. \quad (2)$$

From the conservation of the energy and the L^2 norm, the equation is subcritical for $\sigma_c < 0$ and all H^1 solutions are global and bounded in H^1. The smallest power for which blow up may occur is

$$p_c = 1 + \frac{4}{N}$$

which corresponds to $\sigma_c = 0$ and is referred to as the L^2 critical case. The case $0 < \sigma_c < 1$ is the L^2 super critical and H^1 subcritical case.

Let us recall the simple calculation -the so-called virial law- establishing the existence of blow-up solutions in the critical and supercritical cases (see [19]). For $u_0 \in \Sigma = H^1 \cap \{xu \in L^2\}$, we have the following simple computation:

$$\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx = 4N(p-1)E_0 - \frac{16\sigma_c}{N-2\sigma_c} \int |\nabla u|^2.$$

In particular, we obtain for the L^2 critical and supercritical cases ($\sigma_c \geq 0$):

$$\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx \leq 4N(p-1)E_0.$$

Now, if u_0 has negative energy $E_0 < 0$, this implies that the positive quantity $\int |x|^2 |u(t,x)|^2 dx$ lies below an inverted parabola and has thus to become strictly negative after some time. Hence, the solution cannot exist for all times.

The strength of this blow up proof is that it applies to an large open region of the energy space -up to extra integrability condition. However, it does not provide any explicit description of the singularity formation and of the different possible regimes. The goal of this paper is to present two recent descriptions of blow-up regimes for (1) in the super critical case. We first recall some known results in the critical case.

1.3. The L^2 critical case.
The critical case corresponds to $\sigma_c = 0$ and $p_c = 1 + \frac{4}{N}$, in which case (1) may be rewritten:

$$(NLS) \quad \begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}} u, & (t,x) \in [0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & u_0 : \mathbb{R}^N \to \mathbb{C}. \end{cases} \quad (3)$$

Let Q be the unique positive, radial, nonzero solution to the following equation:

$$\Delta Q - Q + Q^{1+\frac{4}{N}} = 0, \quad Q \in H^1, \quad (4)$$

see [3], [6]. Recall that for u_0 in H^1 with $|u_0|_{L^2} < |Q|_{L^2}$, the corresponding solution $u(t)$ to (3) is global and bounded in H^1 (see [18]). This result is sharp, since there are explicit blow-up solutions with $|u_0|_{L^2} = |Q|_{L^2}$. Indeed, consider the pseudo-conformal transform: if $u(t,x)$ is a solution to (1), then so is

$$v(t,x) = \frac{1}{|t|^{\frac{N}{2}}} \bar{u}\left(\frac{1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}}.$$

Applying the pseudo-conformal transform to the solution to (3) given by $u(t,x) = Q(x)e^{it}$ yields a solution

$$S(t,x) = \frac{1}{|t|^{\frac{N}{2}}} Q\left(\frac{x}{t}\right) e^{i\frac{|x|^2}{4t} - \frac{i}{t}}, \ |S(t)|_{L^2} = |Q|_{L^2}, \quad (5)$$

blowing up at $t = 0$ with the blow-up speed:

$$|\nabla S(t)|_{L^2} \sim \frac{1}{|t|}.$$

In the case $|u_0|_{L^2} > |Q|_{L^2}$, two blow-up regimes have been exhibited. The first one has been constructed by Bourgain and Wang [1] in dimension $N = 1, 2$. The authors construct solutions $u(t)$ to (3) which blow up in finite time and behave locally like the explicit blow-up solution $S(t)$ given by (5). In particular, these solutions have the same blow-up speed as $S(t)$:

$$|\nabla u(t)|_{L^2} \sim \frac{1}{T - t}.$$

Note that these solutions are never observed numerically and are thus believed to be unstable.

A second type of blow-up regime is the so-called 'log-log' regime which is characterized by the following blow-up speed:

$$|\nabla u(t)|_{L^2} \sim \left(\frac{\log|\log(T - t)|}{T - t} \right)^{\frac{1}{2}}.$$

It has been exhibited by Perelman [13] in dimension $N = 1$ and further extensively studied by Merle and Raphael in the series of papers [7], [8], [14], [9], [10], [11] where a complete description of this stable blow up dynamics is given together with sharp classification results in dimension $N \leq 5$. In particular, it leads to a stable blow-up dynamic.

We recall below the existence of the log-log regime obtained in the work of Merle and Raphaël.

Theorem 1 (Existence of a stable log-log regime, [7], [8], [9], [14], [10], [11])**.** *Let $N \leq 5$. There exists a universal constant $\alpha^* > 0$ such that the following holds true. For any initial data $u_0 \in H^1$ with small super-critical mass*

$$|Q|_{L^2} < |u_0|_{L^2} < |Q|_{L^2} + \alpha^* \tag{6}$$

and nonpositive Hamiltonian $E(u_0) < 0$, the corresponding solution to (3) blows up in finite time $0 < T < +\infty$ according to the following blowup dynamics: there exist geometrical parameters $(\lambda(t), x(t), \gamma(t)) \in \mathbb{R}_+^* \times \mathbb{R}^N \times \mathbb{R}$ and an asymptotic residual profile $u^* \in L^2$ such that:

$$u(t) - \frac{1}{\lambda^{\frac{N}{2}}(t)} Q\left(\frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)} \to u^* \quad in \;\; L^2.$$

The blowup point converges at blowup time:

$$x(t) \to x(T) \in \mathbb{R}^N \quad as \;\; t \to T,$$

the blowup speed is given by the log-log law

$$\lambda(t) \sqrt{\frac{\log|\log(T - t)|}{T - t}} \to \sqrt{2\pi} \quad as \;\; t \to T, \tag{7}$$

and the residual profile satisfies:

$$u^* \in L^2 \quad but \;\; u^* \notin L^p, \quad \forall p > 2.$$

More generally, the set of initial data satisfying (6) and such that the corresponding solution to (1) blows up in finite time with the log-log law (7) is open in H^1.

We now turn to the supercritical case.

2. Two blow-up dynamics in the super critical case

The explicit description of blow up dynamics in the super critical setting is mostly open. We present below two blow-up regimes that have been exhibited recently. Interestingly, they both rely on the log-log analysis [10] which allowed Merle and Raphael to derive the sharp log-log law in the critical case.

2.1. Blow up on a sphere for the quintic NLS in dimension $N \geq 2$.

We consider the quintic nonlinear Schrödinger equation with radial data:

$$\begin{cases} iu_t = -\partial_r^2 u - \dfrac{N-1}{r}\partial_r u - |u|^4 u, & (t,r) \in [0,T) \times \mathbb{R}^+, \\ u(0,r) = u_0(r), & u_0 : \mathbb{R}^+ \to \mathbb{C}, \end{cases} \tag{8}$$

which is super critical for $N \geq 2$. The existence and *radial* stability of self similar solutions blowing up on an asymptotic blow up sphere -and not a blow up point- is proved by Raphael [15] for $N = 2$, and by Raphael, Szeftel [16] for $N \geq 3$:

Theorem 2 (Existence and stability of a solution blowing up on a sphere in $\mathbb{R}^N$). *Let Q be the one dimensional ground state solution to (4), explicitly*

$$Q(x) = \left(\frac{3}{\mathrm{ch}^2(x)} \right)^{\frac{1}{4}}.$$

There exists an open subset $\mathcal{O} \subset H_{rad}^N(\mathbb{R}^N)$ such that the following holds true. Let $u_0 \in \mathcal{O}$, then the corresponding solution $u(t)$ to (8) blows up in finite time $0 < T < +\infty$ according to the following dynamics. There exist $\lambda(t) > 0$, $r(t) > 0$ and $\gamma(t) \in \mathbb{R}$ such that

$$u(t,r) - \frac{1}{\lambda(t)^{\frac{1}{2}}} Q\left(\frac{r - r(t)}{\lambda(t)} \right) e^{i\gamma(t)} \to u^*(r), \quad \text{in } L^2 \text{ as } t \to T. \tag{9}$$

Here the radius of the singular circle converges

$$r(t) \to r(T) > 0 \quad \text{as } t \to T \tag{10}$$

and

$$\lambda(t) \left(\frac{\log|\log(T-t)|}{T-t} \right)^{\frac{1}{2}} \to \frac{\sqrt{2\pi}}{|Q|_{L^2}} \quad \text{as } t \to T. \tag{11}$$

Moreover, $\frac{N-1}{2}$ derivatives propagate outside the singularity:

$$\forall R > 0, \quad u^* \in H^{\frac{N-1}{2}}\left(|r - r(T)| > R \right). \tag{12}$$

Note that Theorem 2 includes the energy critical case $N = 3$ and energy super critical problems for $N \geq 4$. Let us briefly sketch the strategy of the proof. One can expect that if the singularity formation happens along the circle $r = 1$, then

$$\left| \frac{\partial_r u}{r} \right| \sim |\partial_r u| << |\partial_r^2 u|$$

and the leading order blow up dynamics should be given by the one dimensional quintic (NLS) which is critical, and for which a stable log-log dynamics is known. We then choose initial conditions which are already close to the one dimensional log-log blow-up. The above heuristic together with the fact that

the log-log regime is stable keeps things under control near the singularity, and controls the fact that the singularity stays away from the origin.

Then, the difficulty is to control the critical norm of $u(t)$ near the origin. More precisely, we need to prove

$$|u|_{H^{\frac{N-1}{2}}(r \leq \frac{1}{2})} \ll 1.$$

Thus, we face a problem of propagation of regularity outside of the blow-up sphere. In dimension $N = 2$ [15], one has to control $|u|_{H^{\frac{1}{2}}(r \leq \frac{1}{2})}$ which is achieved by exploiting the properties of the log-log regime together with the smoothing effect for (NLS). In dimensions $N \geq 3$, controlling $|u|_{H^{\frac{N-1}{2}}(r \leq \frac{1}{2})}$ requires a delicate bootstrap procedure, and we refer to [16] for the details.

2.2. **Stable self similar blow up for slightly supercritical NLS.** While the blow-up solutions of Theorem 2 display a stability with respect to radial perturbations, they are believed to be unstable by non radial perturbations (see the numerical computations in [2]). In fact, it has long been conjectured according to numerical simulations, see [17] and references therein, that the generic blow up dynamics in the super critical setting -at least for p near p_c- should be of self similar type.

Let Q_p be the unique positive, radial, nonzero solution to the following equation:

$$\Delta Q_p - Q_p + Q_p^p = 0, \quad Q_p \in H^1. \tag{13}$$

The following result by Merle, Raphaël, Szeftel [12] establishes the existence of a stable self similar regime in the energy space for slightly super critical (NLS). We again rely on the log-log analysis [10] to somehow *bifurcate* from the critical value $p = p_c$.

Theorem 3 (Existence and stability of a self similar blow up regime). *Let $1 \leq N \leq 5$. There exists $p^* > p_c$ such that for all $p \in (p_c, p^*)$, there exists $\delta(p) > 0$ with $\delta(p) \to 0$ as $p \to p_c$ and an open set $\mathcal{O}$ in H^1 of initial data such that the following holds true. Let $u_0 \in \mathcal{O}$, then the corresponding solution to (1) blows up in finite time $0 < T < +\infty$ according to the following dynamics: there exist geometrical parameters $(\lambda(t), x(t), \gamma(t)) \in \mathbb{R}_+^* \times \mathbb{R}^N \times \mathbb{R}$ and an excess of mass $\varepsilon(t) \in H^1$ such that:*

$$\forall t \in [0, T), \quad u(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)}[Q_p + \varepsilon(t)]\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \tag{14}$$

with

$$|\nabla \varepsilon(t)|_{L^2} \leq \delta(p). \tag{15}$$

The blowup point converges at blowup time:

$$x(t) \to x(T) \in \mathbb{R}^N \quad as \ \ t \to T, \tag{16}$$

and the blow up speed is self similar:

$$\lambda(t) \sim \sqrt{T - t} \quad as \ \ t \to T. \tag{17}$$

Remark 4. *Note that the mechanism of blow-up is stable since the set $\mathcal{O}$ of initial conditions is open in H^1. Also, note that we do not prove asymptotic stability which would correspond to ε converging towards 0 instead of just being bounded as in (15). In fact, we even do not prove orbital stability since the bound in (15) depends only on p and not on the size of ε at $t = 0$.*

In the rest of the paper, we sketch the proof of Theorem 3. We refer to [12] for the details.

3. Approximate self-similar solutions

Our aim in this section is to construct suitable approximate solutions to (1). Let us make the following general ansatz:

$$u(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)} v\left(t, \frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \qquad (18)$$

and introduce the rescaled time

$$\frac{ds}{dt} = \frac{1}{\lambda^2(t)},$$

then u is a solution to (1) if and only if v solves:

$$i\partial_s v + \Delta v - v - i\frac{\lambda_s}{\lambda}\Lambda v + v|v|^{p-1} = (\gamma_s - 1)v + i\frac{x_s}{\lambda} \cdot \nabla v, \qquad (19)$$

where $\Lambda = \frac{2}{p-1} + y \cdot \nabla$. Let us fix

$$\gamma_s = 1, \quad x_s = 0, \quad -\frac{\lambda_s}{\lambda} = b(s), \quad b_s = 0.$$

Note that this corresponds to a self similar regime since $\lambda\lambda_t = -b$ which together with $b_s = 0$ yields $\lambda(t) = C\sqrt{T - t}$ for some constant $C > 0$. We look for solutions of the form:

$$v(y) = Q_b(y)$$

where the unknown is the mapping $b \to Q_b$. The self similar equation becomes:

$$\Delta Q_b - Q_b + ib\Lambda Q_b + Q_b|Q_b|^{p-1} = 0. \qquad (20)$$

Let us perform the conformal change of variables

$$P_b = Q_b e^{\frac{ib|y|^2}{4}},$$

then a simple algebra leads to:

$$\Delta P_b - P_b - i\sigma_c b P_b + \frac{1}{4}b^2|y|^2 P_b + P_b|P_b|^{p-1} = 0. \qquad (21)$$

Since we are in a slightly supercritical case, σ_c is small, and the idea is to treat $\sigma_c b P_b$ in (21) as an error term. Thus, we look for P_b solution to:

$$\Delta P_b - P_b + \frac{1}{4}b^2|y|^2 P_b + P_b|P_b|^{p-1} = 0. \qquad (22)$$

Note that the linear operator $-\Delta + 1 - \frac{b^2|y|^2}{4}$ in (22) is the same as in the critical case. In particular, its structure is well known. It is coercive in the region $|y| < \frac{2}{b}$, whereas in $|y| \geq 2/b$, it induces oscillations:

$$|P_b(y)| \sim |y|^{-\frac{N}{2}} \tag{23}$$

so that P_b does not belong to L^2. As in the critical case, we truncate the solution near the turning point $|y| = \frac{2}{b}$. Going back to Q_b, we obtain an approximate solution to (20):

$$\Delta Q_b - Q_b + ib\Lambda Q_b + Q_b|Q_b|^{p-1} = O\left(e^{-\frac{\pi}{2b}} + b\sigma_c\right) \tag{24}$$

where the first error term comes from the truncation near $|y| = \frac{2}{b}$, and the second one comes from the fact that we treat $\sigma_c b P_b$ in (21) as an error term.

Remark 5. *In reality, we need a slightly more precise approximate solution of the type $Q_b + \sigma_c T_b$. To keep the exposition simple, we drop the correction term $\sigma_c T_b$ and refer to [12] for the details. Also, the error term generated by the truncation near $|y| = \frac{2}{b}$ is not exactly $O(e^{-\frac{\pi}{2b}})$. Again, we ignore this fact for the sake of clarity.*

4. Modulation theory

Let $u(t)$ solution to (1) with maximum life time interval $[0, T)$, $0 < T \leq +\infty$. Using the regularity $u \in \mathcal{C}([0, T), H^1)$ and standard modulation theory, and provided the open set of initial data $\mathcal{O}$ has been appropriately chosen, we can find a small interval $[0, T^*)$ such that for all $t \in [0, T^*)$, $u(t)$ admits a unique geometrical decomposition

$$u(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)}(Q_{b(t)} + \varepsilon)\left(t, \frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \tag{25}$$

where uniqueness follows from the freezing of orthogonality conditions: $\forall t \in [0, T^*]$,

$$\left(\varepsilon_1(t), |y|^2\Sigma\right) + \left(\varepsilon_2(t), |y|^2\Theta\right) = 0, \tag{26}$$

$$\left(\varepsilon_1(t), y\Sigma\right) + \left(\varepsilon_2(t), y\Theta\right) = 0, \tag{27}$$

$$\left(\varepsilon_2(t), \Lambda^2\Sigma\right) - \left(\varepsilon_1(t), \Lambda^2\Theta\right) = 0, \tag{28}$$

$$\left(\varepsilon_2(t), \Lambda\Sigma\right) - \left(\varepsilon_1(t), \Lambda\Theta\right) = 0, \tag{29}$$

where we have denoted:

$$\varepsilon = \varepsilon_1 + i\varepsilon_2, \quad Q_b = \Sigma + i\Theta,$$

in terms of real and imaginary parts. This decomposition is essentially a an application of the implicit function theorem (see [7], [8] for related statements).

Note that the size of T^* is a priori not under control. The goal will be to show that we may choose $T = T^*$, and that we have the following control on $[0, T)$: $0 < b_1 \leq b \leq b_2 \ll 1$, $\lambda(t) \sim \sqrt{T - t}$, $|\nabla \varepsilon|_{L^2} \lesssim e^{-\frac{\pi}{2b}}$ and $x(t) \to x(T)$. This corresponds to (14)-(17) and will therefore conclude the proof of Theorem 3.

5. **The modulation equations**

We start by showing that the control of b and ε immediately yields the selfsimilar behavior $\lambda \sim \sqrt{T-t}$ and the convergence of the translation parameter $x(t) \to x(T)$. To this end, we compute the modulation equations for the scaling and the translation parameter. First, let us look at the equation satisfied by ε. In view of the definition of v (18) and the decomposition of u (25), we have $v = Q_b + \varepsilon$. Now, replacing v by $Q_b + \varepsilon$ in (19) and using the fact that Q_b is an approximate solution (24) yields:

$$i\partial_s\varepsilon + M(\varepsilon) + R(\varepsilon) + \mathcal{P} + O\left(e^{-\frac{\pi}{2b}} + b\sigma_c\right), \tag{30}$$

where M is a linear operator, $R(\varepsilon)$ contains terms that are at least quadratic in ε, and $\mathcal{P}$ contains all the parameters $b_s, \frac{\lambda_s}{\lambda} + b, x_s, \gamma_s$. The modulation equations consist in taking the inner product of (30) by the orthogonality directions (26)-(29) in order to estimate the parameters. The orthogonality conditions remove the $i\partial_s\varepsilon$ term so that the parameters $\mathcal{P}$ are controlled by terms involving ε and the remainder term $O\left(e^{-\frac{\pi}{2b}} + b\sigma_c\right)$. In particular, we obtain for the scaling and the translation parameter:

$$\left|\frac{\lambda_s}{\lambda} + b\right| \lesssim \left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}} + e^{-\frac{\pi}{2b}} + b\sigma_c, \tag{31}$$

and

$$\left|\frac{x_s}{\lambda}\right| \lesssim \left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}} + e^{-\frac{\pi}{2b}} + b\sigma_c. \tag{32}$$

Assume now that we have proved the following control for b and ε:

$$0 < b_1 \leq b \leq b_2 \ll 1 \quad \text{and} \quad \int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \lesssim e^{-\frac{\pi}{b}}. \tag{33}$$

Then, (31) and (33) imply $\lambda\lambda_t = \frac{\lambda_s}{\lambda} \sim -b$ which after integration yields the self-similar behavior $\lambda \sim \sqrt{T-t}$. Also, (32), (33) and the self-similar behavior of λ imply $|x_t| = \left|\frac{x_s}{\lambda^2}\right| \ll \frac{1}{\lambda} \sim \frac{1}{\sqrt{T-t}}$ which after integration yields the convergence of the scaling parameter $x(t) \to x(T)$.

Thus, we have reduced the proof of Theorem 3 to the control of b and ε. This is achieved by a bootstrap method. We assume $0 < b_1 \leq b \leq b_2 \ll 1$ and $\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \leq C e^{-\frac{\pi}{b}}$ on $[0, T^*)$ for some possibly small $T^* > 0$ and we improve on the constants to show that in fact $T = T^*$. The proof relies on two monotonicity formulas. These monotonicity formulas are obtained using the same procedure as in the critical case and the point is to track the structure of the new terms.

6. **First monotonicity formula**

To obtain our first monotonicity formula, we compute the modulation equation for b. This corresponds to taking the imaginary part of the inner product of the equation of ε (30) with ΛQ_b. This computes b_s in function of a remainder term and ε.

Since we look for a monotonicity formula, we would like the terms in ε to have a sign. This is obviously not the case for the linear term. Fortunately,

the linear term in ε corresponds to the linearization of the energy around Q_b and can be replaced by the energy of Q_b and terms at least quadratic in ε by injecting the conservation of energy. Since the energy of Q_b is degenerate, this computes b_s in function of a remainder term and terms at least quadratic in ε.

Now, we are done provided the quadratic form in ε has a sign. This is the case since the negative directions of the quadratic form are controlled using the orthogonality conditions (26)-(29) (which is the main motivation for their choice) together with the conservation of energy and momentum. Finally, we arrive at the following monotonicity formula:

$$b_s \geq c_1 \left(\sigma_c + \int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - \frac{1}{c_1} e^{-\frac{\pi}{b}}, \qquad (34)$$

for some constant $c_1 > 0$.

Remark 6. *The new term with respect to the critical case is $c_1 \sigma_c$. The crucial observation is that this term turns out to have the good sign for our analysis.*

Remark 7. *The last term in (34) is not exactly $e^{-\frac{\pi}{b}}$. We ignore this fact for the sake of clarity and refer to [12] for the details.*

7. Second monotonicity formula

The monotonicity formula (34) will allow us to bound b from below. We now need a second monotonicity formula to bound b from above. We again follow the analysis in the critical case. The first step consists in choosing an improved approximate solution taking into account the behavior at infinity of the true self-similar solution. Indeed, remember that we have truncated Q_b near $|y| = \frac{2}{b}$ and therefore completely neglected the oscillating behavior of the true self similar solution taking place in the region $|y| \geq \frac{2}{b}$. Thus, we now consider a new approximate self-similar profile $\hat{Q}_b$ where we now truncate at $|y| \sim A$ with $A \gg \frac{2}{b}$. We refer to [12] for the precise choice of A which turns out to be exponentially decreasing in $-\frac{1}{b}$.

We now rerun the procedure used to obtain the first monotonicity formula where Q_b is replaced by $\hat{Q}_b$ and ε by $\hat{\varepsilon} = \varepsilon + Q_b - \hat{Q}_b$. The non L^2 tail of the true self-similar solution (see (23)) implies a corresponding decay for $\hat{Q}_b$ in the region $A \leq |y| \leq 2A$. In turn, this forces us to control terms of type $\int_A^{2A} |\varepsilon|^2$.

In fact, we are able to control $b \int_A^{2A} |\varepsilon|^2$ via a localization in space of the L^2 conservation law. Thus, we multiply everything by b. Finally, we arrive at the following monotonicity formula:

$$-\{\mathcal{J}\}_s \geq c_2 b \left(e^{-\frac{\pi}{b}} + \int |\nabla \hat{\varepsilon}|^2 + \int |\hat{\varepsilon}|^2 e^{-|y|} \right) - \frac{b}{c_2} \sigma_c \qquad (35)$$

where $c_2 > 0$ is a constant and $\mathcal{J}$ is an expression depending on b and ε with the following behavior:

$$\mathcal{J} \sim b^2. \qquad (36)$$

Note that the new term in (35) with respect to the critical case is $\frac{b}{c_2} \sigma_c$.

Remark 8. *The first term in the right-hand side of* (35) *is not exactly* $e^{-\frac{\pi}{b}}$. *We ignore this fact for the sake of clarity and refer to* [12] *for the details.*

8. **Control of b and ε**

We now use the first and the second monotonicity formulas to control b and ε. For simplicity, we will assume that we may divide (35) by b and obtain together with (36):

$$-b_s \geq c_3 \left(e^{-\frac{\pi}{b}} + \int |\nabla \hat{\varepsilon}|^2 + \int |\hat{\varepsilon}|^2 e^{-|y|} \right) - \frac{1}{c_3} \sigma_c, \qquad (37)$$

for some constant $c_3 > 0$. We now use (34) and (37) and the sign of the ε and $\hat{\varepsilon}$ terms to obtain:

$$c_1 \sigma_c - \frac{1}{c_1} e^{-\frac{\pi}{b}} \leq b_s \leq -c_3 e^{-\frac{\pi}{b}} + \frac{1}{c_3} \sigma_c. \qquad (38)$$

(38) immediately yields upper and lower bounds for b of type:

$$0 < b_1 \leq b \leq b_2 \ll 1 \quad \text{with} \quad \sigma_c \sim e^{-\frac{\pi}{b}}. \qquad (39)$$

Furthermore, using again the first monotonicity formula (34), the second version of the second monotonicity formula (37) and the bound (39) on b, we obtain:

$$\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \lesssim e^{-\frac{\pi}{b}} + \sigma_c \lesssim e^{-\frac{\pi}{b}}. \qquad (40)$$

(39) and (40) are the wanted estimates for b and ε which concludes the proof of Theorem 3.

Remark 9. *We obtain the convergence of the blow-up point* (16), *the self-similar speed* (17), *and the fact that* $\sigma_c \sim e^{-\frac{\pi}{b}}$ *(see* (39)*). All these properties are in accordance with the numerical simulations (see* [17]*).*

Remark 10. *Exact self-similar solutions (i.e. solutions to* (20)*) have been exhibited by Koppel and Landman,* [5], *for slightly super critical exponent using geometrical ODE techniques. These self similar solutions belong to* $\dot{H}^1 \cap L^{p+1}$ *but always miss* L^2 *and hence the physically relevant space* H^1. *Moreover, the construction of the self similar solution is delicate enough that it is not clear at all how this object should generate a stable self similar blow up dynamics. The strength of our method is to avoid the use of such delicate objects. Instead of obtaining a sharp description of the profile in the decomposition of* u (25)*, we use a rough profile* $Q_{b(t)}$. *In particular, note that we do neither prove the convergence of* b *to a limit as* $t \to T$, *nor the convergence of* ε *to 0. In fact,* (39) *and* (40) *do not exclude possible oscillations of both* $b(t)$ *and* $\varepsilon(t)$.*

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