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A Two-Particle Quantum System with Zero-Range Interaction

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Abstract

We study a two-particle quantum system given by a test particle interacting in three dimensions with a harmonic oscillator through a zero-range potential. We give a rigorous meaning to the Schrödinger operator associated with the system by applying the theory of quadratic forms and defining suitable families of self-adjoint operators. Finally we fully characterize the spectral properties of such operators.

1 Introduction

Zero-range (or point) interactions were introduced in the '30s in [F] and [KP] as simplified models to describe the interactions between low-energy particles and target nuclei in scattering processes: The range of the interaction is assumed to be so short that one can approximate the potential, at least formally¹, by a Dirac delta function $\sum_i \delta(\vec{x} - \vec{x}_i)$ where \vec{x} is the position of the test particle and \vec{x}_i , $i = 1, \dots, N$, the interaction centers.

The main motivation behind the computational use of such interacting models is that one can obtain nontrivial interesting results depending on a minimal set of parameters, as, e.g., the interaction strength and the positions of the nuclei. Indeed the solutions to the Schrödinger equation are almost explicit and the relevant quantities can be at least approximated with good precision. It should be stressed however that in the physics literature all the computations are done by applying the perturbation theory and considering $\delta(\vec{x} - \vec{x}_i)$ as a small perturbation of the free Hamiltonian, which of course is not the case and the perturbation series diverges.

From the mathematical point of view, point interactions attracted some attention already in the '60s, when Berezin and Faddeev first gave in [BF] a rigorous definition of a formal one-particle operator with a zero-range potential and in the last years many interesting results on this topic have been proven. We refer to the monograph [AGH-KH] and references therein for further details.

In this article we study a two-particle Schrödinger operator which is formally given by the expression

$$H_\alpha = H_0 + \text{“}\alpha\delta(\vec{x} - \vec{y})\text{”}, \quad (1.1)$$

$$H_0 = -\frac{1}{2}\Delta_x + H_{\text{osc}}(y), \quad H_{\text{osc}}(y) = -\frac{1}{2}\Delta_y + \frac{1}{2}y^2 - \frac{3}{2}, \quad (1.2)$$

where $\vec{x}, \vec{y} \in \mathbb{R}^3$ and we have set the harmonic frequency of the oscillator equal to 1 and subtracted the ground state energy of the harmonic oscillator for convenience.

Such a model is major simplification of the formal Hamiltonian used to describe the interaction between a test particle and several target nuclei, where only one target is kept: The problem of the

¹In physics literature such a formal potential is often called pseudo-potential or Fermi pseudo-potential.

mathematical definition of the Schrödinger operator describing a test particle interacting through a zero-range potential with an array of N harmonic oscillators (the so called Rayleigh gas) in three dimensions was indeed addressed in [DFinT] but it remains unsolved except for the simple case $N = 1$. The two-particle system above was studied again in [CDF], where the spectral properties of the rigorous counterpart of (1.1) are investigated. The reason to consider a so simple model is that, despite its simplicity, most of the technical issues associated with the N particle model already occur in the analysis of (1.1), so that there is some hope to find out useful results to apply to the original problem. In this note we give a rigorous meaning to (1.1) and study its spectral properties. Most of the results are already stated in [DFinT] and [CDF] but the presentation here is self-contained and cleaner.

In order to give a rigorous meaning to the above expression, one can follow several alternative strategies: The usual one (see [AGH-KH]) is to consider the operator $\tilde{H}_0 = H_0$ on the domain

$$\mathcal{D}(\tilde{H}_0) = \{\Psi \in \mathcal{D}(H_0) \mid \mathcal{P}\Psi = 0\}, \quad (1.3)$$

where we denote by $\mathcal{D}(H_0)$ the self-adjointness domain of H_0 , i.e.²,

$$\mathcal{D}(H_0) = \{\Psi \in \mathcal{H}^2(\mathbb{R}^6) \mid y^2\Psi \in L^2(\mathbb{R}^6)\}, \quad (1.4)$$

and by \mathcal{P} the projection onto the plane $\Pi = \{(\vec{x}, \vec{y}) \in \mathbb{R}^6 \mid \vec{x} = \vec{y}\}$, i.e., $\mathcal{P} : L^2(\mathbb{R}^6) \rightarrow L^2(\mathbb{R}^3)$ and³

$$(\mathcal{P}\Psi)(\vec{x}) = \Psi(\vec{x}, \vec{x}). \quad (1.5)$$

It is not difficult to realize that \mathcal{D} is not a domain of self-adjointness for the operator H_0 . More precisely H_0 is symmetric on \mathcal{D}_0 and it admits self-adjoint extensions, since the deficiency indexes equal: Indeed the equation $H_0^*\Psi = \mp i\Psi$ is solved by any function of form

$$\Psi(\vec{x}, \vec{y}) = (\mathcal{G}_{\sqrt{\mp i}} q)(\vec{x}, \vec{y}),$$

where the operator \mathcal{G}_λ is given by

$$(\mathcal{G}_\lambda q)(\vec{x}, \vec{y}) \equiv \int_{\Pi} d\vec{x}' q(\vec{x}') G_\lambda(\vec{x}, \vec{y}; \vec{x}', \vec{x}'), \quad (1.6)$$

$q \in L^2(\mathbb{R}^3)$ and (see⁴ the Appendix in [DFinT]),

$$G_\lambda(\vec{x}, \vec{y}; \vec{x}', \vec{y}') \equiv (H_0 + \lambda)^{-1}(\vec{x}, \vec{y}; \vec{x}', \vec{y}') = \frac{1}{2^{\frac{3}{2}}\pi^3} \int_0^1 d\nu \frac{\nu^{\lambda-1}}{(1-\nu^2)^{\frac{3}{2}} |\ln \nu|^{\frac{3}{2}}} \exp \left\{ -\frac{1-\nu}{2(1+\nu)} (y^2 + y'^2) - \frac{(\vec{x} - \vec{x}')^2}{2 \ln \frac{1}{\nu}} - \frac{\nu(\vec{y} - \vec{y}')^2}{1-\nu^2} \right\}. \quad (1.7)$$

Note that the deficiency indexes are equal to ∞ , since there is a solution to the deficiency equation for any function q such that $\mathcal{G}_{\sqrt{\mp i}} q \in L^2(\mathbb{R}^6)$. Hence the family of self-adjoint extensions of H_0 contains an infinite number of self-adjoint operators which are actually labeled by (self-adjoint) operators acting on $L^2(\Pi)$. We refer to [AGH-KH] and [P] for additional details about this approach.

An alternative approach (see, e.g., [DFigT]) consists in applying a renormalization procedure to the singular quadratic form given by $\langle \Psi | H_\alpha | \Psi \rangle$, i.e.,

$$\tilde{F}_\gamma[\Psi] = F_0[\Psi] + \gamma \int_{\mathbb{R}^6} d\vec{x} d\vec{y} \delta(\vec{x} - \vec{y}) |\Psi(\vec{x}, \vec{y})|^2, \quad F_0[\Psi] = \langle \Psi | H_0 | \Psi \rangle, \quad (1.8)$$

²We denote by \mathcal{H}^m the Sobolev space of order m . A point in \mathbb{R}^3 is always denoted by a vector as, e.g., \vec{x} , whereas $x \in \mathbb{R}^+$ stands for the norm of the vector.

³Actually the operator \mathcal{P} is well defined only on the subspace of functions belonging to $\mathcal{H}^m(\mathbb{R}^6)$ for $m > 3/2$ by Sobolev trace theorem.

⁴The different power of ν in the integral with respect to the formula in [DFinT] is due to the subtraction of $\frac{3}{2}$ in (1.2).

and then defining the rigorous counterpart of (1.1) as the unique self-adjoint operator associated with so obtained closed quadratic form. It is indeed clear that the above expression is not well defined for any Ψ in the domain of the unperturbed form F_0 because the restriction of Ψ to the plane Π has not to belong to $L^2(\Pi)$. However by formally renormalizing the coupling constant γ , one can obtain a well defined quadratic form which is closed and bounded from below. We stress that in general this approach does not allow to recover all the self-adjoint extensions above but only some minimal subfamily of them.

In Section 2 we describe in details the renormalization of the quadratic form \tilde{F}_γ as well as some fundamental properties of the renormalized form. Section 3 is devoted to the analysis of the domain and spectrum of the associated family of self-adjoint operators.

2 The Quadratic Form

In this Section we shall investigate the properties of the quadratic form

$$F_\alpha[\Psi] = \mathcal{F}^\lambda[\phi_\lambda] + \Phi_\alpha^\lambda[q] - \lambda\|\Psi\|_2^2, \quad \mathcal{F}^\lambda[\phi] = F_0[\phi] + \lambda\|\phi\|_2^2, \quad (2.1)$$

where

$$\Phi_\alpha^\lambda[q] = \int_{\mathbb{R}^3} d\vec{x} (\alpha + a^\lambda(x)) |q(\vec{x})|^2 + \frac{1}{2} \int_{\mathbb{R}^6} d\vec{x} d\vec{x}' G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') |q(\vec{x}) - q(\vec{x}')|^2, \quad (2.2)$$

$$a^\lambda(x) = \frac{1}{(4\pi)^{\frac{3}{2}}} \left\{ \frac{1}{2} + \int_0^1 d\nu \frac{1}{(1-\nu)^{\frac{3}{2}}} \left[1 - \frac{8\nu^{\lambda-1}(1-\nu)^{\frac{3}{2}}}{[(1+\nu^2)|\ln \nu| + 1 - \nu^2]^{\frac{3}{2}}} \exp \left\{ -\frac{(1-\nu^2)|\ln \nu| + 2(1-\nu)^2}{2[(1+\nu^2)|\ln \nu| + 1 - \nu^2]} x^2 \right\} \right] \right\}. \quad (2.3)$$

The form F_α is defined on the domain

$$\mathcal{D}(F_\alpha) = \{ \Psi \in L^2(\mathbb{R}^6) \mid \exists q \in \mathcal{D}(\Phi_\alpha^\lambda) \text{ s.t. } \Psi = \phi_\lambda + \mathcal{G}_\lambda q, \phi_\lambda \in \mathcal{D}(F_0) \}, \quad (2.4)$$

where $\lambda > 0$ is a positive parameter and⁵

$$\mathcal{D}(\Phi_\alpha^\lambda) = \{ q \in L^2(\mathbb{R}^3) \mid \Phi_\alpha^\lambda[q] < \infty \}. \quad (2.5)$$

2.1 Heuristic Derivation of F_α

We start the analysis of the quadratic form (2.1) by showing an heuristic derivation of it via a renormalization of the naive singular form (1.8), which can be rewritten as

$$\begin{aligned} \tilde{F}_\gamma[\Psi] = F_0[\Psi - \mathcal{G}_\lambda q] + \lambda\|\Psi - \mathcal{G}_\lambda q\|_2^2 - \lambda\|\Psi\|_2^2 + \langle (H_0 + \lambda)\mathcal{G}_\lambda q | \Psi \rangle + \langle \Psi | (H_0 + \lambda)\mathcal{G}_\lambda q \rangle \\ - \langle (H_0 + \lambda)\mathcal{G}_\lambda q | \mathcal{G}_\lambda q \rangle + \gamma \int_{\Pi} d\vec{x} |\Psi(\vec{x}, \vec{x})|^2. \end{aligned}$$

By setting $\phi_\lambda(\vec{x}, \vec{y}) \equiv \Psi(\vec{x}, \vec{y}) - (\mathcal{G}_\lambda q)(\vec{x}, \vec{y})$ according to (2.4) and

$$q(\vec{x}) = -\gamma\Psi(\vec{x}, \vec{x}), \quad (2.6)$$

the expression above becomes

$$\tilde{F}_\gamma[\Psi] = F_0[\phi_\lambda] + \lambda\|\phi_\lambda\|_2^2 - \lambda\|\Psi\|_2^2 + \tilde{\Phi}_\gamma[q], \quad \tilde{\Phi}_\gamma[q] = \int_{\Pi} d\vec{x} q^*(\vec{x}) \left[-\frac{q(\vec{x})}{\gamma} - (\mathcal{G}_\lambda q)(\vec{x}, \vec{x}) \right]. \quad (2.7)$$

⁵The condition $q \in L^2(\mathbb{R}^3)$ is actually not a restriction as it will be clear by Proposition 2.1.

Note that the above expression is only formal since $\mathcal{G}_\lambda q$ diverges on the plane Π because of the singularity of the free Green function. However given some small neighborhood U_ε of Π , one can decompose $\mathcal{G}_\lambda q$ as

$$(\mathcal{G}_\lambda q)(\vec{x}, \vec{y}) = q(\vec{x}) \zeta_\varepsilon(\vec{x}, \vec{y}) + \tau_q(\vec{x}, \vec{y}), \quad (2.8)$$

where ζ_ε is the singular part of $\mathcal{G}_\lambda \chi_\varepsilon$, χ_ε denoting the characteristic function of the set U_ε , and τ_q remains bounded as $\vec{x} \rightarrow \vec{y}$. The goal is then a renormalization of the coupling constant to cancel the singular part ζ_ε and, in view of this, we set

$$(\mathcal{G}_\lambda^{\text{ren}} q)(\vec{x}) \equiv \lim_{\vec{y} \rightarrow \vec{x}} \tau_q(\vec{x}, \vec{y}). \quad (2.9)$$

Then we consider a tubular neighborhood of Π given for instance by $\Pi_\delta = \{(\vec{x}, \vec{y}) \in \mathbb{R}^6 \mid |\vec{x} - \vec{y}| \leq \sqrt{2} \delta\}$ and take some suitable approximation $q_\delta(\vec{x}, \vec{y})$ of the charge such that $q_\delta \rightarrow q$ as $\delta \rightarrow 0$. The singular part of (2.7) is

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_{\Pi_\delta} d\vec{x} d\vec{y} q_\delta^*(\vec{x}, \vec{y}) \left[-\frac{q_\delta(\vec{x}, \vec{y})}{\gamma_\delta(\vec{x}, \vec{y})} - (\mathcal{G}_\lambda q_\delta)(\vec{x}, \vec{y}) \right] = \\ \lim_{\delta \rightarrow 0} \int_{\Pi_\delta} d\vec{x} d\vec{y} q_\delta^*(\vec{x}, \vec{y}) \left[-\frac{q_\delta(\vec{x}, \vec{y})}{\gamma_\delta(\vec{x}, \vec{y})} - q_\delta(\vec{x}, \vec{y}) \zeta_\delta(\vec{x}, \vec{y}) \right] - \int_{\Pi_\delta} d\vec{x} d\vec{y} q_\delta^* \tau_\delta(\vec{x}, \vec{y}), \end{aligned}$$

where we have used the decomposition (2.8) associated with the neighborhood Π_δ and replaced the coupling parameter γ with some function $\gamma_\delta(\vec{x}, \vec{y})$. The key point in the renormalization is then the choice

$$-\frac{1}{\gamma_\delta(\vec{x}, \vec{y})} = \zeta_\delta(\vec{x}, \vec{y}) + \alpha, \quad (2.10)$$

for some $\alpha \in \mathbb{R}$. Note that because of the divergence of ζ , the original coupling parameter γ has to go to zero as $\delta \rightarrow 0$. With this choice the form $\tilde{\Phi}_\gamma$ can be simply written as

$$\alpha \int_{\Pi} d\vec{x} |q(\vec{x})|^2 - \int_{\Pi} d\vec{x} q^*(\vec{x}) (\mathcal{G}_\lambda^{\text{ren}} q)(\vec{x}), \quad (2.11)$$

and the second term can be expressed in a more convenient form: Since the renormalized quantity can be written as

$$\begin{aligned} \int_{\Pi} d\vec{x} q^*(\vec{x}) (\mathcal{G}_\lambda^{\text{ren}} q)(\vec{x}) = \lim_{\vec{y} \rightarrow \vec{x}} \left[\int_{|\vec{x}' - \vec{x}| > \varepsilon} d\vec{x}' q^*(\vec{x}) q(\vec{x}') G_\lambda(\vec{x}, \vec{y}; \vec{x}', \vec{x}') + \right. \\ \left. \int_{|\vec{x}' - \vec{x}| < \varepsilon} d\vec{x}' q^*(\vec{x}) (q(\vec{x}') - q(\vec{x})) G_\lambda(\vec{x}, \vec{y}; \vec{x}', \vec{x}') + \int_{\Pi} d\vec{x} |q(\vec{x})|^2 \tau_{\chi_\varepsilon}(\vec{x}, \vec{y}) \right] = \\ \int_{\mathbb{R}^6} d\vec{x} d\vec{x}' q^*(\vec{x}) (q(\vec{x}') - q(\vec{x})) G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') + \int_{\Pi} d\vec{x} |q(\vec{x})|^2 \left[\tau_{\chi_\varepsilon}(\vec{x}, \vec{x}) + \int_{|\vec{x}' - \vec{x}| > \varepsilon} d\vec{x}' G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') \right] = \\ -\frac{1}{2} \int_{\mathbb{R}^6} d\vec{x} d\vec{x}' |q(\vec{x}) - q(\vec{x}')|^2 G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') - \int_{\mathbb{R}^3} d\vec{x} a_\lambda(\vec{x}) |q(\vec{x})|^2, \quad (2.12) \end{aligned}$$

where τ_{χ_ε} is the regular part of $\mathcal{G}_\lambda \chi_\varepsilon$ and

$$a_\lambda(\vec{x}) = -\lim_{\vec{y} \rightarrow \vec{x}} \left[\tau_{\chi_\varepsilon}(\vec{x}, \vec{y}) + \int_{|\vec{x}' - \vec{x}| > \varepsilon} d\vec{x}' G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') \right]. \quad (2.13)$$

In the final expression in (2.12) the singularity of the Green function is manifestly treated by multiplying $G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}')$ by $|q(\vec{x}) - q(\vec{x}')|^2$, which vanishes on the plane $\vec{x} = \vec{x}'$. It remains thus to analyze a_λ :

After the change of coordinates $\vec{\xi} = \frac{1}{\sqrt{2}}(\vec{x} - \vec{y})$ and $\vec{\eta} = \frac{1}{\sqrt{2}}(\vec{x} + \vec{y})$, we have to study the asymptotic behavior for small ξ of the expression $\mathcal{G}_{\lambda\chi_\varepsilon}$, which can be written as

$$(\mathcal{G}_{\lambda\chi_\varepsilon})(\vec{\xi}, \vec{\eta}) = \int_{U_\varepsilon} d\vec{\eta}' G_\lambda(\vec{\xi}, \vec{\eta}; 0, \vec{\eta}'),$$

with $\vec{\eta}' = \sqrt{2} \vec{x}'$. Since the right hand side is independent of the shape of U_ε , we can choose $U_\varepsilon = \{|\vec{\eta}' - \vec{\eta}| < \varepsilon\}$, so that the above expression becomes after a change of coordinates

$$\begin{aligned} (\mathcal{G}_{\lambda\chi_\varepsilon})(\vec{\xi}, \vec{\eta}) &= \frac{1}{\pi^3} \int_0^1 d\nu \frac{\nu^{\lambda-1}}{(1-\nu^2)^{\frac{3}{2}} |\ln \nu|^{\frac{3}{2}} m(\nu)^{\frac{3}{2}}} \cdot \\ &\quad \exp \left\{ -\frac{1-\nu}{4(1+\nu)} \left[\eta^2 + (\vec{\eta} - \vec{\xi})^2 \right] - \frac{1}{4} \left[\frac{1}{|\ln \nu|} + \frac{2\nu}{1-\nu^2} \right] \xi^2 \right\} \cdot \\ &\quad \int_{|\vec{\eta}'| < \varepsilon \sqrt{m(\nu)}} d\vec{\eta}' e^{-\eta'^2} \exp \left\{ -\frac{2\vec{\eta}'}{\sqrt{m(\nu)}} \cdot \left[\frac{(1-\nu)\vec{\eta}}{2(1+\nu)} + \frac{\vec{\xi}}{2|\ln \nu|} - \frac{\nu\vec{\xi}}{2(1-\nu^2)} \right] \right\}, \end{aligned}$$

where $m(\nu)$ stands for the quantity

$$m(\nu) = \frac{1+\nu^2}{1-\nu^2} + \frac{1}{|\ln \nu|}. \quad (2.14)$$

The above expression allows to investigate the singularity of the Green function: By setting $\nu = 1 - \nu'$ and $\nu' = \xi^2 \mu$, one gets

$$\begin{aligned} (\mathcal{G}_{\lambda\chi_\varepsilon})(\vec{\xi}, \vec{\eta}) &= \frac{1}{\pi^3 \xi} \int_0^{\frac{1}{\xi^2}} d\mu \frac{f_1(1-\xi^2\mu)}{\mu^{\frac{3}{2}}} \exp \left\{ -\frac{\xi^2\mu}{4(2-\xi^2\mu)} \left[\eta^2 + (\vec{\eta} - \vec{\xi})^2 \right] \right\} \exp \left\{ -\frac{f_2(1-\xi^2\mu)}{4\mu} \right\} \cdot \\ &\quad \int_{|\vec{\eta}'| < \varepsilon \sqrt{m(1-\xi^2\mu)}} d\vec{\eta}' \exp \left\{ -\eta'^2 - 2\vec{\eta}' \cdot \left[\frac{\xi^2\mu\vec{\eta}}{2\sqrt{m(1-\xi^2\mu)}(2-\xi^2\mu)} + \frac{f_3(1-\xi^2\mu)\vec{\xi}}{2\xi^2\mu\sqrt{m(1-\xi^2\mu)}} \right] \right\}, \end{aligned} \quad (2.15)$$

with

$$f_1(\nu) \equiv \frac{\nu^{\lambda-1}}{(1+\nu)^{\frac{3}{2}} (|\ln \nu|)^{\frac{3}{2}} m(\nu)^{\frac{3}{2}}} \xrightarrow{\nu \rightarrow 1} \frac{1}{8}, \quad (2.16)$$

$$f_2(\nu) \equiv \frac{1-\nu}{|\ln \nu|} + \frac{2\nu}{1+\nu} \xrightarrow{\nu \rightarrow 1} 2, \quad f_3(\nu) \equiv \frac{1-\nu}{|\ln \nu|} - \frac{2\nu}{1+\nu} \xrightarrow{\nu \rightarrow 1} 0, \quad (2.17)$$

where we have used $\lim_{\nu \rightarrow 1} (1-\nu)m(\nu) = 2$. Therefore as $\xi \rightarrow 0$

$$(\mathcal{G}_{\lambda\chi_\varepsilon})(\vec{\xi}, \vec{\eta}) \simeq \frac{1}{8\pi^3 \xi} \int_0^\infty d\mu \frac{1}{\mu^{\frac{3}{2}}} \exp \left\{ -\frac{1}{2\mu} \right\} \int_{\mathbb{R}^3} d\vec{\eta}' e^{-\eta'^2} = \frac{c}{\xi},$$

i.e., one recovers the usual coulombian singularity of the Green function of the Laplacian in three dimensions. For convenience we can write the expression above also as

$$(\mathcal{G}_{\lambda\chi_\varepsilon})(\vec{\xi}, \vec{\eta}) \simeq \frac{1}{8\pi^{\frac{3}{2}}} \int_0^\infty d\mu \frac{1}{\mu^{\frac{3}{2}}} \exp \left\{ -\frac{\xi^2}{2\mu} \right\} = \frac{1}{8\pi^{\frac{3}{2}}} \left[\int_0^1 d\mu \frac{1}{\mu^{\frac{3}{2}}} \exp \left\{ -\frac{\xi^2}{2\mu} \right\} + \frac{1}{2} \right], \quad (2.18)$$

where we have separated the singular contribution (the first term in the expression above) from the regular one, which converges as $\xi \rightarrow 0$ to $1/2$. Therefore from (2.13) and (2.15), one obtains

$$\begin{aligned} a_\lambda(\vec{x}) &= \frac{1}{(4\pi)^{\frac{3}{2}}} \left\{ \frac{1}{2} + \lim_{\xi \rightarrow 0} \left[\int_0^1 d\mu \frac{1}{\mu^{\frac{3}{2}}} \exp \left\{ -\frac{\xi^2}{2\mu} \right\} - \int_{\mathbb{R}^3} d\vec{x}' G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') \right] \right\} = \\ &\quad \frac{1}{(4\pi)^{\frac{3}{2}}} \left\{ \frac{1}{2} + \int_0^1 d\nu \frac{1}{(1-\nu)^{\frac{3}{2}}} \left[1 - 8f_1(\nu) \exp \left\{ -\left[\frac{1-\nu}{2(1+\nu)} - \frac{(1-\nu)^2}{4(1+\nu)^2 m(\nu)} \right] \eta^2 \right\} \right] \right\}. \end{aligned} \quad (2.19)$$

Finally (2.7), (2.7), (2.11), (2.12) and (2.19) yield the expression of the quadratic form (2.1)-(2.5). Note that the quantity between square brackets in (2.3) goes to zero as $\nu \rightarrow 1$ for any finite \vec{x} , so that the integral in ν is actually finite: Indeed the exponential converges to 1 and

$$\lim_{\nu \rightarrow 1} \frac{8\nu^{\lambda-1}(1-\nu)^{\frac{3}{2}}}{(1+\nu^2)|\ln \nu| + 1 - \nu^2} = 1.$$

Therefore the function a_λ is well defined and bounded for any finite \vec{x} and diverges as $x \rightarrow \infty$: Setting $\nu = 1 - x^{-2}\mu$, one has, as $x \rightarrow \infty$,

$$a_\lambda(x) \simeq \frac{x}{(4\pi)^{\frac{3}{2}}} \int_0^{x^2} d\mu \frac{1 - e^{-\mu/2}}{\mu^{\frac{3}{2}}} \simeq c x. \quad (2.20)$$

2.2 Rigorous Definition of the Schrödinger Operator H_α

In the first part of this Section we investigate the definition (2.1) and prove that the form F_α is closed and bounded from below. To this purpose we first need to show that (2.1) is well posed and, for instance, the decomposition

$$\Psi = \phi_\lambda + \mathcal{G}_\lambda q, \quad (2.21)$$

is well defined and unique for any given $\Psi \in L^2(\mathbb{R}^6)$ and $\lambda > 0$.

We start however by stating a crucial result about the natural domain $\mathcal{D}(\Phi_\alpha^\lambda)$. In the following we shall denote by \hat{q} the Fourier transform of the function $q(\vec{x})$, i.e.,

$$\hat{q}(\vec{k}) \equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} d\vec{x} e^{-i\vec{k}\cdot\vec{x}} q(\vec{x}). \quad (2.22)$$

Proposition 2.1 (Domain of Φ_α^λ) *For any $\lambda > 0$, one has*

$$\mathcal{D}(\Phi_\alpha^\lambda) = \left\{ q \in \mathcal{H}^{1/2}(\mathbb{R}^3) \mid \hat{q} \in \mathcal{H}^{1/2}(\mathbb{R}^3) \right\}. \quad (2.23)$$

and the form Φ_α^λ is closed and bounded from below on the domain above.

Proof: Before characterizing the domain $\mathcal{D}(\Phi_\alpha^\lambda)$, we first prove that Φ_α^λ is coercive on Φ_α^λ a domain for λ sufficiently large, which implies the second part of the Proposition. By direct inspection of (2.2) one has the bound

$$\Phi_\alpha^\lambda[q] \geq \int_{\mathbb{R}^3} d\vec{x} (\alpha + a_\lambda(x)) |q(\vec{x})|^2 \geq \left(\alpha + \inf_{\vec{x} \in \mathbb{R}^3} a_\lambda(x) \right) \|q\|_2^2, \quad (2.24)$$

but $a_\lambda(x)$ is an increasing function of x , so that

$$\inf_{\vec{x} \in \mathbb{R}^3} a_\lambda(x) = a_\lambda(0) = \frac{1}{(4\pi)^{\frac{3}{2}}} \left\{ \frac{1}{2} + \int_0^1 d\nu \frac{1}{(1-\nu)^{\frac{3}{2}}} \left[1 - \frac{8\nu^{\lambda-1}(1-\nu)^{\frac{3}{2}}}{[(1+\nu^2)|\ln \nu| + 1 - \nu^2]^{\frac{3}{2}}} \right] \right\}.$$

Moreover

$$\frac{\partial a_\lambda(0)}{\partial \lambda} = \frac{1}{(4\pi)^{\frac{3}{2}}} \int_0^1 d\nu \frac{8|\ln \nu| \nu^{\lambda-1}}{[(1+\nu^2)|\ln \nu| + 1 - \nu^2]^{\frac{3}{2}}} > 0, \quad (2.25)$$

i.e., $a_\lambda(0)$ is increasing, and $\lim_{\lambda \rightarrow \infty} a_\lambda(0) = +\infty$. Therefore for any $\alpha \in \mathbb{R}$ there exists λ_α , such that for $\lambda > \lambda_\alpha$, the factor on the right hand side of (2.24) is strictly positive. The result for any $\lambda > 0$ is a trivial consequence of coerciveness of the form $\Phi_\alpha^\lambda[q] + \lambda_\alpha \|q\|_2^2$.

In order to prove the claim about the domain $\mathcal{D}(\Phi_\alpha^\lambda)$, we show first that the following upper bound holds true,

$$\Phi_\alpha^\lambda[q] \leq c \left(\|q\|_{\mathcal{H}^{1/2}(\mathbb{R}^3)}^2 + \|\hat{q}\|_{\mathcal{H}^{1/2}(\mathbb{R}^3)}^2 \right). \quad (2.26)$$

By the asymptotics (2.20), it is clear that there exists a constant c such that

$$\int_{\mathbb{R}^3} d\vec{x} a_\lambda(x) |q(x)|^2 \leq c \int_{\mathbb{R}^3} d\vec{x} (1+x) |q(x)|^2 \leq c \|\hat{q}\|_{\mathcal{H}^{1/2}(\mathbb{R}^3)}^2, \quad (2.27)$$

therefore it remains to check the expression

$$\Phi_0^\lambda[q] \equiv \frac{1}{2} \int_{\mathbb{R}^6} d\vec{x} d\vec{x}' G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') |q(\vec{x}) - q(\vec{x}')|^2, \quad (2.28)$$

but using the pointwise bound

$$G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') \leq c \int_0^1 d\nu \nu^{\lambda-1} (1-\nu^2)^{-\frac{3}{2}} |\ln \nu|^{-\frac{3}{2}} \exp \left\{ -\frac{(\vec{x} - \vec{x}')^2}{2 \ln \frac{1}{\nu}} - \frac{\nu(\vec{x} - \vec{x}')^2}{1-\nu^2} \right\},$$

and taking the Fourier transform, we obtain

$$\begin{aligned} \Phi_0^\lambda[q] \leq c \int_{\mathbb{R}^3} d\vec{k} \int_0^1 d\nu \frac{\nu^{\lambda-1}}{(2\nu |\ln \nu| + 1 - \nu^2)^{3/2}} \left\{ 1 - \exp \left[-\frac{(1-\nu^2) |\ln \nu|}{2(1-\nu^2 + 2\nu |\ln \nu|)} k^2 \right] \right\} |\hat{q}(k)|^2 \leq \\ c \int_{\mathbb{R}^3} d\vec{k} (1+k) |\hat{q}(k)|^2 \leq c \|q\|_{\mathcal{H}^{1/2}(\mathbb{R}^3)}^2. \end{aligned} \quad (2.29)$$

Thus (2.26) is proven and it remains to show the corresponding lower bound. The key point is that the form Φ_α^λ can be rewritten by applying the Fourier transform in the following way

$$\Phi_\alpha^\lambda[q] = \int_{\mathbb{R}^3} d\vec{k} (\alpha + \tilde{a}_\lambda(k)) |\hat{q}(\vec{k})|^2 + \frac{1}{2} \int_{\mathbb{R}^6} d\vec{k} d\vec{k}' \tilde{G}_\lambda(\vec{k}; \vec{k}') \left| \hat{q}(\vec{k}) - \hat{q}(\vec{k}') \right|^2, \quad (2.30)$$

where

$$\begin{aligned} \tilde{a}_\lambda(k) \equiv \frac{1}{(4\pi)^{\frac{3}{2}}} \left\{ \frac{1}{2} + \int_0^1 d\nu \frac{1}{(1-\nu)^{\frac{3}{2}}} \left[1 - \frac{8\nu^{\lambda-1} (1-\nu)^{\frac{3}{2}}}{[(1+\nu^2) |\ln \nu| + 1 - \nu^2]^{\frac{3}{2}}} \right. \right. \\ \left. \left. \exp \left\{ -\frac{(1-\nu^2) |\ln \nu| k^2}{2[(1+\nu^2) |\ln \nu| + 1 - \nu^2]} \right\} \right] \right\}, \end{aligned} \quad (2.31)$$

and $\tilde{G}_\lambda(\vec{k}; \vec{k}')$ is a suitable kernel. Note that \tilde{a}_λ is not the Fourier transform of a_λ , as one might expect, but it comes from the extraction of the term containing $|\hat{q}(\vec{k}) - \hat{q}(\vec{k}')|^2$. However it is not difficult to show that the function $\tilde{a}_\lambda(k)$ has the same asymptotic behavior for $k \rightarrow \infty$ as $a^\lambda(x)$, namely $\tilde{a}_\lambda(k) \simeq ck$. Therefore, since $\inf_x a_\lambda(x) \rightarrow \infty$ and $\inf_k \tilde{a}_\lambda(k) \rightarrow \infty$ as $\lambda \rightarrow \infty$, by taking λ sufficiently large, one can find some positive constant c such that at the same time

$$\int_{\mathbb{R}^3} d\vec{x} a_\lambda(x) |q(\vec{x})|^2 \geq c \|\hat{q}\|_{\mathcal{H}^{1/2}(\mathbb{R}^3)}^2, \quad \int_{\mathbb{R}^3} d\vec{k} \tilde{a}_\lambda(k) |\hat{q}(\vec{k})|^2 \geq c \|q\|_{\mathcal{H}^{1/2}(\mathbb{R}^3)}^2. \quad (2.32)$$

The result is then a simple consequence of (2.2) and (2.30) which imply

$$\Phi_\alpha^\lambda[q] \geq \frac{1}{2} \int_{\mathbb{R}^3} d\vec{x} a_\lambda(x) |q(\vec{x})|^2 + \frac{1}{2} \int_{\mathbb{R}^3} d\vec{k} a_\lambda(k) |\hat{q}(\vec{k})|^2. \quad \square$$

We are now able to show that the decomposition $\Psi = \phi_\lambda + \mathcal{G}_\lambda q$ and the form F_α does not depend on λ :

Proposition 2.2 (Decomposition $\Psi = \phi_\lambda + \mathcal{G}_\lambda q$) For any $\lambda > 0$ and $\Psi \in L^2(\mathbb{R}^6)$, there is a unique $q \in \mathcal{D}(\Phi_\alpha^\lambda)$ such that $\Psi = \phi^\lambda + \mathcal{G}_\lambda q$ with $\phi^\lambda \in \mathcal{D}(F_0)$.

Proof: We first observe that if $q \in \mathcal{D}(\Phi_\alpha^\lambda)$ then $\mathcal{G}_\lambda q \in L^2(\mathbb{R}^6)$ for any given $\lambda > 0$: It is indeed sufficient to evaluate

$$\|\mathcal{G}_\lambda q\|_{L^2(\mathbb{R}^6)}^2 = \int_{\mathbb{R}^6} d\vec{x} d\vec{x}' q^*(\vec{x}') q(\vec{x}) T(\vec{x}; \vec{x}'),$$

where the (positive) kernel $T(\vec{x}, \vec{x}')$ can be bounded by

$$T(\vec{x}, \vec{x}') = \frac{1}{8\pi^3} \int_0^1 d\nu \int_0^1 d\mu \left(\frac{\nu}{1-\nu^2} + \frac{\mu}{1-\mu^2} \right)^{-3} \nu^{\lambda-1} (1-\nu^2)^{-\frac{3}{2}} |\ln \nu|^{-\frac{3}{2}} \mu^{\lambda-1} (1-\mu^2)^{-\frac{3}{2}} |\ln \mu|^{-\frac{3}{2}} \exp \left\{ -\frac{|\vec{x} - \vec{x}'|^2}{4\left(\frac{1-\nu^2}{\nu} + \frac{1-\mu^2}{\mu}\right)} \right\}.$$

One can then apply Schur test (see, e.g., [HS]) to get

$$\begin{aligned} \|\mathcal{G}_\lambda q\|_{L^2(\mathbb{R}^6)}^2 &\leq \|q\|_2^2 \sup_{\vec{x}' \in \mathbb{R}^3} \int_{\mathbb{R}^3} d\vec{x} T(\vec{x}, \vec{x}') \leq \\ &\frac{\|q\|_2^2}{8\pi^{\frac{3}{2}}} \int_0^1 d\nu \int_0^1 d\mu \nu^{\lambda-1} \mu^{\lambda-1} |\ln \nu|^{-\frac{3}{2}} |\ln \mu|^{-\frac{3}{2}} \left(\frac{\nu}{1-\nu^2} + \frac{\mu}{1-\mu^2} \right)^{-\frac{3}{2}} \leq c \|q\|_2^2. \end{aligned} \quad (2.33)$$

The above calculation thus guarantees that for any $q \in \mathcal{D}(\Phi_\alpha^\lambda)$ and $\lambda > 0$, $\mathcal{G}_\lambda q \in L^2(\mathbb{R}^6)$, but, as we are going to see, under the same hypothesis $\mathcal{G}_\lambda q \notin \mathcal{D}(F_0)$, unless $q = 0$: Here we present only a formal proof of the result which can however be made rigorous by modifying⁶ the form F_0 in a neighborhood Π_δ of the plane Π and then showing that the expression diverges as $\delta \rightarrow 0$. By using the distributional identity

$$[(H_0 + \lambda)\mathcal{G}_\lambda q](\vec{x}, \vec{y}) = q(\vec{x})\delta(\vec{x} - \vec{y}),$$

one can calculate

$$\begin{aligned} F_0[\mathcal{G}_\lambda q] &= \int_{\mathbb{R}^6} d\vec{x} d\vec{x}' q^*(\vec{x}') q(\vec{x}) G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') = \\ &\int_{\mathbb{R}^6} d\vec{x} d\vec{x}' q^*(\vec{x}') q(\vec{x}) [G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') - G_\lambda^{\text{ren}}(\vec{x}, \vec{x}; \vec{x}', \vec{x}')] + \int_{\mathbb{R}^6} d\vec{x} d\vec{x}' q^*(\vec{x}') q(\vec{x}) G_\lambda^{\text{ren}}(\vec{x}, \vec{x}; \vec{x}', \vec{x}') = \\ &\int_{\mathbb{R}^6} d\vec{x} d\vec{x}' q^*(\vec{x}') q(\vec{x}) [G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') - G_\lambda^{\text{ren}}(\vec{x}, \vec{x}; \vec{x}', \vec{x}')] - \Phi_\alpha^\lambda[q], \end{aligned}$$

where the regularization is performed in same way as described in (2.11). Now it is clear that for any charge $q \in \mathcal{D}(\Phi_\alpha^\lambda)$ the second term on the right hand side of the above expression is always bounded, whereas the first term diverges by definition. Thus $F_0[q] = +\infty$, $\mathcal{G}_\lambda q \notin \mathcal{D}(F_0)$ and the decomposition (2.21) is well posed.

The uniqueness can be very easily proven by observing that, if by absurd there exists $\tilde{q} \in \mathcal{D}(\Phi_\alpha^\lambda)$ such that $\Psi = \tilde{\phi}_\lambda + \mathcal{G}_\lambda \tilde{q}$ with $\tilde{\phi}_\lambda \in \mathcal{D}(F_0)$, then $\tilde{\phi}_\lambda - \phi_\lambda = \mathcal{G}_\lambda q - \mathcal{G}_\lambda \tilde{q} \notin \mathcal{D}(F_0)$, which implies $\tilde{\phi}_\lambda - \phi_\lambda = \mathcal{G}_\lambda q - \mathcal{G}_\lambda \tilde{q} = 0$ since the left hand side of the above expression is by definition in $\mathcal{D}(F_0)$. Therefore $\tilde{q} = q$ since the operator \mathcal{G}_λ has empty kernel. \square

Two other important properties of the domain $\mathcal{D}(F_\alpha)$ are formulated in the Proposition below:

⁶For instance, as in [DFinT], one can simply cut the integration over Π_δ , which contains the singular contribution.

Proposition 2.3 (Domain of F_α) *The form F_α and its domain $\mathcal{D}(F_\alpha)$ do not depend on λ . Moreover for any $\Psi \in \mathcal{D}(F_\alpha)$,*

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_{|\vec{x}-\vec{y}|=\delta} d\vec{x} \Psi(\vec{x}, \vec{y}) = q(\vec{y}), \quad (2.34)$$

in the norm topology in $L^2(\Pi)$.

Remark: The above Proposition shows that q is the coefficient of the singular part (i.e., the part not in the domain $\mathcal{D}(F_0)$ of the unperturbed form) of any wave function $\Psi \in \mathcal{D}(F_\alpha)$. The fact that such a singular part can be expressed as $\mathcal{G}_\lambda q$, which resembles the electrostatic potential generated by a charge q , motivates the name “charge” for q .

Proof: The fact that the domain $\mathcal{D}(F_\alpha)$ is independent of λ simply follows from the observation that, if $\Psi = \phi_\lambda + \mathcal{G}_\lambda q$ then also $\Psi = \phi_{\lambda'} + \mathcal{G}_{\lambda'} q$ for $\lambda' \neq \lambda$, $\lambda' > 0$: Indeed, since $\mathcal{G}_\lambda q - \mathcal{G}_{\lambda'} q \in \mathcal{D}(F_0)$, one can set $\phi_{\lambda'} = \phi_\lambda + \mathcal{G}_\lambda q - \mathcal{G}_{\lambda'} q$, which implies the statement above.

It remains to show that the form F_α itself does not depend on λ , but we are going to prove that

$$\mathcal{F}^\lambda[\Psi] - \mathcal{F}^{\lambda'}[\Psi] - (\lambda - \lambda') \|\Psi\|_2^2 = \Phi_\alpha^{\lambda'}[q] - \Phi_\alpha^\lambda[q]. \quad (2.35)$$

The left hand side of the expression above can be written as

$$\mathcal{F}^\lambda[\Psi] - \mathcal{F}^{\lambda'}[\Psi] - (\lambda - \lambda') \|\Psi\|_2^2 = (\lambda' - \lambda) \langle \mathcal{G}_\lambda q | \mathcal{G}_{\lambda'} q \rangle, \quad (2.36)$$

where we have used $H_0(\mathcal{G}_{\lambda'} q - \mathcal{G}_\lambda q) = \lambda \mathcal{G}_\lambda q - \lambda' \mathcal{G}_{\lambda'} q$, which follows from the first resolvent identity. Similarly⁷

$$a_{\lambda'}(\vec{x}) - a_\lambda(\vec{x}) + \int_{\mathbb{R}^3} d\vec{x}' (G_{\lambda'}(\vec{x}, \vec{x}; \vec{x}', \vec{x}') - G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}')) = 0,$$

which yields

$$\Phi_\alpha^{\lambda'}[q] - \Phi_\alpha^\lambda[q] = \int_{\mathbb{R}^3} d\vec{x} q^*(\vec{x}) [(\mathcal{G}_{\lambda'} - \mathcal{G}_\lambda) q](\vec{x}, \vec{x}), \quad (2.37)$$

and the result is a direct consequence of the resolvent identity.

Concerning the second part of the Proposition, we first show that, for any $\lambda > 0$, the regular part ϕ_λ of the wave function Ψ gives no contribution in the integral (2.34), i.e.,

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \left\| \int_{|\vec{x}-\vec{y}|=\delta} d\vec{x} \phi_\lambda(\vec{x}, \vec{y}) \right\|_{L^2(\Pi)} = 0.$$

Indeed by Sobolev inequality in \mathbb{R}^3 $\|f\|_6 \leq \|\nabla f\|_2$ and, setting $\phi_{\lambda, \vec{y}}(\vec{x}) \equiv \phi_\lambda(\vec{x}, \vec{y})$, we have for almost every $\vec{y} \in \mathbb{R}^3$

$$\begin{aligned} \frac{1}{\delta} \left| \int_{|\vec{x}-\vec{y}|=\delta} d\vec{x} \phi_\lambda(\vec{x}, \vec{y}) \right| &\leq c\sqrt{\delta} \|\phi_{\lambda, \vec{y}}\|_{L^4(\mathbb{R}^3)} \leq c\sqrt{\delta} \|\phi_{\lambda, \vec{y}}\|_{L^6(\mathbb{R}^3)}^{\frac{3}{4}} \|\phi_{\lambda, \vec{y}}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \leq \\ &c\sqrt{\delta} \|\nabla_{\vec{x}} \phi_{\lambda, \vec{y}}\|_{L^2(\mathbb{R}^3)}^{\frac{3}{4}} \|\phi_{\lambda, \vec{y}}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \leq c\sqrt{\delta} \|\phi_{\lambda, \vec{y}}\|_{\mathcal{H}^1(\mathbb{R}^3)}, \end{aligned}$$

and taking the L^2 norm in \vec{y} of both sides

$$\frac{1}{\delta} \left\| \int_{|\vec{x}-\vec{y}|=\delta} d\vec{x} \phi_\lambda(\vec{x}, \vec{y}) \right\|_{L^2(\Pi)} \leq c\sqrt{\delta} \|\phi_\lambda\|_{\mathcal{H}^1(\mathbb{R}^6)} \xrightarrow{\delta \rightarrow 0} 0.$$

⁷Note that the quantity inside the integral is regular, since the Green function singularities are cancelled by the difference.

To complete the proof it remains to show that

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_{|\vec{x}-\vec{y}|=\delta} d\vec{x} (\mathcal{G}_\lambda q)(\vec{x}, \vec{y}) = q(\vec{y}),$$

in L^2 norm. We start by calculating⁸

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_{\mathbb{R}^3} d\vec{y}' \int_{|\vec{x}-\vec{y}|=\delta} d\vec{x} G_\lambda(\vec{x}, \vec{y}; \vec{y}', \vec{y}') &= \frac{1}{2^{\frac{3}{2}} \pi^3} \lim_{\delta \rightarrow 0} \int_0^{\frac{1}{\delta^2}} d\mu \frac{(1 - \delta^2 \mu)^{\lambda-1}}{\mu^{\frac{3}{2}} (2 - \delta^2 \mu)^{\frac{3}{2}}} \left[\frac{\delta^2}{|\ln(1 - \delta^2 \mu)|} \right]^{\frac{3}{2}} \\ &\int_{\mathbb{R}^3} d\vec{y}'' \int_{|\vec{x}'|=1} d\vec{x}' \exp \left\{ -\frac{\delta^2 \mu (y^2 + (\vec{y} + \delta \vec{y}'')^2)}{2(2 - \delta^2 \mu)} - \frac{\delta^2 (\vec{x}' - \vec{y}'')^2}{2|\ln(1 - \delta^2 \mu)|} - \frac{(1 - \delta^2 \mu) y''^2}{\mu(2 - \delta^2 \mu)} \right\} = \\ &\frac{1}{8\pi^3} \int_0^\infty d\mu \frac{1}{\mu^3} \int_{\mathbb{R}^3} d\vec{y} \int_{|\vec{x}|=1} d\vec{x} \exp \left\{ -\frac{(\vec{x} - \vec{y})^2}{2\mu} - \frac{y^2}{2\mu} \right\} = \frac{1}{2\sqrt{\pi}} \int_0^\infty d\mu \frac{1}{\mu^{\frac{3}{2}}} e^{-\frac{1}{4\mu}} = 1, \end{aligned} \quad (2.38)$$

where we have exchanged the integrals with the limit $\delta \rightarrow 0$ by dominated convergence. Therefore we have the identity

$$q(\vec{y}) - \lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_{|\vec{x}-\vec{y}|=\delta} d\vec{x} (\mathcal{G}_\lambda q)(\vec{x}, \vec{y}) = \lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_{|\vec{x}-\vec{y}|=\delta} d\vec{x} \int_{\mathbb{R}^3} d\vec{y}' G_\lambda(\vec{x}, \vec{y}; \vec{y}', \vec{y}') (q(\vec{y}) - q(\vec{y}')), \quad (2.39)$$

and, in order to complete the proof, it suffices to observe

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^3} d\vec{y} \left| \frac{1}{\delta} \int_{|\vec{x}-\vec{y}|=\delta} d\vec{x} \int_{\mathbb{R}^3} d\vec{y}' G_\lambda(\vec{x}, \vec{y}; \vec{y}', \vec{y}') (q(\vec{y}) - q(\vec{y}')) \right|^2 \leq \\ \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^3} d\vec{y} \left[\int_{\mathbb{R}^3} d\vec{y}' T_\delta(\vec{y}; \vec{y}') |q(\vec{y}) - q(\vec{y}')| \right]^2, \end{aligned} \quad (2.40)$$

where the kernel $T_\delta(\vec{y}; \vec{y}')$ is the convolution kernel

$$T_\delta(\vec{y}; \vec{y}') = \frac{1}{2^{\frac{3}{2}} \pi^3 \delta} \int_{|\vec{x}-\vec{y}|=\delta} d\vec{x} \int_0^1 d\nu \nu^{\lambda-1} (1 - \nu^2)^{-\frac{3}{2}} |\ln \nu|^{-\frac{3}{2}} \exp \left\{ -\frac{|\vec{x} - \vec{y}'|^2}{2|\ln \nu|} - \frac{\nu |\vec{y} - \vec{y}'|^2}{1 - \nu^2} \right\}. \quad (2.41)$$

Indeed it is not difficult to show that the integral operator with kernel T_δ converges as $\delta \rightarrow 0$ to the identity operator: The integral of $T_\delta(\vec{y})$ is independent of δ and it converges pointwise to 0 in any compact set not containing the origin, which implies $T_\delta(\vec{y}) \rightarrow \delta(\vec{y})$ in the sense of distributions. The right hand side of (2.40) thus vanishes, since one can exchange the integral with the limit $\delta \rightarrow 0$ again by dominated convergence. \square

The main result about the quadratic form F_α is the following

Theorem 2.1 (Closure of F_α) *The quadratic form F_α is closed and bounded from below on the domain $\mathcal{D}(F_\alpha)$.*

Proof: By Propositions 2.2 and 2.3, the domain of the form is well defined and independent of λ . Therefore it is sufficient to show that the quadratic form $F_\alpha + \lambda \|\Psi\|_2^2$ is closed and positive for λ sufficiently large.

Positivity is a trivial consequence of the explicit expression of the form F_0 , which is the expectation value of a positive operator, together with Proposition 2.1 (see also (2.24) and the discussion right after). It

⁸We apply the change of variables $\delta \vec{y}'' = \vec{y}' - \vec{y}$ and $\delta \vec{x}' = \vec{x} - \vec{y}$.

remains then to prove closure but, because of Proposition 2.1, it is sufficient to note that $\mathcal{F}^\lambda[\phi_\lambda] \geq \lambda \|\phi_\lambda\|_2^2$, i.e., the form \mathcal{F}^λ is coercive and then closed. \square

The family of self-adjoint operators associated with the quadratic form F_α are given in the following Proposition. We denote by Γ^λ the self-adjoint operator associated with Φ_α^λ , i.e.,

$$\langle q | \Gamma^\lambda | q \rangle \equiv \Phi_\alpha^\lambda[q] - \alpha \|q\|_2^2. \quad (2.42)$$

Proposition 2.4 (Operator H_α) *The self-adjoint operator H_α associated with the form F_α is given by*

$$\mathcal{D}(H_\alpha) = \{ \Psi \in L^2(\mathbb{R}^6) \mid \exists q \in \mathcal{D}(\Gamma^\lambda) \text{ s.t. } \Psi = \phi_\lambda + \mathcal{G}_\lambda q, \phi_\lambda \in \mathcal{D}(H_0), (\alpha + \Gamma^\lambda)q = \mathcal{P}\phi_\lambda \}, \quad (2.43)$$

$$(H_\alpha + \lambda)\Psi = (H_0 + \lambda)\phi_\lambda. \quad (2.44)$$

Moreover for any $\Psi \in L^2(\mathbb{R}^6)$, the resolvent of H_α can be expressed as

$$(H_\alpha + \lambda)^{-1}\Psi = (H_0 + \lambda)^{-1}\Psi + \mathcal{G}_\lambda q, \quad (2.45)$$

where $\lambda > 0$ and $q \in \mathcal{D}(\Gamma^\lambda)$ solves the equation

$$(\alpha + \Gamma^\lambda)q = \mathcal{P}(H_0 + \lambda)^{-1}\Psi. \quad (2.46)$$

Proof: By definition a function Ψ belongs to the domain \mathcal{D} if there exists some $\chi = H_\alpha \Psi \in L^2(\mathbb{R}^3)$ such that $F_\alpha[\Psi, \Xi] = \langle \chi | \Xi \rangle$ for any $\Xi \in \mathcal{D}(F_\alpha)$. The action and domain of H_α easily follow. \square

It is interesting to notice that the family of self-adjoint operators H_α , $\alpha \in \mathbb{R}$, contains only self-adjoint extensions of the operator \tilde{H}_0 introduced in (1.3). Moreover such extensions are local in the following sense: If $\Psi \in \mathcal{D}(H_\alpha)$ and $\Psi = 0$ in some open set Ω , then $H_\alpha \Psi = 0$ in the same open set Ω . Similarly the boundary condition associated with H_α , i.e., $(\alpha + \Gamma^\lambda)q = \mathcal{P}\phi_\lambda$ is local by construction, since the value of the charge $q(\vec{x})$ is proportional to $\Psi(\vec{x}, \vec{x})$. All these features can be summed up by saying that H_α is a local point interaction Hamiltonian.

We stress however that the family H_α by no means recovers all the possible self-adjoint extension of \tilde{H}_0 (see, e.g., [P]): For instance one could take into account more general extensions labeled by functions $\alpha(\vec{x})$ over the plane Π . In this context H_α , $\alpha \in \mathbb{R}$, are minimal extensions of \tilde{H}_0 . The unperturbed hamiltonian H_0 belongs to the family and is recovered by setting $\alpha = +\infty$.

3 Spectral Analysis

As pointed out in the Introduction, zero-range or point interaction Hamiltonians prove to be very useful in order to explicitly investigate the main physical features of the (toy) model as, in particular, the spectral properties of the system. As we are going to see the energy spectrum of H_α is basically inherited from the operator Γ^λ (2.42). We start then by investigating the spectral properties of Γ^λ :

Proposition 3.1 (Spectral Analysis of Γ^λ) *For any $\lambda > 0$ (2.42) defines an unbounded self-adjoint operator Γ^λ with domain*

$$\mathcal{D}(\Gamma^\lambda) = \left\{ q \in L^2(\mathbb{R}^3) \mid q \in \mathcal{H}^{1/2}(\mathbb{R}^3), \hat{q} \in \mathcal{H}^{1/2}(\mathbb{R}^3) \right\}. \quad (3.1)$$

Moreover there exists some λ_0 such that $\Gamma^\lambda > 0$ for any $\lambda > \lambda_0$.

The spectrum of Γ^λ is discrete, i.e., $\sigma(\Gamma^\lambda) = \sigma_{\text{pp}}(\Gamma^\lambda)$, and denoting by $\gamma_n(\lambda)$, $n \in \mathbb{N}$, its eigenvalues arranged in an increasing order ($\lim_{n \rightarrow \infty} \gamma_n(\lambda) = +\infty$), one has:

- (i) For any $n \in \mathbb{N}$, $\gamma_n(\lambda)$ is an non-decreasing function of $\lambda \in \mathbb{R}^+$,

(ii) $\lim_{\lambda \rightarrow 0} \gamma_0(\lambda) = -\infty$,

(iii) *There exists a finite constant c such that $\gamma_n(\lambda) > -c$ for any $\lambda \in \mathbb{R}^+$ and $n \geq 1$.*

Proof: The fact that Γ^λ is a self-adjoint operator as well as the domain characterization are trivial consequences of (2.42).

By (2.2) and (2.42), Γ^λ can be decomposed as $\Gamma^\lambda = a^\lambda + \Gamma_0^\lambda$, where a^λ is the multiplication operator by the function $a^\lambda(x)$ and Γ_0^λ is the self-adjoint operator associated with the positive quadratic form

$$\langle q | \Gamma_0^\lambda | q \rangle = \frac{1}{2} \int_{\mathbb{R}^3} d\vec{x}' G_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') |q(\vec{x}) - q(\vec{x}')|^2. \quad (3.2)$$

In order to prove the first statement, it is then sufficient to notice that $a^\lambda(x) \geq a^\lambda(0)$ and $\lim_{\lambda \rightarrow \infty} a^\lambda(0) = \infty$, so that by (2.25) for λ sufficiently large the operator is positive.

The discreteness of $\sigma(\Gamma^\lambda)$ follows from compactness of the domain (3.1) (see, e.g., Theorem XIII.64 in [RS4]): By Rellich's criterion (Theorem XIII.65 in [RS4]) one immediately has that $\mathcal{D}(\Gamma^\lambda)$ is a compact subset of $L^2(\mathbb{R}^3)$, since the $\mathcal{H}^{1/2}$ norm of both q and \hat{q} are bounded.

Monotonicity of the eigenvalues as functions of λ is a consequence of monotonicity of the form Φ_α^λ with respect to λ .

The second statement can be proven by showing that there is a trial function $q \in \mathcal{D}(\Gamma^\lambda)$ such that $\langle q | \Gamma^\lambda | q \rangle \rightarrow -\infty$ as $\lambda \rightarrow 0$, since the result is then a consequence of the Min-Max theorem: Denoting by $\psi_0(\vec{x})$ the ground state of the 3d harmonic oscillator $-\frac{1}{2}\Delta_x + \frac{1}{2}x^2$, one has

$$\begin{aligned} \langle \psi_0 | \Gamma^\lambda | \psi_0 \rangle &= \frac{1}{(4\pi)^{\frac{3}{2}}} \left\{ \frac{1}{2} + \int_0^1 d\nu \frac{1}{(1-\nu)^{\frac{3}{2}}} \left[1 - \frac{8\sqrt{2}\nu^{\lambda-1}(1-\nu)^{\frac{3}{2}} [(1+3\nu^2)|\ln \nu| + 4\nu(1-\nu)]^{\frac{1}{2}}}{(1+\nu^2)|\ln \nu| + 1 - \nu^2} \right] \right\} \leq \\ & c_1 - c_2 \int_0^{1/2} d\nu \frac{\nu^{\lambda-1}(\nu + |\ln \nu|)}{1 + |\ln \nu|} \xrightarrow{\lambda \rightarrow 0} -\infty. \end{aligned}$$

The last inequality is proven by showing that the expectation value of the operator Γ^λ on any q^\perp belonging to the subspace orthogonal to ψ_0 remains bounded from below. Let then q^\perp be any normalized function in $\mathcal{D}(\Gamma^\lambda)$ such that $\langle q^\perp | \psi_0 \rangle = 0$: We first notice that in the expectation value of Γ^λ (see also (2.2)), we can restrict the integration in ν in a^λ as well as in G_λ to the interval $[0, e^{-1}]$, since the remainder is uniformly bounded in λ as $\lambda \rightarrow 0$. After the restriction we can expand the square $|q^\perp(\vec{x}) - q^\perp(\vec{x}')|$, since everything is now bounded, and exploit the cancellation to get

$$\langle q^\perp | \Gamma^\lambda | q^\perp \rangle \geq \left[-c + \frac{1}{(4\pi)^{\frac{3}{2}}} \int_0^{e^{-1}} d\nu \frac{1}{(1-\nu)^{\frac{3}{2}}} \right] \|q^\perp\|_2^2 - \int_{\mathbb{R}^6} d\vec{x}d\vec{x}' \bar{G}_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') q^\perp(\vec{x})^* q^\perp(\vec{x}'), \quad (3.3)$$

where \bar{G}_λ stands for the Green function (1.7) where the integration in ν has been restricted to $[0, e^{-1}]$. The first term is again uniformly bounded whereas the only singular contribution is contained inside the second term but it is multiplied by a projector onto the subspace spanned by ψ_0 : Denoting by $k_\nu(\vec{x}, \vec{x}')$ the kernel

$$k_\nu(\vec{x}; \vec{x}') = \exp \left\{ -\frac{1-\nu}{2(1+\nu)} (x^2 + x'^2) - \frac{(\vec{x} - \vec{x}')^2}{2|\ln \nu|} - \frac{\nu(\vec{x} - \vec{x}')^2}{1-\nu^2} \right\}, \quad (3.4)$$

one has

$$\begin{aligned} & \int_{\mathbb{R}^6} d\vec{x}d\vec{x}' \bar{G}_\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') q^\perp(\vec{x})^* q^\perp(\vec{x}') = \\ & c \int_0^{e^{-1}} d\nu \frac{\nu^{\lambda-1}}{(1-\nu^2)^{\frac{3}{2}} |\ln \nu|^{\frac{3}{2}}} \int_{\mathbb{R}^6} d\vec{x}d\vec{x}' k_\nu(\vec{x}; \vec{x}') q^\perp(\vec{x})^* q^\perp(\vec{x}') \leq c \int_0^{e^{-1}} d\nu \frac{\nu^{\lambda-1}}{(1-\nu^2)^{\frac{3}{2}}} \langle q^\perp | k_\nu | q^\perp \rangle. \end{aligned} \quad (3.5)$$

Note that we have used the restriction of the integration domain in order to estimate $|\ln \nu| \leq 1$. On the other hand one can easily prove the upper bound

$$\left| \int_0^{e^{-1}} d\nu \nu^{\lambda-1} (1-\nu^2)^{-\frac{3}{2}} \langle q^\perp | k_\nu | q^\perp \rangle - \pi^{\frac{3}{2}} \langle q^\perp | (H_{\text{osc}} + \lambda - 3/2)^{-1} | q^\perp \rangle \right| \leq c \int_0^{e^{-1}} d\nu (1-\nu^2)^{-\frac{3}{2}} \int_{\mathbb{R}^6} d\vec{x} d\vec{x}' [\bar{k}_\nu(\vec{x}; \vec{x}') - k_\nu(\vec{x}; \vec{x}')] q^\perp(\vec{x})^* q^\perp(\vec{x}'), \quad (3.6)$$

where \bar{k}_ν stands for the kernel associated with the harmonic oscillator (see, e.g., [BC]),

$$\bar{k}_\nu(\vec{x}; \vec{x}') = \exp \left\{ -\frac{1-\nu}{2(1+\nu)} (x^2 + x'^2) - \frac{\nu(\vec{x} - \vec{x}')^2}{1-\nu^2} \right\}.$$

By using the bound

$$|\bar{k}_\nu(\vec{x}; \vec{x}') - k_\nu(\vec{x}; \vec{x}')| \leq \left| 1 - \exp \left\{ -\frac{(\vec{x} - \vec{x}')^2}{2|\ln \nu|} \right\} \right| \leq c \frac{(\vec{x} - \vec{x}')^2}{|\ln \nu|},$$

the above quantity can be easily estimated as follows

$$\begin{aligned} & \int_0^{e^{-1}} d\nu (1-\nu^2)^{-\frac{3}{2}} \int_{\mathbb{R}^6} d\vec{x} d\vec{x}' (\bar{k}_\nu(\vec{x}; \vec{x}') - k_\nu(\vec{x}; \vec{x}')) q^\perp(\vec{x})^* q^\perp(\vec{x}') \leq \\ & c \int_0^{e^{-1}} d\nu (1-\nu^2)^{-\frac{3}{2}} \int_{\mathbb{R}^6} d\vec{x} d\vec{x}' x^2 \bar{k}_\nu(\vec{x}; \vec{x}') |q^\perp(\vec{x})| |q^\perp(\vec{x}')| \leq \\ & c \langle |q^\perp| | H_{\text{osc}} (H_{\text{osc}} + \lambda - 3/2)^{-1} | |q^\perp| \rangle \leq c \|q^\perp\|_2^2. \end{aligned}$$

Altogether (3.3), (3.5) and (3.6) imply

$$\lim_{\lambda \rightarrow 0} \langle q^\perp | \Gamma^\lambda | q^\perp \rangle \geq -c_1 \|q^\perp\|_2^2 - c_2 \lim_{\lambda \rightarrow 0} \langle q^\perp | (H_{\text{osc}} + \lambda - 3/2)^{-1} | q^\perp \rangle \geq -c \|q^\perp\|_2^2, \quad (3.7)$$

since for any function q^\perp orthogonal to the ground state ψ_0 of the harmonic oscillator

$$\langle q^\perp | (H_{\text{osc}} + \lambda - 3/2)^{-1} | q^\perp \rangle \leq (\lambda + 1)^{-1} \|q^\perp\|_2^2.$$

In conclusion the expectation value of Γ^λ in the subspace orthogonal to ψ_0 is bounded from below and then the last estimate is a straightforward consequence of the Min-Max theorem. \square

The spectral properties of the operator H_α follow from the above Proposition via the charge equation (2.46):

Theorem 3.1 (Negative Spectrum of H_α) *For any $\alpha \in \mathbb{R}$, the discrete spectrum $\sigma_{\text{pp}}(H_\alpha)$ is not empty and contains a number $N(\alpha)$ of negative eigenvalues $-E_0(\alpha) \leq -E_1(\alpha) \leq \dots \leq 0$. The corresponding eigenvectors are given by*

$$\Psi_n(\vec{x}, \vec{y}) = (\mathcal{G}^{E_n} q_n)(\vec{x}, \vec{y}), \quad (3.8)$$

where q_n is a solution to the homogeneous equation

$$\alpha q_n + \Gamma^{E_n} q_n = 0. \quad (3.9)$$

Moreover there exists $\alpha_0 \in \mathbb{R}$ such that, if $\alpha > \alpha_0$, $N(\alpha) = 1$ and $N(\alpha) \simeq c|\alpha|^6$ as $\alpha \rightarrow -\infty$. The ground state energy has the following asymptotic behavior: $E_0 \simeq c\alpha^{-1}$ as $\alpha \rightarrow +\infty$ and $E_0 \simeq c\alpha^2$ as $\alpha \rightarrow -\infty$.

Proof: We first derive the integral equation equivalent to the eigenvalue problem. Let Ψ_E be a solution to $H_\alpha \Psi_E = -E \Psi_E$, for some $E > 0$, by (2.44) this equation is equivalent to $(H_0 + E) \phi^\lambda = (\lambda - E) \mathcal{G}_\lambda q$ and the first resolvent identity yields $\phi^\lambda = \mathcal{G}^E q - \mathcal{G}_\lambda q$, which implies $\Psi_E = \mathcal{G}^E q$. On the other hand Ψ_E belongs to the domain of Γ^λ and has to satisfy the boundary condition (2.46), so that

$$\alpha q + \Gamma^E q = 0. \quad (3.10)$$

Therefore there is a one-to-one correspondence between the negative eigenvalues of H_α and non trivial solutions to the homogeneous equation above. In other words $-E$ is an eigenvalue of H_α , if and only if 0 is an eigenvalue of $\alpha + \Gamma^E$.

All the remaining properties of the eigenvalues are direct consequences of Proposition 3.1. For instance since the spectrum of Γ^E is discrete, one can project (3.10) onto its eigenvectors and obtain the algebraic equation

$$\alpha + \gamma_n(E) = 0. \quad (3.11)$$

The eigenvalue equation is thus equivalent to find some $n \in \mathbb{N}$ and $E > 0$ solving the above equation. Note that by the properties of γ_n and in particular $\lim_{E \rightarrow 0} \gamma_0(E) = -\infty$ and $\lim_{n \rightarrow \infty} \gamma_n(E) = +\infty$, there always is a solution to (3.11). On the other hand points (ii) and (iii) in Proposition 3.1 imply that for any $\alpha > \alpha_0 = -\inf_{n>0} \gamma_n(0)$, there is only one solution to (3.11). Moreover the number $N(\alpha)$ of the eigenvalues is bounded from below by the cardinality of the set $\{n \in \mathbb{N} \mid \gamma_n(0) \leq -\alpha\}$, so that any upper bound on $\gamma_n(0)$ provides a lower bound for $N(\alpha)$. Due to the monotonicity in λ of $\gamma_n(\lambda)$, (2.26), (2.3), (2.31) and (2.32), one has the asymptotics $\langle q \mid \Gamma^\lambda \mid q \rangle \simeq \langle q \mid \sqrt{H_{\text{osc}}} \mid q \rangle$ for large λ . We can then use the eigenvalue distribution of the square root of the harmonic oscillator to estimate the asymptotics of $N(\alpha)$ as $\alpha \rightarrow -\infty$.

In order to complete the asymptotic analysis for $\alpha \rightarrow +\infty$, it is sufficient to notice that the λ -dependence of a^λ implies $\gamma_0(\lambda) \simeq -c\lambda^{-1}$ as $\lambda \rightarrow 0$, and then $E_0 = \mathcal{O}(\alpha^{-1})$ as $\alpha \rightarrow +\infty$. Such an argument also applies to the asymptotics $\alpha \rightarrow -\infty$ and, since $\gamma_0(\lambda) \simeq c\sqrt{\lambda}$ as $\lambda \rightarrow +\infty$, $E_0 = \mathcal{O}(\alpha^2)$ as $\alpha \rightarrow -\infty$. \square

An interesting consequence of the above Theorem is the existence of a bound state for any $\alpha \in \mathbb{R}$, in particular even if $\alpha > 0$ and there is no bound state for the “reduced” system, which is given by a particle interacting with a fixed center. In this case the harmonic oscillator strength is assumed to be infinite, so that the system becomes the one-particle model described by the formal hamiltonian $-\frac{1}{2}\Delta_{\vec{x}} + \alpha\delta(\vec{x})$. The associated self-adjoint extensions h_α (see, e.g., [AGH-KH]) are labeled by a real parameter $\alpha \in \mathbb{R}$ and the spectrum $\sigma(h_\alpha)$ is purely absolutely continuous if $\alpha \geq 0$. On the opposite H_α has at least one bound state even if $\alpha \geq 0$ and this is due to the presence of the harmonic oscillator which compensates the “repulsive” force of the zero-range interaction.

The positive part of $\sigma(H_\alpha)$ can be analysed by exploiting the explicit expression of the resolvent (2.45).

Theorem 3.2 (Positive Spectrum of H_α) *The essential spectrum of H_α is equal to $[0, +\infty)$ and $\sigma_{\text{ess}}(H_\alpha) = \sigma_{\text{ac}}(H_\alpha) = \mathbb{R}^+$, i.e., the singular spectrum of H_α is empty. The wave operators $\Omega_\pm(H_\alpha^\omega, H_0^\omega)$ exist and are complete.*

Proof: We start by noticing that (2.45) can be rewritten by using (2.46) as

$$(H_\alpha + \lambda)^{-1} - (H_0 + \lambda)^{-1} = \mathcal{G}_\lambda (\alpha + \Gamma^\lambda)^{-1} \mathcal{G}_\lambda^*, \quad (3.12)$$

where $\mathcal{G}_\lambda^* : L^2(\mathbb{R}^6) \rightarrow L^2(\mathbb{R}^3)$, $\mathcal{G}_\lambda^* = \mathcal{P}(H_0 + \lambda)^{-1}$. Of course the above expression makes sense if and only if $\alpha + \Gamma^\lambda$ is invertible. We fix then λ sufficiently large so that $(\alpha + \Gamma^\lambda)^{-1}$ exists and defines a bounded operator.

Therefore in order to prove the first part of the Theorem, i.e., $\sigma_{\text{ess}}(H_\alpha) = \mathbb{R}^+$, it is sufficient to show that the operator on the right side of (3.12) is a compact operator, since then the result is a consequence

of Weyl's Theorem (see Theorem XIII.14 in [RS4]). Such a result is actually a by-product of the stronger statement

$$\mathcal{G}_\lambda^* (H_0 + \lambda)^{-k} \mathcal{G}_\lambda \in \mathcal{B}_p(L^2(\mathbb{R}^3), L^2(\mathbb{R}^3)), \quad \forall p > \frac{3}{k + \frac{1}{2}}, \quad (3.13)$$

where $k \in \mathbb{N}$ and \mathcal{B}_p stands for the Schatten ideal of order p . We refer to [S] and [CDF] for the theory of Schatten ideals and the detailed proof of the above result respectively.

Now we claim that by (3.13) $\mathcal{G}_\lambda \in \mathcal{B}_p(L^2(\mathbb{R}^3), L^2(\mathbb{R}^6))$ with $p > 12$, since, setting $k = 0$ and using Hölder inequality in Schatten ideals, (3.13) implies $\mathcal{G}_\lambda^* \mathcal{G}_\lambda \in \mathcal{B}_p(L^2(\mathbb{R}^3), L^2(\mathbb{R}^3))$, $p > 6$. Indeed denoting by g_n^2 , $n \in \mathbb{N}$, the singular values associated with $\mathcal{G}_\lambda^* \mathcal{G}_\lambda$, one has $\{g_n\} \in \ell_p$ for $p > 6$ and by a standard argument (see, e.g., the proof of Theorem VI.17 in [RS1]), one can show that $\{g_n\}$ are the singular values of \mathcal{G}_λ , which yields the result. Note that this also implies that $\mathcal{G}_\lambda^* \in \mathcal{B}_p(L^2(\mathbb{R}^6), L^2(\mathbb{R}^3))$, $p > 6$ and both operators are compact. Since $(\Gamma^\lambda + \alpha)^{-1}$ is a bounded operator, by Hölder inequality $\mathcal{G}_\lambda (\Gamma^\lambda + \alpha)^{-1} \mathcal{G}_\lambda^* \in \mathcal{B}_p(L^2(\mathbb{R}^6), L^2(\mathbb{R}^3))$ with $p > 6$ and in particular it is a compact operator.

Moreover (3.13) can be used to show that $[(H_\alpha + \lambda)^{-1}]^4 - [(H_0 + \lambda)^{-1}]^4$ is a trace class operator for some $\lambda > 0$ and existence and completeness of wave operators are thus proven as in Corollary 3 of Theorem XI.11 in [RS3]: The explicit computation of the above difference yields

$$[(H_\alpha + \lambda)^{-1}]^4 - [(H_0 + \lambda)^{-1}]^4 = \sum_{1 \leq m \leq 4} \iota (\mathcal{G}_\lambda (\Gamma^\lambda + \alpha)^{-1} \mathcal{G}_\lambda^*)^m (H_0 + \lambda)^{-4+m}, \quad (3.14)$$

where the ι in the sum denotes the fact that it contains several copies of the same product term where the order of the single factors are exchanged. It is however clear that irrespective of the order of factors, one can apply Hölder inequality together with (3.13) (taking $k = 3$) to any term in the above expression and show that it is a trace class operator.

In order to conclude the proof, we have to show that $\sigma_{\text{sing}}(H_\alpha) = \emptyset$, i.e., the operator has no singular continuous spectrum, but this can be proven by using the limiting absorption principle (see, e.g., Theorem XIII.19 in [RS4]), i.e., by showing that for any interval $[a, b] \subset \mathbb{R}^+$,

$$\sup_{0 < \varepsilon < 1} \int_a^b d\lambda \left| \Im \langle \Psi | (H_\alpha - \lambda + i\varepsilon)^{-1} | \Psi \rangle \right|^p < \infty, \quad (3.15)$$

for any $\Psi \in L^2(\mathbb{R}^6)$ and some $1 < p < \infty$. First of all we notice that the definition (2.45) together with (2.46) can be easily extended to any $\lambda \in \mathbb{Z} \setminus \mathbb{R}$ and in particular to $\lambda \pm i\varepsilon$: A simple computation gives

$$\Im \langle \Psi | (H_\alpha - \lambda + i\varepsilon)^{-1} | \Psi \rangle = \Im \langle \Psi | (H_0 - \lambda + i\varepsilon)^{-1} | \Psi \rangle + \Im \langle q | \Gamma^{-\lambda+i\varepsilon} | q \rangle,$$

where $q \in \mathcal{D}(\Gamma^\lambda)$ solves (2.46). On the other hand the operator H_0 certainly satisfies (3.15) because $\sigma(H_0) = \sigma_{\text{ac}}(H_0) = \mathbb{R}^+$ (see, e.g., Theorem XIII.20 in [RS4]). Therefore by restricting the analysis to any $1 < p \leq 2$, we find that it suffices to show that

$$\sup_{0 < \varepsilon < 1} \int_a^b d\lambda \left| \Im \langle q | \Gamma^{-\lambda+i\varepsilon} | q \rangle \right|^p < \infty, \quad (3.16)$$

for any $1 < p \leq 2$ and $q \in \mathcal{D}(\Gamma^\lambda)$. Now by (2.42), (2.3) and (2.19), one has

$$\Im \langle q | \Gamma^{-\lambda+i\varepsilon} | q \rangle = - \int_{\mathbb{R}^6} d\vec{x} d\vec{x}' q^*(\vec{x}') \Im [G_{-\lambda+i\varepsilon}(\vec{x}, \vec{x}; \vec{x}', \vec{x}')] q(\vec{x}).$$

Note that because of the cancellation of the singular term in a^λ , the above quantity is well defined. Moreover it can be rewritten by using an alternative expression (see also [CDF]) for the “free” Green function, i.e.,

$$G_\lambda(\vec{x}, \vec{y}; \vec{x}', \vec{y}') = \frac{1}{4\pi^3} \sum_{\vec{n} \in \mathbb{N}^3} \int_{\mathbb{R}^3} d\vec{k} \frac{e^{i\vec{k} \cdot (\vec{x}' - \vec{x})}}{k^2 + 2n + 2\lambda} \psi_{\vec{n}}(\vec{y}) \psi_{\vec{n}}(\vec{y}'),$$

where ψ_n is the n -th eigenfunction of the 3d harmonic oscillator and we have used the notation $\vec{n} = (n_1, n_2, n_3)$ and $n = n_1 + n_2 + n_3$. It is then easy to calculate

$$|\Im \langle q | \Gamma^{-\lambda+i\varepsilon} | q \rangle| = \varepsilon \sum_{\vec{n} \in \mathbb{N}^3} \int_{\mathbb{R}^3} d\vec{k} \frac{|\tilde{q}_{\vec{n}}(\vec{k})|^2}{(k^2 + 2n - 2\lambda)^2 + \varepsilon^2}, \quad (3.17)$$

where $\tilde{q}_{\vec{n}}(\vec{k})$ stands for the Fourier transform of the product $q(\vec{x})\psi_{\vec{n}}(\vec{x})$. In order to simplify the proof we decompose the sum in the above expression into two terms, i.e.,

$$\sum_{\vec{n} \in \mathbb{N}^3} \int_{\mathbb{R}^3} d\vec{k} \frac{|\tilde{q}_{\vec{n}}(\vec{k})|^2}{(k^2 + 2n - 2\lambda)^2 + \varepsilon^2} \leq \sum_{n \leq [a]} \int_{\mathbb{R}^3} d\vec{k} \frac{|\tilde{q}_{\vec{n}}(\vec{k})|^2}{(k^2 + 2n - 2\lambda)^2 + \varepsilon^2} + \sum_{n > [a]} \int_{\mathbb{R}^3} d\vec{k} \frac{|\tilde{q}_{\vec{n}}(\vec{k})|^2}{k^4 + \varepsilon^2},$$

where we denoted by $[a]$ the integer part of a . Using the completeness of the harmonic oscillator eigenfunctions one can show that $\sum_{\vec{n} \in \mathbb{N}^3} |\tilde{q}_{\vec{n}}(\vec{k})|^2 = \|q\|_2^2$. On the other hand Cauchy-Schwarz inequality and normalization of ψ_n give the pointwise estimate⁹ $|\tilde{q}_{\vec{n}}(\vec{k})| \leq \|q\|_2$. The estimate (3.17) finally becomes

$$\begin{aligned} |\Im \langle q | \Gamma^{-\lambda+i\varepsilon} | q \rangle| &\leq c\varepsilon \left[\int_{\mathbb{R}^3} d\vec{k} \frac{1}{(k^2 + 2[a] - 2\lambda)^2 + \varepsilon^2} + \frac{1}{\sqrt{\varepsilon}} \right] \|q\|_2^2 \leq \\ &c \left[\frac{\pi}{\sqrt{2}} \left(1 - \frac{2(\lambda - [a])}{\sqrt{4(\lambda - [a])^2 + \varepsilon^2}} \right)^{\frac{1}{2}} + \sqrt{\varepsilon} \right] \leq c, \end{aligned}$$

which implies the result. □

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⁹The estimate can easily be improved by a factor $(1+n)^{-1/4}$ by exploiting the domain characterization (3.1) and the fact that the $H^{1/2}$ norm of q is bounded.

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