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Introduction

In these notes we summarize the results obtained in [9] where we develop an infinite dimensional version of the Vey theorem and apply it to study the Birkhoff normal forms of integrable Hamiltonian PDEs near an equilibrium point. In his celebrated paper [11] J. Vey proved a local version of the Liouville-Arnold theorem which in the case of an elliptic singular point can be stated as follows. [1] Consider the standard symplectic linear space $(\mathbb{R}^{2n}_x, \omega_0)$, $\omega_0 = \sum_{j=1}^n dx_j \wedge dx_{n+j}$. Let $H(x) = O(|x|^2)$ be a germ of an analytic function [2] and V_H be the corresponding Hamiltonian vector field. It has a singularity at zero and we assume that in a suitable neighbourhood $\mathcal{O}$ of the origin, H has n commuting analytic integrals $H_1 = H, H_2, \dots, H_n$ such that $H_j(x) = O(|x|^2)$ for each j, the quadratic forms $d^2H_j(0)$, $1 \le j \le n$, are linearly independent and for all sufficiently small numbers $\delta_1, \dots, \delta_n$ we have $\{x : H_j(x) = \delta_j \; \forall j\} \Subset \mathcal{O}$. Then in the vicinity of the origin exist special symplectic analytic coordinates $\{y_1, \dots, y_{2n}\}$ (Birkhoff coordinates) in which the hamiltonians $H_r(x)$ may be written as $H_r(x) = \hat{H}_r(I_1, \dots, I_n)$, $I_j = \frac{1}{2}(y_j^2 + y_{n+j}^2)$, where $\hat{H}_1, \dots, \hat{H}_n$ are germs of analytic functions on $\mathbb{R}^n$. This theorem was further developed and generalised in [2, 4, 3, 12]. In [2, 3] Eliasson suggested a constructive proof of the theorem, which applies both to smooth and analytic hamiltonians and may be generalised to infinite-dimensional systems. In [9] we use Eliasson's approach to get an infinite-dimensional version of Vey's theorem applicable to integrable Hamiltonian PDEs. Namely, we consider the l_2-space h^0, formed by sequences $u = (u_1^+, u_1^-, u_2^+, u_2^-, \dots)$,

[1] Vey's result applies as well to hyperbolic singular points and to singular points of mixed type.

[2] Here and everywhere below 'a germ' means a germ at zero of a function or a map, defined in the vicinity of the origin.

provide it with the symplectic form $\omega_0 = \sum_{j=1}^{\infty} du_j^+ \wedge du_j^-$, and include h^0 in a scale $\{h^j, j \in \mathbb{R}\}$ of weighted l_2-spaces. Let us take any space h^m, $m \geq 0$, and in a neighbourhood $\mathcal{O}$ of the origin in h^m consider commuting analytic hamiltonians $I_1, I_2, \dots$. We assume that $I_j = O(\|u\|_m^2) \geq 0 \;\forall\, j$ and that this system of functions is *κ-regular*, $\kappa \geq 0$, in the following sense: There are analytic maps $F_j : \mathcal{O} \to \mathbb{R}^2$, $j \geq 1$, such that $I_j = \frac{1}{2}|F_j|^2$ and

i) the map $F = (F_1, F_2, \dots) : \mathcal{O} \to h^m$ is an analytic diffeomorphism on its image,

ii) $dF(0) = \mathrm{id}$ and the mapping $F - \mathrm{id}$ analytically maps $\mathcal{O} \to h^{m+\kappa}$ (i.e., $F - \mathrm{id}$ is κ-smoothing). Moreover, for any $u \in \mathcal{O}$ the linear operator $dF(u)^* - \mathrm{id}$ continuously maps h^m to $h^{m+\kappa}$.

We also make some mild assumptions concerning Cauchy majorants for the maps $F - \mathrm{id}$ and $dF(u)^* - \mathrm{id}$, see Section 1. The main result of [9] is the following theorem:

Theorem 1. *Let the system of commuting analytic functions $I_1, I_2, \dots$ on $\mathcal{O} \subset h^m$ is regular. Then there are analytic maps $F'_j : \mathcal{O}' \to \mathbb{R}^2$, defined on a suitable neighbourhood $0 \in \mathcal{O}' \subset \mathcal{O}$, such that the map $F' = (F'_1, F'_2, \dots) : \mathcal{O}' \to h^m$ satisfies properties i), ii), it is a symplectomorphism, the functions $I'_j = \frac{1}{2}|F'_j|^2$ commute and their joint level-sets define the same foliation of $\mathcal{O}'$ as level-sets of the original functions I_j. In particular, each I_j is an analytic function of the variables $I'_1, I'_2, \dots$.*

See Section 1, Theorem 2 for a more detailed statement of the result.

Theorem 1 applies to study an integrable Hamiltonian PDE in the vicinity of an equilibrium. As an example we consider in [9] the KdV equation under zero-meanvalue periodic boundary conditions

$$\dot{u}(t,x) = \frac{1}{4} u_{xxx} + 6uu_x, \quad x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}, \quad \int_0^{2\pi} u\,dx = 0. \tag{1}$$

We show that Theorem 1 with $\kappa = 1$ applies to (1) and guarantees the existence of local Birkhoff coordinates in a neighbourhood of the origin. The integrating transformation we get in this way has the form: "identity plus 1-smoothing analytic map", see Section 2, Theorem 4 for the precise statement.

1 Main result

Consider a scale of Hilbert spaces $\{h^m,\, m \in \mathbb{R}\}$. A space h^m is formed by *complex* sequences $u = (u_j \in \mathbb{C},\, j \geq 1)$ and is regarded as a *real* Hilbert space with the Hilbert norm

$$\|u\|_m^2 = \sum_{j \geq 1} j^{2m} |u_j|^2. \tag{2}$$

We will denote by $\langle \cdot, \cdot \rangle$ the scalar product in h^0: $\langle u, v \rangle = \sum u_j \cdot v_j = \mathrm{Re} \sum u_j \bar{v}_j$. For any linear operator $A : h^m \to h^n$ we will denote by $A^* : h^{-n} \to h^{-m}$ the operator, conjugated to A with respect to this scalar product.

We next introduce some notions related to the infinite-dimensional techniques needed for our arguments. We consider germs or real-analytic maps

$$F : \mathcal{O}_\delta(h^m) \to h^n, \quad F(0) = 0,$$

where $\mathcal{O}_\delta(h^m) = \{u \in h^m \,\big|\, \|u\|_m < \delta\}$ and $\delta > 0$ depends on F. Abusing language we will say that F is an analytic germ $F : h^m \to h^n$. Any analytic germ $F = (F^1, F^2, \dots)$ can be written as an absolutely and uniformly convergent series

$$F^j(u) = \sum_{N=1}^{\infty} F_N^j(u), \quad F_N^j(u) = \sum_{|\alpha|+|\beta|=N} A^j_{\alpha\beta} u^\alpha \bar{u}^\beta, \tag{3}$$

where $\alpha,\, \beta \in \mathbb{Z}_+^\infty$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We will write that $F(u) = O(u^l)$ if in (3) $F_N^j(u) = 0$ for $N < l$ and all j.

Clearly,

$$|F(u)| \leq \underline{F}(|u|), \quad \underline{F}^j(|u|) = \sum_{N=1}^{\infty} \sum_{|\alpha|+|\beta|=N} |A^j_{\alpha\beta}| |u|^{\alpha+\beta} \leq \infty.$$

Here $|F(u)| = (|F^1(u)|, |F^2(u)|, \dots)$, $|u| = (|u_1|, |u_2|, \dots)$ and $|u|^{\alpha+\beta} = \prod |u_j|^{\alpha_j+\beta_j}$. The inequality is understood component-wise.

Definition 1. An analytic germ F as above is called normally analytic (n.a.) if $\underline{F}$ defines a germ of a real analytic map $h_R^m \to h_R^n$, where the space h_R^m is formed by real sequences (u_j), given the norm (2). That is, each N-homogeneous map $\underline{F}_N^j(v) = \sum_{|\alpha|+|\beta|=N} |A^j_{\alpha\beta}| v^{\alpha+\beta}$, where $v \in h_R^m$, satisfies $\|\underline{F}_N(v)\|_n \leq CR^N \|v\|_m^N$ for suitable $C,\, R > 0$.

Take any $m \ge 0$ and $\kappa \ge 0$.

Definition 2. A n.a. germ $F : h^m \to h^{m+\kappa}$ belongs to the class $\mathfrak{A}_{m,\kappa}$ if $F = O(u^2)$ and the adjoint map $dF(u)^* v$ is such that

$$\underline{dF(|u|)^*|v|} = \Phi(|u|)|v|, \tag{4}$$

where the linear map $\Phi(|u|) = \Phi_F(|u|) \in \mathcal{L}(h_R^m, h_R^{m+\kappa})$ has non-negative matrix elements and defines an analytic germ $|u| \mapsto \Phi(|u|)$, $h_R^m \to \mathcal{L}(h_R^m, h_R^{m+\kappa})$.

The notion of a n.a. germ naturally formalizes the method of Cauchy majorants, is well known and was exploited before to calculate normal forms for nonlinear PDE and other infinite-dimensional systems. In particular, Nikolenko [10] used it to get a version of the Poincaré normal form theorem for dissipative PDE, while Bambusi and Grébert [1] applied it to calculate partial Birkhoff normal forms for some Hamiltonian PDE. In the same time the notion of the class $\mathfrak{A}_{m,\kappa}$ may be new. It is not difficult to check that the set of germs (id+$\mathfrak{A}_{m,\kappa}$) form a division ring with respect to taking a composition of germs (see [9]), this fact is one of the basic points of our analysis.

We will write elements of the spaces h^m as $u = (u_k \in \mathbb{C},\ k \ge 1)$, $u_k = u_k^+ + iu_k^-$, $u_k^\pm \in \mathbb{R}$, and provide h^m, $m \ge 0$, with a symplectic structure by means of the two-form $\omega_0 = \sum du_k^+ \wedge du_k^-$. This form may be written as $\omega_0 = idu \wedge du$. Here and below for any antisymmetric (in h^0) operator J we denote by $Jdu \wedge du$ the 2-form

$$(Jdu \wedge du)(\xi, \eta) =< J\xi, \eta > . \tag{5}$$

The form ω_0 is exact, $\omega_0 = d\alpha_0$, where

$$\alpha_0 = \frac{1}{2} \sum u_k^+ du_k^- - \frac{1}{2} \sum u_k^- du_k^+.$$

By $\{H_1, H_2\}$ we will denote the Poisson brackets of functionals H_1 and H_2, corresponding to ω_0: $\{H_1, H_2\}(u) = \langle i\nabla H_1(u), \nabla H_2(u)\rangle$. Functionals H_1 and H_2 *commute* if $\{H_1, H_2\} = 0$.

Theorem 2. *Assume that for some $m \ge 0$ there exists a real analytic germ $\Psi : h^m \to h^m$ such that*

i) $d\Psi(0) = id$ and $(\Psi - id) \in \mathfrak{A}_{m,\kappa}$ for some $\kappa \ge 0$;

ii) the functionals $I_j(\Psi(u)) = \frac{1}{2}\left|\Psi^j(u)\right|^2$, $j \geq 1$, *commute with each other.*

Then there exists a germ $\Psi^+ : h^m \to h^m$ *which satisfies i), ii) with the same* κ, *and such that*

a) foliation of the vicinity of the origin in h^m *by the sets*

$$\{\left|\Psi^j\right|^2 = const_j,\ \forall j\}; \tag{6}$$

is the same as by the sets $\{\left|\Psi^{+j}\right|^2 = const_j,\ \forall j\}$.

b) the germ Ψ^+ *is symplectic:* ${\Psi^+}^*\omega_0 = \omega_0$.

Remarks. 1) The sets, forming the foliation (6), are tori of dimension $\#\{const_j > 0\}$, which is $\leq \infty$.

2) By the item a) of the theorem each $I_j(\Psi(u))$ is a function of the vector $I^+ = \{I_j^+ = \frac{1}{2}|\Psi^{+j}|^2, j \geq 1\}$. In fact, I_j is an analytic function of I^+ with respect to the norm $\|I^+\| = \sum |I_j^+| j^{2m}$. E.g., see the proof of Lemma 3.1 in [7].

3) The map Ψ^+ is obtained from Ψ in a constructive way, independent from m.

4) The theorem above is an infinite-dimensional version of Theorem C in [3] which is the second step in Eliasson's proof of the Vey theorem. At the first step he *proves* that any n commuting integrals $H_1, \dots, H_n$ as in Introduction can be written in the form ii). In difference with his work we have to *assume* that the integrals are of the form ii), where the maps $\Psi_1, \Psi_2, \dots$ have additional properties, specified in i). Fortunately, we can check i) and ii) for some important infinite-dimensional systems.

2 Application to the KdV equation

Consider the KdV equation (1). This equation can be viewed as a Hamiltonian system in any Sobolev spaces H_0^m, $m \geq 1$, of zero-meanvalue functions on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, equipped with the symplectic form $\nu\big(u(\cdot), v(\cdot)\big) = -\int_{S^1} (\partial/\partial x)^{-1} u(x) \cdot v(x)\, dx$. The corresponding hamiltonian has the form:

$$h_{KdV}(u) = \int \Big(-\frac{1}{8}\, u_x^2 + u^3 \Big)\, dx.$$

It will be convenient for us to renormalize the symplectic space (H_0^m,ν) to the canonical space (h^m,ω_0). To do this we write any $u(x)\in H_0^m$ as Fourier series, $u(x)=\pi^{-1/2}\sum_{s=1}^\infty(u_s^+\cos sx-u_s^-\sin sx)$, and consider the map

$$T:u(x)\mapsto v=(v_1^\pm,v_2^\pm,\dots),\qquad v_j^\pm=u_j^\pm j^{-1/2}\quad\forall\, j.$$

Then $T:H_0^m\to h^{m+1/2}$ is an isomorphism for any m, and $T^*\omega_0=\nu$.

To apply Theorem 2 we need a way to construct germs of analytic maps $\Psi:h^m\to h^m$ which satisfy i) and ii). For Lax-integrable PDEs this can be done by using spectral characteristics of the associated Lax operator. The corresponding construction goes back to the work of Kappeler [5] (see also [7], pp. 42-44) and in the case of the KdV equation can be summarized as follows. The Lax operator for the KdV hierarchy is the Sturm-Liouville operator $L_u=-\partial^2/\partial x^2-u(x)$. Consider this operator on the interval $[0,4\pi]$ with the periodic boundary conditions. Its spectrum is discrete and consists of simple or double eigenvalues $\{\lambda_j,\ j\ge 0\}$, tending to infinity:

$$\lambda_0<\lambda_1\le\lambda_2<\dots,\quad \lambda_j\to\infty\ \text{ as }\ j\to\infty.$$

Let γ_j, $j\ge1$, be the lengths of spectral gaps of L_u: $\gamma_j=\lambda_{2j}-\lambda_{2j-1}$. It is well known that $\gamma_j^2(u)$, $j\ge1$, are commuting analytic functionals which are integrals of motion for (1), as well as for other equations from the KdV hierarchy, see [6]. In [5] T. Kappeler suggested a way to use the spectral theory of the operator L_u to construct germs of analytic maps $\Psi^j:h^{1/2}\to\mathbb{R}^2$, $j\ge1$, such that $\frac12|\Psi^j(v)|^2=\frac{\pi}{2j}\gamma_j^2(T^{-1}v)$. In [9] we show that the map $\Psi=(\Psi^1,\Psi^2,\dots)$ meets assumptions i), ii) of Theorem 2 with $\kappa=1$:

Theorem 3. *For any $m\ge1/2$, Ψ defines a real-analytic germ $\Psi:h^m\to h^m$ such that*

i) $d\Psi(0)=\mathrm{id}$ and $(\Psi-\mathrm{id})\in\mathfrak{A}_{m,1}$;

ii) for any $j\ge1$ and $v\in h^m$ we have $\frac12|\Psi^j(v)|^2=\frac{\pi}{2j}\gamma_j(u)^2$, where $u(x)=\frac{1}{\sqrt\pi}\operatorname{Re}\sum_{j=1}^\infty\sqrt{j}\,v_je^{ijx}$.

Combining Theorems 2 and 3 we get

Theorem 4. *For any $m\ge0$ there exists a germ of an analytic symplectomorphism $\overline\Psi:(H_0^m,\nu)\to(h^{m+1/2},\omega_0)$, $d\overline\Psi(0)=T$, such that*

a) $\overline\Psi-T$ defines a germ of an analytic mapping $H_0^m\to h^{m+3/2}$;

b) each γ_j^2, $j\ge1$, is an analytic function of the vector $\bar I=(\frac12|\overline\Psi^j(u)|^2,j\ge 1)$. Similar, a hamiltonian of any equation from the KdV hierarchy is an *analytic function of $\bar{I}$ (provided that m is so big that this hamiltonian is analytic on the space H_0^m);*

c) the maps $\overline{\Psi}$, corresponding to different m, agree. That is, if $\overline{\Psi}_{m_j}$ corresponds to $m = m_j$, $j = 1, 2$, then $\overline{\Psi}_{m_1} = \overline{\Psi}_{m_2}$ on $h^{\max(m_1,m_2)}$.

Assertion b) of the theorem means that the map $\overline{\Psi}$ puts KdV (and other equations from the KdV hierarchy) to the Birkhoff normal form.

In a number of publications, starting with [5], T. Kappeler with collaborators established existence of a global analytic symplectomorphism

$$\Psi : (H_0^m, \nu) \to (h^{m+1/2}, \omega_0), \quad d\Psi(0) = T,$$

which satisfies assertion b) of Theorem 4, see in [6]. Our work shows that a local version of Kappeler's result follows from Vey's theorem. What is more important, it specifies the result by stating that a local transformation which integrates the KdV hierarchy may be chosen '1-smoother than its linear part'. This specification is crucial to study qualitative properties of perturbed KdV equations, e.g. see [8].

Although we believe that 1-smoothing is optimal, we have only a partial result in this direction:

Proposition 1. *Assume that there exists a real-analytic germ $\Psi : H_0^m \to h^{m+1/2}$ $\forall\, m \geq 0$, $d\Psi(0) = T$, such that:*

a) for each $m \geq 0$, $\Psi - T$ defines a germ of analytic mapping $H_0^m \to h^{m+1/2+\kappa}$ with some $\kappa \geq 0$;

b) the hamiltonian h_{KdV} of the KdV equation is a function of the variables $\frac{1}{2}\left|\Psi^j(u)\right|^2$, $j \geq 1$, only.

Then $\kappa \leq 3/2$.

3 Proof of the main theorem

In this section we sketch the proof of Theorem 2, the details can be found in [9]. The proof is constructive and follows the scheme suggested in Section VI of [3]. To overcome corresponding infinite-dimensional difficulties we check recursively that all involved transformations of the phase-space h^m are in $id + \mathfrak{A}_{m,\kappa}$.

Denote

$$G = \Psi^{-1}, \quad \omega_1 = G^* \omega_0, \quad \alpha_1 = G^* \alpha_0.$$

We have $\omega_1 = d\alpha_1$.

For $j \geq 1$ and $\tau \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ define the rotation $\Phi_j^\tau : h^m \to h^m$ as the linear transformation of vectors $(u_1, u_2, \dots)$ which does not change components u_l, $l \neq j$, and multiplies u_j by $e^{i\tau}$.

Our goal is to find a transformation $\Theta : h^m \to h^m$ which satisfies i) of Theorem 2, reduces α_1 to α_0 and ω_1 to ω_0 and such that the maps $I_j \circ \Theta$, $j \geq 1$, are rotation invariant:

$$I_j \circ \Theta \circ \Phi_j^\tau = I_j \circ \Theta, \quad \forall j, k, \tau.$$

Then the mapping $\Psi^+ = \Theta \circ \Psi$ would satisfy the required properties. Following [3] we construct such Θ in two steps.

Step 1. At this step we achieve that the average in angles of the form ω_1 equal to ω_0.

For any function $f(u)$ we define its averaging with respect to j-th angle as

$$M_j f(u) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(\Phi_j^t u) dt,$$

and define its averaging in all angles as

$$M f(u) = (M_1 M_2 \dots) f(u) = \int\limits_{\mathbb{T}^\infty} f(\Phi^\theta u) d\theta,$$

where $d\theta$ is the Haar measure on $\mathbb{T}^\infty$ and $\Phi^\theta u = (\Phi_1^{\theta_1} \circ \Phi_2^{\theta_2} \circ \dots) u$. For a form α we define $M_j \alpha$ ad $M\alpha$ similarly. That is

$$M_j \alpha(u) = \frac{1}{2\pi} \int_0^{2\pi} ((\Phi_j^t)^* \alpha)(u) dt,$$

and $M\alpha = (M_1 M_2 \dots)\alpha$.

Let us consider the equation

$$(\omega_0 + s(M\omega_1 - \omega_0)) \rfloor Z_s = -M(\alpha_1 - \alpha_0).$$

It defines a non-autonomous vector field $Z_s \in \mathfrak{A}_{m,\kappa}$ for $0 \leq s \leq 1$. One can show that the flow map φ^s of Z_s belongs to $id + \mathfrak{A}_{m,\kappa}$, commutes with the rotations Φ_j^τ, $j \geq 1$, $\tau \in S^1$, and pulls $\omega_0 + s(M\omega_1 - \omega_0)$ back to ω_0. Therefore,

$$\omega_0 = (\varphi^1)^* M\omega_1 = M(\varphi^1)^* \omega_1.$$

Set $\bar{\Psi} = (\varphi^1)^{-1} \circ \Psi$. Then $\bar{\Psi}$ satisfies assumptions i), ii) and in addition $M\big((\bar{\Psi}^*)^{-1}\omega_0\big) = \omega_0$. Since φ^1 commutes with the rotations, then $\bar{\Psi}$ satisfies assertion a) of Theorem 2.

We re-denote back $\bar{\Psi} = \Psi$. Then

iii) $M\omega_1 = \omega_0$ for $\omega_1 = \big(\bar{\Psi}^*)^{-1}\omega_0$.

Step 2. Now we prove the theorem, assuming that Ψ meets i) – iii). Let χ_j be the vector field generating the rotations Φ_j^τ, $\tau \in \mathbb{R}$: $\chi_j(v) = (0, \dots, iv_j, 0, \dots)$. Denote $h_j(v) = (\alpha_1 - \alpha_0, \chi_j)$. We first construct a functional $f : h^m \to \mathbb{R}$ such that

$$(df, \chi_j) \equiv (iv_j \cdot \nabla_{v_j}) f(v) = h_j(v), \quad j \geq 1. \tag{7}$$

Using ii), iii) it is not difficult to check that h_j, $j \geq 1$, satisfy the following compatibility conditions

$$\chi_k(h_j) = \chi_j(h_k), \quad \forall j, k,$$

$$Mh_j = 0, \quad \forall j.$$

Under these conditions system (7) is solvable and its solution can be obtained by the explicit formulas which are due J. Moser (see [3]):

$$f(v) = \sum_{l=1}^{\infty} f_l(v), \quad f_l = M_1 \dots M_{l-1} L_l h_l, \tag{8}$$

where L_j is given by

$$L_j g(v) = \frac{1}{2\pi} \int_0^{2\pi} t\, g(\Phi_j^t(v)) dt.$$

One can show that series (8) converges and defines a n. a. germ $f : h^m \to \mathbb{R}$ that satisfies (7). Moreover, the germ $v \mapsto Y(v) = \nabla_v f(v)$, $h^m \to h^{m+\kappa}$, is n.a. and $Y(v) = O(v)$.

We are now in position to complete the proof of Theorem 2. Let Z_s be the solution of

$$(\omega_0 + s(\omega_1 - \omega_0)) \rfloor Z_s = -(\alpha_1 - \alpha_0 - df),$$

and let φ^s be the flow map of Z_s for $0 \le s \le 1$. One has: $\varphi^s - id : h^m \to h^{m+\kappa}$ is n.a., $\varphi^s - id = O(v^2)$, $(\varphi^s)^*(\omega_0 + s(\omega_1 - \omega_0)) = \omega_0$. In particular, the 1-time map φ^1 pulls ω_1 back to ω_0:

$$(\varphi^1)^*\omega_1 = \omega_0. \tag{9}$$

We now define Ψ^+ as

$$\Psi^+ = (\varphi^1)^{-1} \circ \Psi.$$

Then clearly, $\Psi^+ : h^m \to h^m$ is n.a., $d\Psi^+(0) = id$ and $d\Psi^+ - id : h^m \to h^{m+\kappa}$ is n.a. as well. Also by (9), ${\Psi^+}^* \omega_0 = \omega_0$.

Furthermore, it follows from (7) that

$$(\omega_0 + s(\omega_1 - \omega_0))(Z_s, \chi_j) = 0, \quad \forall j \ge 1,$$

which implies that $I_j \circ \varphi^s = I_j$, $j \ge 1$. As a consequence,

$$I_j \circ \Psi^+ = I_j \circ \Psi, \quad j \ge 1.$$

Finally, the property $\Psi^+ - id \in \mathfrak{A}_{m,\kappa}$ follows from the fact that Ψ^+ is symplectic and $d\Psi^+ - id : h^m \to h^{m+\kappa}$ is n.a. This completes the proof of the theorem.

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