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# STABLE BLOW UP DYNAMICS FOR THE CRITICAL CO-ROTATIONAL WAVE MAPS AND EQUIVARIANT YANG-MILLS PROBLEMS

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ABSTRACT. This note summarizes the results obtained in [30]. We exhibit stable finite time blow up regimes for the energy critical co-rotational Wave Map with the $\mathbb{S}^2$ target in all homotopy classes and for the equivariant critical $SO(4)$ Yang-Mills problem. We derive sharp asymptotics on the dynamics at blow up time and prove quantization of the energy focused at the singularity.

## 1. Introduction

We summarize the results obtained in [30] where we study the dynamics of two critical problems: the $(2+1)$-dimensional Wave Map and the $(4+1)$-dimensional Yang-Mills equations. These problems admit non trivial static solutions (topological solitons) which have been extensively studied in the literature both from the mathematical and physical point of view, see e.g. [2],[3],[11],[42]. The static solutions for the (WM) are harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^2 \subset \mathbb{R}^3$ satisfying the equation

$$-\Delta\Phi = \Phi|\nabla\Phi|^2$$

They are explicit solutions of the $O(3)$ nonlinear $\sigma$-model of isotropic plane ferromagnets. For the (YM) equations a particularly interesting class of static solutions is formed by (anti)self-dual instantons, satisfying the equations

$$F = \pm * F$$

for the curvature $F$ of an $so(4)$-valued connection over $\mathbb{R}^4$. The 4-dimensional euclidean Yang-Mills theory forms a basis of the Standard Model of particle physics and its special static solutions played an important role as pseudoparticle models in Quantum Field Theory.

The geometry of the moduli space of static solutions has been a subject of a thorough investigation, see e.g. [41],[1],[11]. In particular, the moduli spaces are incomplete due to the scale invariance property of both problems. This gave rise to a plausible scenario of singularity formation in the corresponding time dependent equation which has been studied heuristically, numerically and very recently from a mathematical point of view, [5],[12],[18],[19],[31],[21] and references therein.

The focus of this paper is the investigation of special classes of solutions to the critical $(2+1)$-dimensional (WM) and the critical $(4+1)$-dimensional (YM) describing a **stable** (in a fixed co-rotational class) and **universal** regime in which an open set of initial data leads to a finite time formation of singularities.

The Wave Map problem for a map $\Phi : \mathbb{R}^{2+1} \to \mathbb{S}^2 \subset \mathbb{R}^3$ is described by a nonlinear hyperbolic evolution equation

$$\partial_t^2\Phi - \Delta\Phi = \Phi\left(|\nabla\Phi|^2 - |\partial_t\Phi|^2\right)$$

with initial data $\Phi_0 : \mathbb{R}^2 \to \mathbb{S}^2$ and $\partial_t\Phi|_{t=0} = \Phi_1 : \mathbb{R}^2 \to T_{\Phi_0}\mathbb{S}^2$. We will study the problem under an additional assumption of co-rotational symmetry, which can be described as follows. Parametrizing the target sphere with the Euler angles $\Phi = (\Theta, u)$ we assume that the solution has a special form

$$\Theta(t,r,\theta) = k\theta, \qquad u(t,r,\theta) = u(t,r)$$

with an integer constant $k \geq 1$ – homotopy index of the map $\Phi(t,\cdot) : \mathbb{R}^2 \to \mathbb{S}^2$. Under such symmetry assumption the full wave map system reduces to the one dimensional semilinear wave equation:

$$\partial_t^2 u - \partial_r^2 u - \frac{\partial_r u}{r} + k^2\frac{\sin(2u)}{2r^2} = 0, \quad k \geq 1, \quad (t,r) \in \mathbb{R}\times\mathbb{R}_+, \quad k \in \mathbb{N}^*. \tag{1.1}$$

Similarly, the equivariant reduction, given by the ansatz,

$$A_\alpha^{ij} = (\delta_\alpha^i x^j - \delta_\alpha^j x^i)\frac{1-u(t,r)}{r^2},$$

of the $(4+1)$-dimensional Yang-Mills system

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta],$$
$$\partial_\beta F^{\alpha\beta} + [A_\beta, F^{\alpha\beta}] = 0, \qquad \alpha,\beta = 0,...,3$$

for the $so(4)$-valued gauge potential $A_\alpha$ and curvature $F_{\alpha\beta}$, leads in the semilinear wave equation:

$$\partial_t^2 u - \partial_r^2 u - \frac{\partial_r u}{r} - \frac{2u(1-u^2)}{r^2} = 0, \quad (t,r) \in \mathbb{R}\times\mathbb{R}_+. \tag{1.2}$$

The problems (1.1) and (1.2) can be unified by an equation of the form

$$\begin{cases} \partial_t^2 u - \partial_r^2 u - \frac{\partial_r u}{r} + k^2\frac{f(u)}{r^2} = 0, \\ u_{|t=0} = u_0, \quad (\partial_t u)_{|t=0} = v_0 \end{cases} \quad \text{with} \quad f = gg' \tag{1.3}$$

and

$$g(u) = \begin{cases} \sin(u), \quad k \in \mathbb{N}^* \ \text{ for } \ (WM) \\ \frac{1}{2}(1-u^2), \quad k = 2 \ \text{ for } \ (YM). \end{cases}$$

(1.3) admits a conserved energy quantity

$$E(u,\partial_t u) = \int_{\mathbb{R}^2}\left((\partial_t u)^2 + |\partial_r u|^2 + k^2\frac{g^2(u)}{r^2}\right)$$

which is left invariant by the scaling symmetry

$$u_\lambda(t,r) = u(\frac{t}{\lambda},\frac{r}{\lambda}), \quad \lambda > 0.$$

The minimizers of the energy functional can be explicitly obtained as

$$Q(r) = 2\tan^{-1}(r^k) \ \text{ for } \ (WM), \quad Q(r) = \frac{1-r^2}{1+r^2} \ \text{ for } \ (YM), \tag{1.4}$$

and their rescalings which constitute the moduli space of stationary solutions in the given corotational homotopy class.

A sufficient condition for the global existence of solutions to (1.3) was established in the pioneering works by Christodoulou-Tahvildar-Zadeh [8], Shatah-Tahvildar-Zadeh [33], Struwe [36]. It can be described as folllows: for smooth initial data $(u_0, v_0)$ with $E(u_0,v_0) < E(Q)$, the corresponding solution to (1.3) is global in time and decays to zero, see also [10]. More precisely, it was shown that if a singularity is formed at time $T < +\infty$, then energy must concentrate at $r = 0$ and $t = T$. This concentration must happen strictly inside the backward light cone from $(T, 0)$, that is if the scale of concentration is $\lambda(t)$, then

$$\frac{\lambda(t)}{T-t} \to 0 \quad \text{as} \quad t \to T. \tag{1.5}$$

Note that the case $\lambda(t) = T - t$ would correspond to self-similar blow up which is therefore ruled out. Finally, a universal blow up profile may be extracted in rescaled variables, at least on a sequence of times:

$$u(t_n, \lambda(t_n)r) \to Q \quad \text{in} \quad H^1_{loc} \quad \text{as} \quad n \to +\infty. \tag{1.6}$$

These results hold for more general targets for (WM) with $Q$ being a non trivial harmonic map. In particular, this implies the global existence and propagation of regularity for the corotational (WM) problem with targets admitting no non trivial harmonic map from $\mathbb{R}^2$. Very recently, in a series of works [38],[39],[34],[35],[17], this result has been remarkably extended to the full (WM) problem without the assumpion of corotational symmetry, hence completing the program developed in [14],[13],[40],[37],[16].

These works leave open the question of existence and description of singularity formation in the presence of non trivial harmonic maps, or the instanton for the (YM). This long standing question has first been addressed through some numerical and heuristic works in [4], [5], [12]. Côte [9] has shown instability of $Q$ for the $k = 1$ (WM) and (YM). A rigorous evidence of singularity formation has been recently given via two different approaches. In [31], Rodnianski and Sterbenz study the (WM) system for a large homotopy number $k \geq 4$ and prove the existence of *stable* finite time blow up dynamics. These solutions behave near blow up time according to the decomposition

$$u(t, r) = (Q + \varepsilon)(t, \frac{r}{\lambda(t)}) \quad \text{with} \quad \|\varepsilon, \partial_t \varepsilon\|_{\dot{H}^1 \times L^2} \ll 1 \tag{1.7}$$

with a lower bound on the concentration:

$$\lambda(t) \to 0 \quad \text{as} \quad t \to T \quad \text{with} \quad \lambda(t) \geq \frac{T-t}{|\log(T-t)|^{\frac{1}{4}}}. \tag{1.8}$$

In [18], [19], Krieger, Schlag and Tataru consider respectively the (WM) system for $k = 1$ and the (YM) equation and exhibit finite time blow up solutions which satisfy (1.7) with

$$\lambda(t) = \begin{cases} (T-t)^{\nu} & \text{for (WM) with} \quad k = 1, \\ (T-t)|\log(T-t)|^{-\nu} & \text{for (YM)} \end{cases} \tag{1.9}$$

for any chosen $\nu > \frac{3}{2}$. This continuum of blow up solutions are believed to be non-generic.

## 2. **Statement of the result**

The following theorem gives a complete description of a stable singularity formation for the (WM) for all homotopy classes and the (YM) in the presence of corotational/equivariant symmetry near the harmonic map/instanton, and is the main result of [30].

**Theorem 2.1** (Stable blow up dynamics of co-rotational Wave Maps and Yang-Mills)**.** *Let $k \geq 1$. Let $\mathcal{H}^2_a$ denote the affine Sobolev space* (2.7)*. There exists a set $\mathcal{O}$ of initial data which is open in $\mathcal{H}^2_a$ and a universal constant $c_k > 0$ such that the following*

*holds true. For all* $(u_0, v_0) \in \mathcal{O}$, *the corresponding solution to* (1.3) *blows up in finite time* $0 < T = T(u_0, v_0) < +\infty$ *according to the following universal scenario:*
(i) Sharp description of the blow up speed*: There exists* $\lambda(t) \in \mathcal{C}^1([0,T), \mathbb{R}_+^*)$ *such that:*

$$u(t, \lambda(t)y) \to Q \quad in \quad H^1_{r,loc} \quad as \quad t \to T \tag{2.1}$$

*with the following asymptotics:*

$$\lambda(t) = c_k(1 + o(1)) \frac{T - t}{|\log(T - t)|^{\frac{1}{2k-2}}} \quad as \quad t \to T \quad for \quad k \geq 2, \tag{2.2}$$

$$\lambda(t) = (T - t)e^{-\sqrt{|\log(T-t)|+O(1)}} \quad as \quad t \to T \quad for \quad k = 1. \tag{2.3}$$

*Moreover,*

$$b(t) = -\lambda_t(t) = \frac{\lambda(t)}{T - t}(1 + o(1)) \to 0 \quad as \quad t \to T.$$

(ii) Quantization of the focused energy*: Let* $\mathcal{H}$ *be the energy space* (2.6)*, then there exist* $(u^*, v^*) \in \mathcal{H}$ *such that the following holds true. Pick a smooth cut off function* $\chi$ *with* $\chi(y) = 1$ *for* $y \leq 1$ *and let* $\chi_{\frac{1}{b(t)}}(y) = \chi(b(t)y)$*, then:*

$$\lim_{t \to T} \left\| u(t,r) - \left(\chi_{\frac{1}{b(t)}} Q\right)\left(\frac{r}{\lambda(t)}\right) - u^*, \partial_t \left[u(t,r) - \left(\chi_{\frac{1}{b(t)}} Q\right)\left(\frac{r}{\lambda(t)}\right) - v^*\right]\right\|_{\mathcal{H}} = 0. \tag{2.4}$$

*Moreover, there holds the quantization of the focused energy:*

$$E_0 = E(u, \partial_t u) = E(Q, 0) + E(u^*, v^*). \tag{2.5}$$

This theorem thus gives a complete description of a stable blow up regime for all homotopy numbers $k \geq 1$. Stable blow up solutions in $\mathcal{O}$ decompose into a singular part with a universal structure and a regular part which has a strong limit in the scale invariant space. Moreover, the amount of energy which is focused by the singular part is a universal quantum independent of the Cauchy data.

**Remark 2.2.** The energy space $H$ corresponds to the norm:

$$\|(\varepsilon, \sigma)\|^2_{\mathcal{H}} = \int \left[\sigma^2 + (\partial_y \varepsilon)^2 + \frac{\varepsilon^2}{y^2}\right] \tag{2.6}$$

The $H^2$ type affine space $\mathcal{H}^2_a$ introduced in theorem 2.1 is explicitely:

$$\mathcal{H}^2_a = \mathcal{H}^2 + Q \tag{2.7}$$

with

$$\|(\varepsilon, \sigma)\|^2_{\mathcal{H}^2} = \|(\varepsilon, \sigma)\|^2_{\mathcal{H}} + \int \left[(\partial^2_y \varepsilon)^2 + \frac{(\partial_y \varepsilon)^2}{y^2} + (\partial_y \sigma)^2 + \frac{\sigma^2}{y^2}\right] \quad \text{for} \quad k \geq 2, \tag{2.8}$$

$$\|(\varepsilon, \sigma)\|^2_{\mathcal{H}^2} = \|(\varepsilon, \sigma)\|^2_{\mathcal{H}} + \int \left[(\partial^2_y \varepsilon)^2 + (\partial_y \sigma)^2 + \frac{\sigma^2}{y^2}\right] + \int_{y \leq 1} \frac{1}{y^2}\left(\partial_y \varepsilon - \frac{\varepsilon}{y}\right)^2 \quad \text{for} \quad k = 1.$$

This fixes in particular the boundary condition for $u \in \mathcal{H}^2_a$:

$$\lim_{r \to 0} u(t,r) = \begin{cases} 0 & \text{for} \quad (WM), \\ 1 & \text{for} \quad (YM) \end{cases}, \quad \lim_{r \to +\infty} u(t,r) = \begin{cases} \pi & \text{for} \quad (WM), \\ -1 & \text{for} \quad (YM) \end{cases}$$

*Comments on the result*

*1.* $k = 1$ *case*: In the $k \geq 2$ and (YM) case, the blow up speed $\lambda(t)$ is to leading order universal ie independent of initial data. On the contrary, in the $k = 1$ case, the presence of the $e^{O(1)}$ factor in the blow up speed seems to suggest that the law is not entirely universal and has an additional degree of freedom depending on the initial data. In general, the analysis of the $k = 1$ and to some extend $k = 2$ problems is more involved. In particular for $k = 1$, the instability direction $r\partial_r Q$ driving the singularity formation misses the $L^2$ space logarithmically. This anomalous logarithmic growth is fundamental in determining the blow up rate. On the other hand, this anomaly also adversely influences the size of the radiation term which implies that there is only a logarithmic difference between the leading order and the radiative corrections. This requires a very precise analysis and a careful track of all logarithmic gains and losses. In the case of larger k, these gains are polynomial and hence the effect of radiation is more easily decoupled from the leading order behavior.

*2. Regularity of initial data*: The open set $\mathcal{O}$ of initial data described in the theorem contains an open subset of $C^\infty$ data coinciding with $Q$ for all sufficiently large values of $r \geq R$. As a consequence, the main result of the paper in particular describes singularity formation in solutions arising from *smooth* initial data. This should be compared with the results in [18],[19] where solutions, specifically constructed to exhibit the blow up behavior given by the rates in (1.9), lead to the initial data of limited regularity dependent on the value of the parameter $\nu$ and degenerating as $\nu \to \frac{3}{2}$.

*3. Comparison with the $L^2$ critical (NLS)*: This theorem as stated can be compared to the description of the stable blow up regime for the $L^2$ critical (NLS)

$$iu_t + \Delta u + u|u|^{\frac{4}{N}} = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^N, \quad N \geq 1,$$

see Perelman [28] and the series of papers by Merle and Raphaël [23], [24], [29], [25], [26], [27]. There is a conceptual analogy between the mechanisms of a stable regime singularity formation for the critical (WM) and (YM) problems and the $L^2$ critical (NLS) problem. For the latter problem the sharp blow up speed and the quantization of the blow up mass is derived in [25], [26], [27]. The concentration occurs on an almost self-similar scale

$$\lambda(t) \sim \sqrt{\frac{2\pi(T-t)}{\log|\log(T-t)|}} \quad \text{as} \quad t \to T.$$

In both (WM), (YM) and the $L^2$ critical (NLS) problems self-similar singularity formation is corrected by subtle interactions between the ground state and the radiation parts of the solution. The precise nature of these interactions, affecting the blow up laws, depends in a very sensitive fashion on the asymptotic behavior of the ground state: polynomially decaying to the final value for the (WM) and (YM) and exponentially decaying for the (NLS), see also [20] for related considerations. This dependence becomes particularly apparent upon examining the blow up rates for the (WM) problem in different homotopy classes parametrized by $k$. For $k = 1$ the harmonic map approaches its constant value at infinity at the slowest rate, which leads to the strongest deviation of the corresponding blow up rate from the self-similar law.

*4. Least energy blow up solutions*: The importance of the $k = 1$ case for the (WM) problem is due to the fact that the $k = 1$ ground state is the least energy harmonic map:

$$E(Q) = 4\pi k.$$

A closer investigation of the structure of $Q$ for $k \geq 2$ shows that this configuration corresponds to the accumulation of $k$ topological charges at the origin $r = 0$. For the full, non-symmetric problem, we expect such configurations to split under a generic perturbation into a collection of $k = 1$ harmonic maps and lead to a different dynamics driven by the evolution of each of the $k = 1$ ground states and their interaction.

From this point the stability of the least energy $k = 1$ configuration under generic non-symmetric perturbations is an important remaining problem.

## 3. **Strategy of the proof**

Let us sketch the main ingredients of the proof of Theorem 2.1.

**Step 1** The family of approximate self similar profiles.

We start with the construction of suitable approximate self-similar solutions in the fashion related to the approach developed in [24], [26]. Following the scaling invariance of (1.3), we pass to the self-similar variables and look for a one parameter family of self similar solutions dependent on a small -scaling invariant- parameter $b > 0$:

$$u(t, r) = Q_b(y), \quad y = \frac{r}{\lambda(t)}, \quad \lambda(t) = b(T - t).$$

This transformation maps (1.3) into the self-similar equation:

$$-\Delta v + b^2 D\Lambda Q_b + k^2 \frac{f(v)}{y^2} = 0 \tag{3.1}$$

where the differential operators $\Lambda, D$ are given by

$$\Lambda f = y \cdot \nabla f, \quad Df = f + \Lambda f.$$

A well known class of exact solutions are given by the explicit profiles:

$$Q_b(r) = Q\left(\frac{r}{1 + \sqrt{1 - b^2 r^2}}\right), \quad r \leq \frac{1}{b}.$$

These solutions were used by Côte to prove that $Q$ is unstable for both (WM) and (YM), [9]. A direct inspection however reveals that these have infinite energy due to a logarithmic divergence on the backward light cone

$$r = (T - t) \text{ equivalently } y = \frac{1}{b}.$$

This situation is exactly the same for the $L^2$ critical (NLS), [24], and reveals the critical nature of the problem. Note that in higher dimensions finite energy self-similar solutions can be shown to exist thus providing explicit blow up solutions to the Wave Map and Yang-Mills equations, [32], [7].

In order to find *finite energy* suitable approximate solutions to (3.1) in the vicinity of the ground state $Q$ we introduce a formal expansion

$$Q_b = Q + \Sigma_{i=1}^p b^{2i} T_i.$$

Substituting the ansatz into the self-similar equation (3.1), we get at the order $b^{2i}$ an equation of the form:

$$HT_i = F_i \tag{3.2}$$

where

$$H = -\Delta + k^2 \frac{f'(Q)}{y^2} \tag{3.3}$$

is obtained by linearizing (3.1) on $Q$ (setting $b=0$) and $F_i$ is a nonlinear expression. The solvability of (3.2) requires that $F_i$ is orthogonal to the kernel of $H$, which is explicit by the variational characterization of $Q$:

$$Ker(H) = \text{span}(\Lambda Q) \tag{3.4}$$

and hence the orthogonality condition:

$$(F_i, \Lambda Q) = 0. \tag{3.5}$$

While the condition (3.5) seems at first hand to be a very nonlinear condition, it can be proved to hold due to the specific algebra of the $H^1$ critical problem and its connection to the Pohozaev identity. In fact, if $Q_b^{(p)} = Q + \Sigma_{i=1}^p b^{2i} T_i$ is the expansion of the profile to the order $p$, then (3.5) holds as long as the Pohozaev computation is valid:

$$\left(-\Delta Q_b^{(p)} + b^2 D\Lambda Q_b^{(p)} + k^2 \frac{f(Q_b^{(p)})}{y^2}, D\Lambda Q_b^{(p)}\right) \tag{3.6}$$

$$= \lim_{R\to+\infty}\left[\frac{b^2}{2}|r\Lambda Q_b^{(p)}(R)|^2 + \frac{k^2}{2}|g(Q_b^{(p)}(R)|^2\right] = 0. \tag{3.7}$$

By a direct computation, $F_1 \sim D\Lambda Q \sim \frac{1}{y^k}$ as $y \to +\infty$ and at each step, the inversion of (3.2) dampens the decay of $T_{i+1}$ at infinity by an extra $y^2$ factor, and hence the validity of (3.7) comes under question after $p$ steps, for as $y \to \infty$:

$$T_p(y) \sim \frac{c_k}{y} \quad \text{for} \quad p = \frac{k-1}{2}, \quad k \quad \text{odd}, \tag{3.8}$$

$$T_p(y) \sim c_k \quad \text{for} \quad p = \frac{k}{2}, \quad k \quad \text{even}. \tag{3.9}$$

In fact (3.8), (3.9) will result in a *universal nontrivial* flux type contribution to (3.6). Moreover, $T_p$ is the first term which gives an infinite contribution to the energy of the approximate self-similar profile $Q_b^{(p)}(\frac{r}{\lambda(t)})$. $T_p$ is the *radiation* term which becomes dominant in the region $y \geq \frac{1}{b}$ – exterior to the backward light cone from a singularity at the point $(T,0)$. We therefore stop the asymptotic expansion at $p$[1] and localize constructed profiles by connecting $Q_b$ to the constant $a = Q(+\infty)$, which is also an exact self-similar solution:

$$P_{B_1} = \chi_{B_1} Q_b + (1-\chi_{B_1})a, \quad B_1 = \frac{|\text{log}b|}{b} >> \frac{1}{b} \tag{3.10}$$

where $\chi_{B_1} = 1$ for $y \leq B_1$, $\chi_{B_1} = 0$ for $y \geq 2B_1$. $P_{B_1}$ satisfies an approximate self-similar equation of the form:

$$-\Delta P_{B_1} + b^2 D\Lambda P_{B_1} + k^2 \frac{f(P_{B_1})}{y^2} = \Psi_{B_1} \tag{3.11}$$

where $\Psi_{B_1}$ is very small inside the light cone $y \leq \frac{1}{b}$ but encodes a slow decay near $B_1$ induced by the cut off function and the radiative behavior of $T_p$ at infinity.

**Step 2** The $H^2$ type bound.

[1] We will in fact also need the next term $T_{p+1}$ in the expansion, what will be made possible thanks to a subtle cancellation

Let now $u(t,r) \in \mathcal{H}^2_a$ be the solution to (1.3) for a suitably chosen initial data close enough to $Q$. Given the profile $P_{B_1}$, we introduce, with the help of the standard modulation theory, a decomposition of the wave:

$$u(t,r) = P_{B_1(t)}(\frac{r}{\lambda(t)}) + w(t,r)$$

or alternatively

$$u(t,r) = (P_{B_1(t)} + \varepsilon)(s,y), \quad y = \frac{r}{\lambda(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda}$$

with $B_1$ given by (3.10) and where we have **set** the relation

$$b(s) = -\frac{\lambda_s}{\lambda} = -\lambda_t. \tag{3.12}$$

The decomposition is complemented by the orthogonality condition[2]

$$\forall s > 0, \quad (\varepsilon(s), \Lambda Q) = 0$$

as is natural from (3.4). Our first main claim is the derivation of a *pointwise in time bound* on $\varepsilon$

$$\|\varepsilon\|_{\tilde{H}} \lesssim b^{k+1} \tag{3.13}$$

in a certain *weighted* Sobolev space $\tilde{\mathcal{H}}$. Bounds related to (3.13) but for a weaker norm than $\tilde{\mathcal{H}}$ and with $b^{k+1}$ replaced by $b^4$ were derived in [31] for higher homotopy classes $k \geq 4$. They were a consequence of the proof of energy and Morawetz type estimates for the corresponding nonlinear problem satisfied by $w$. The linear part of the equation for $w$ is given by the expression

$$\partial_t^2 w + H_\lambda w$$

with the Hamiltonian

$$H_\lambda = -\Delta + \frac{f'(Q_\lambda)}{r^2} \tag{3.14}$$

Special variational nature of $Q$, discovered in [2], provides an important factorization property for $H_\lambda$:

$$H_\lambda = A^*_\lambda A_\lambda, \qquad A_\lambda = -\partial_r + k\frac{g'(Q_\lambda)}{r}. \tag{3.15}$$

It arises as a consequence of the fact that[3] $Q$ represents the co-rotational global minimum of energy $V[\Phi]$ in a given topological class of maps $\Phi : \mathbb{R}^2 \to \mathbb{S}^2$ of degree $k$:

$$V[\Phi] \;=\; \frac{1}{2}\int_{\mathbb{R}^2} (\nabla_x\Phi \cdot \nabla_x\Phi)\; dx,$$

which can be factorized using the notation $\epsilon_{ij}$ for the antisymmetric tensor on two indices, as follows:

$$\begin{aligned} V[\Phi] \;&=\; \frac{1}{4}\int_{\mathbb{R}^2} \Big[(\partial_i\Phi \pm \epsilon_i{}^j\Phi \times \partial_j\Phi)\cdot(\partial^i\Phi \pm \epsilon^{ij}\Phi \times \partial_j\Phi)\Big]\; dx \\ &\qquad \pm\; \frac{1}{2}\int_{\mathbb{R}^2} \epsilon^{ij}\Phi\cdot(\partial_i\Phi \times \partial_j\Phi)\; dx\;, \\ &=\; \frac{1}{4}\int_{\mathbb{R}^2} \Big[(\partial_i\Phi \pm \epsilon_i{}^j\Phi \times \partial_j\Phi)\cdot(\partial^i\Phi \pm \epsilon^{ij}\Phi \times \partial_j\Phi)\Big]\; dx \;\pm\; 4\pi k \end{aligned} \tag{3.16}$$

[2]The actual orthogonality condition is defined with respect to a cut-off version of $\Lambda Q$.

[3]We restrict this discussion to the (WM) case. Similar considerations also apply to the (YM) problem, [6]

from which it is immediate that an absolute minimum of the energy functional $V[\Phi]$ in a given topological sector $k$ must be a solution of the equation:

$$\partial_i\Phi \pm {\epsilon_i}^j \Phi \times \partial_j \Phi \;=\; 0\ . \tag{3.17}$$

The ground state $Q$ is precisely the representation of the unique co-rotational solution of (3.17).

In [31] factorization (3.15) gave the basis for the $H^2$ and Morawetz type bounds for $w$, obtained by conjugating the problem for $w$ with the help of the operator $A_\lambda$, so that

$$A_\lambda H_\lambda w = \tilde H_\lambda (A_\lambda w)$$

with $\tilde H_\lambda = A_\lambda A_\lambda^*$, and exploiting the space-time repulsive properties of $\tilde H_\lambda$ to derive the energy and Morawetz estimates for $A_\lambda w$. Simultaneous use of pointwise in time energy bounds and space-time Morawetz estimates however runs into difficulties in the cases $k = 1, 2$, which become seemingly insurmountable for $k = 1$.

A new approach is introduced in [30], still based on the factorization of $H_\lambda$, yet relying **only** on the appropriate *energy estimates* for the associated Hamiltonian $\tilde H_\lambda$, which retains its repulsive properties even in the most difficult cases of $k = 1, 2$. We note that $\|\varepsilon\|_{\tilde H}$ norm introduced above can be conveniently written in the form

$$\|\varepsilon\|^2_{\tilde H} = \lambda^2 (\tilde H_\lambda A_\lambda w, A_\lambda w) + \lambda^2 \|(\partial_t w, 0\|^2_H.$$

One difficulty will be that the bound (3.13) *is not sufficient* to derive the sharp blow up speed. The size $b^{k+1}$ in the RHS of (3.13) is sharp and is induced by a very slowly decaying term in $\Psi_{B_1}$ in (3.11), which arises from the localization of the profile $Q_b$. Such terms however are *localized* on $y \sim B_1 >> \frac{1}{b}$ far away from the backward light cone with the vertex at the singularity. Another crucial new feature of our analysis here is a use of *localized* energy identities. It is based on the idea of writing the energy identity in the region bounded by the initial hypersurface $t = 0$ and the hypersurface

$$r = 2\frac{\lambda(t)}{b(t)}, \quad \text{equivalently} \quad y = \frac{2}{b(t)}$$

which, under the bootstrap blow up assumptions, is complete (the point $r = 0$ is reached at the blow up time) and space-like. Such an energy identity effectively restricts the error term $\Psi_{B_1}$ to the region $y \le 2/b$, where it is better behaved, and leads to an improved bound:

$$\|\varepsilon\|_{\tilde H(y\le \frac{2}{b})} \lesssim \frac{b^{k+1}}{|\log b|}. \tag{3.18}$$

Note that the logarithmic gain from (3.13) to (3.18) is typical of the $k = 1$ case and can be turned to a polynomial gain for $k \ge 2$.

**Step 3** The flux computation and the derivation of the sharp law.

The pointwise bounds (3.13), (3.18) are specific to the *almost self-similar regime* we are describing. They are derived by a bootstrap argument, which incidentally requires *only* an upper bound[4] on $|b_s|$. To derive the precise law for $b$ we examine

[4] Such an upper bound is already sufficient to conclude the finite time blow up and establish a lower bound on the concentration scale $\lambda(t)$, see [23], [22] for related considerations.

the equation for $\varepsilon$, which has the following approximate form:

$$\partial_s^2\varepsilon + H_{B_1}\varepsilon = -b_s\Lambda P_{B_1} + \Psi_{B_1} + \text{L.O.T.} \tag{3.19}$$

where $H_{B_1} = -\Delta + k^2\frac{f'(P_{B_1})}{y^2}$. We consider an almost self-similar solution $P_{B_0}$ localized on the scale $B_0 = \frac{c}{b}$ with a well chosen constant $0 < c < 1$ and project this equation onto $\Lambda P_{B_0}$, which is almost in the null space of $H_{B_1}$. The result is an identity of the form:

$$b_s|\Lambda P_{B_0}|^2_{L^2} = (\Psi_{B_1}, \Lambda P_{B_0}) + O(b^{k-1}\|\varepsilon\|_{\tilde{H}(y\leq\frac{2}{b})}). \tag{3.20}$$

The first term in the above RHS yields the leading order flux and tracks the nontrivial contribution of $T_p$ to the Pohozaev integration (3.6):

$$(\Psi_{B_1}, \Lambda P_{B_0}) = -c_k b^{2k}(1 + o(1))$$

for some universal constant $c_k$. This computation can be thought of as related to the derivation of the log-log law in [26]. The $\varepsilon$-term in (3.20) is treated with the help of (3.18), observe that (3.13) alone would not have been enough:

$$O(b^{k-1}\|\varepsilon\|_{\tilde{H}(y\leq\frac{2}{b})}) = o(b^{2k}).$$

Finally, from the behavior

$$\Lambda Q \sim \frac{1}{y^k} \quad \text{as} \quad y \to +\infty$$

and $P_{B_0} \sim Q$ for $b$ small, there holds:

$$|\Lambda P_{B_0}|^2_{L^2} \sim \begin{cases} c_k & \text{for} \quad k \geq 2 \\ c_1|\text{log}b| & \text{for} \quad k = 1 \end{cases}$$

for some universal constant $c_k > 0$. We hence get the following system of ODE's for the scaling law:

$$\frac{ds}{dt} = \frac{1}{\lambda}, \quad b = -\frac{\lambda_s}{\lambda}, \quad b_s = -\begin{cases} c_k(1 + o(1))b^{2k} & \text{for} \quad k \geq 2 \\ (1 + o(1))\frac{b^2}{2|\text{log}b|} & \text{for} \quad k = 1 \end{cases}$$

Its integration yields – for the class of initial data under consideration – the existence of $T < +\infty$ such that $\lambda(T) = 0$ with the laws (2.2), (2.3) near $T$, thus concluding the proof of the sharp asymptotics (2.2), (2.3). The non-concentration of the excess of energy (2.4), (2.5) now follows from the dispersive bounds obtained on the solution, hence concluding the proof of Theorem 2.1.

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