Centre de Mathématiques Laurent Schwartz
ECOLE POLYTECHNIQUE

SEMINAIRE

Equations aux Dérivées Partielles

2008-2009

Nikolay Tzvetkov

Riemannian analogue of a Paley-Zygmund theorem

Séminaire É. D. P. (2008-2009), Exposé n° XV, 14 p.

<http://sedp.cedram.org/item?id=SEDP_2008-2009____A15_0>

U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

Séminaire *Équations aux dérivées partielles*
Centre de mathématiques Laurent Schwartz, 2008-2009
Exposé n° XV, 1-14

RIEMANNIAN ANALOGUE OF A PALEY-ZYGMUND THEOREM

NIKOLAY TZVETKOV

1. Presentation of the results

In this exposé we present two recent works of the author on random series. The first is in collaboration with A. Ayache [1] and concerns L^p properties while the second one, in collaboration with S. Grivaux [3], is devoted to continuity results (roughly L^p with $p \to \infty$). This work also benefited from discussions I had with K. Tzanev.

I am grateful to F. Golse for the kind invitation to give a talk at the February 2009 session of the seminar X-EDP.

1.1. **A Paley-Zygmund theorem.**

Let $f \in L^2(S^1)$, $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Then there exists a sequence $(c_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ such that

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx} \,. \tag{1}$$

It is a very natural question to find under what kind of assumptions on the sequence $(c_n)_{n\in\mathbb{Z}}$ the series (1) defines a continuous function on the circle S^1 (or a L^p, $p > 2$ function). Such kind of conditions are given by the Sobolev embedding theorems. Define for $s \geq 0$, the Sobolev space $H^s(S^1)$ via the norm

$$\|f\|^2_{H^s(S^1)} = \sum_{n\in\mathbb{Z}} (1+|n|)^{2s}|c_n|^2 \,.$$

Of course $H^0(S^1) = L^2(S^1)$ and for $s_1 > s_2$, $H^{s_1}(S^1) \subset H^{s_2}(S^1)$ with a continuous embedding. We have the following statement.

Theorem 1 (Sobolev embeddings)**.** If $f \in H^s(S^1)$ for some $s > 1/2$, then $f \in C(S^1)$. If for $p \in [2,\infty)$, $f \in H^{1/2-1/p}(S^1)$ then $f \in L^p(S^1)$. In particular, if $f \in H^{1/2}(S^1)$ then for every $p \in [2,\infty)$, $f \in L^p(S^1)$. The restriction on the Sobolev regularity is optimal (for example there exists $f \in H^{1/2}(S^1)$ which is not in $C(S^1)$).

Thanks to a remarkable work by Paley-Zygmund [6], the conditions on f in Theorem 1 can be strongly relaxed if one allows random variations of the signs of the coefficients c_n. For instance, for any $f \in L^2(S^1)$, given by (1), the expression

$$\sum_{n\in\mathbb{Z}} \pm c_n e^{inx} \tag{2}$$

belongs to any $L^p(S^1)$, $2 \leq p < \infty$ for almost all choices of the signs $\pm$. A rigorous way to define (2) is to see it as a $L^2(S^1)$ random variable defined by the map $\omega \mapsto F(\omega, x)$ from a probability space $(\Omega, \mathbb{P}, \mathcal{A})$ to $L^2(S^1)$ as

$$F(\omega, x) = \sum_{n\in\mathbb{Z}} h_n(\omega) c_n e^{inx}\,,$$

where $(h_n)_{n\in\mathbb{Z}}$ is a system of independent Bernoulli random variables on a probability space $(\Omega, \mathbb{P}, \mathcal{A})$ (i.e. taking values ± 1 with equal probability). We have that $F(\omega, x)$ is a priori defined as an element of $L^2(\Omega \times S^1)$ and the issue is to find better regularity properties of this object. Here is a precise statement.

Theorem 2 (Paley-Zygmund)**.** If $f \in L^2(S^1)$ is defined by (1) then for all $2 \leq p < \infty$, $F(\omega, x) \in L^p(S^1)$ almost surely (a.s) in ω. If $f \in L^2(S^1)$ is defined by (1) with (c_n) satisfying for some $\alpha > 1$

$$\sum_{n\in\mathbb{Z}} (\log(1 + |n|))^{\alpha} |c_n|^2 < \infty$$

then $F(\omega, x) \in C(S^1)$ a.s. in ω. Moreover the restriction $\alpha > 1$ is sharp in the sens that there exists a sequence $(c_n)_{n\in\mathbb{Z}}$ such that

$$\sum_{n\in\mathbb{Z}} \log(1 + |n|) |c_n|^2 < \infty$$

but (2) defines a continuous function for *no choice* of the signs $\pm$.

One may also show a.s. uniform convergence of the partial sums. We will not insist here on these aspects (essentially due to Kolmogorov) of the analysis. The optimality of the restriction $\alpha > 1$ follows from the fact that 3^n, $n = 0, 1, 2, 3 \cdots$ is a Sidon set in $\mathbb{Z}$, a fact which can be easily proved by a use of Riesz products.

1.2. **A generalization on a Riemannian manifold.** The discussion of the previous section essentially asserts that the Paley-Zygmund theorem gains a.s. a half derivative with respect to the Sobolev embedding at L^∞ level. Indeed the Sobolev embedding condition is that $\sim 1/2$ derivatives of f belong to L^2 while the restriction in Theorem 2 implies that the function f is in all $H^s(S^1)$, $s > 0$. It turns out that this phenomenon has a natural extension if we replace S^1 by a compact Riemannian manifold (M, m) and $(e^{inx})_{n\in\mathbb{Z}}$ by an orthonormal basis of $L^2(M)$ formed by eigenfunctions of the Laplace-Beltrami operator Δ_m associated to the metric m. This phenomenon, combined with some "deterministic analysis", was recently exploited by N. Burq and the author in the analysis of wave equations with data of super-critical regularity [2].

Let (M, m) be a d-dimensional smooth, compact, boundaryless Riemannian manifold. This means that we consider a compact boundaryless differentiable manifold, such that the tangent space T_xM at each point of $x \in M$ is equipped with a scalar product which depends smoothly when one varies the point x. The Laplace-Beltrami operator $\Delta_m : C^\infty(M) \to C^\infty(M)$ is defined by $\Delta_m(f) \equiv \mathrm{div}(\nabla_m f)$, where ∇_m is the Riemannian gradient (i.e. $\langle \nabla_m f, h\rangle_g = df \cdot h, \forall\, h \in TM$) and div denotes the divergence of a vector field on M associated to the volume element induced by m. The operator Δ_m is a symmetric, negative operator with respect to the $L^2(M)$ scalar product and admits a self-adjoint realization with domain the Sobolev space $H^2(M)$. Moreover, since M is compact, thanks to the Reilich theorem $(i+\Delta_m)^{-1}$ is a compact operator and thus Δ_m has a discrete spectrum with no finite accumulation point. We denote by $(\varphi_{\lambda_n})_{n\in\mathbb{N}}$ an orthonormal basis of $L^2(M)$ formed by eigenfunctions of $-\Delta_m$ associated to eigenvalues $(\lambda_n^2)_{n\in\mathbb{N}}$ respectively with $0=\lambda_0^2<\lambda_1^2\leq\lambda_2^2\leq\cdots$ (the constants build the eigenspace associated to λ_0). These eigenvalues λ_n are counted with multiplicities. In local coordinates Δ_m is a second order elliptic operator. In the context of the circle S^1, one simply has $\Delta_m=\frac{d^2}{dx^2}$ and the sequence of eigenvalues is

$$0<1=1<4=4<9=9<16=16<25=25<\cdots$$

We now repeat the discussion of the previous section by replacing the corresponding objects. Let $f\in L^2(M)$. Then there exists a sequence $(c_n)_{n\in\mathbb{N}}\in L^2(\mathbb{N})$ such that

$$f(x)=\sum_{n\in\mathbb{N}}c_n\varphi_{\lambda_n}(x)\,. \tag{3}$$

We say that f belongs to the Sobolev space $H^s(M)$ if $\|f\|_{H^s(M)}$ is finite, where

$$\|f\|^2_{H^s(M)}=\sum_{n\in\mathbb{N}}(1+\lambda_n)^{2s}|c_n|^2\,.$$

One may show that $H^s(M)$ is independent of the choice of the bases φ_{λ_n}, and even independent of the choice of the metric m. We now state the counterpart of Theorem 1 in our new context.

Theorem 3 (Sobolev embeddings)**.** *If $f\in H^s(M)$ for some $s>d/2$, then $f\in C(M)$. If for $p\in[2,\infty)$, $f\in H^{d/2-d/p}(S^1)$ then $f\in L^p(M)$. As in Theorem 1 the restrictions on the Sobolev regularity are optimal.*

Now for $f\in L^2(M)$ defined by (3), we define the random series F by

$$F(\omega,x)=\sum_{n\in\mathbb{N}}h_n(\omega)c_n\varphi_{\lambda_n}(x)\,, \tag{4}$$

where again $(h_n)_{n\in\mathbb{Z}}$ is a system of independent Bernoulli random variables. Here is our L^p riemannian Paley-Zygmund theorem.

Theorem 4 (a.s. improvement of the Sobolev embeddings)**.** *Let $p\in[2,\infty)$. Suppose that there exist $C>0$ and $\delta(p)\geq 0$ such that for every n,*

$$\|\varphi_{\lambda_n}\|_{L^p(M)}\leq C\lambda_n^{\delta(p)} \tag{5}$$

Then the following holds true. If $f\in L^2(M)$, defined by (3) belongs to $H^{\delta(p)}(M)$ then $F(\omega,x)\in L^p(M)$ a.s in ω.

In the case when M is a flat d-dimensional torus, the estimate (5) holds with $\delta(p)=0$ if one considers the "usual" basis of the exponentials. Thus we extend the result of Theorem 2 to the higher dimensional torus. Thanks to the work by Sogge [7], estimate (5) is known to hold with

$$\delta(p)=\begin{cases}\frac{d-1}{4}-\frac{d-1}{2p}, & 2\leq p\leq\frac{2(d+1)}{d-1}\\ \frac{d-1}{2}-\frac{d}{p}, & \frac{2(d+1)}{d-1}\leq p\leq\infty.\end{cases}$$

Therefore Theorem 4 displays a gain of 1/2 derivatives with respect to the Sobolev embedding, for $p \geq 2(d+1)/(d-1)$. Indeed, in order to ensure that a function f belongs to $L^p(M)$ the Sobolev embedding requires that $f \in H^{d/2-d/p}(M)$ while combining Theorem 4 and the Sogge estimates the randomisation $F(\omega, x)$ of a function f belongs to $L^p(M)$ a.s. provided $f \in H^{d/2-d/p-1/2}(M)$, i.e. a half derivative less then the deterministic result ! In the case $2 \leq p \leq 2(d+1)/(d-1)$ the gain with respect to the Sobolev embedding is $\frac{d+1}{2}(\frac{1}{2}-\frac{1}{p})$, a positive number $\leq 1/2$ in the considered range for p.

We next turn to our $C(M)$ riemannian Paley-Zygmund theorem.

Theorem 5. Suppose that there exist $C > 0$ and $\beta \geq 0$ such that for every n, every $x, y \in M$,

$$|\varphi_{\lambda_n}(x)| \leq C(1+\lambda_n)^{\beta}, \quad |\varphi_{\lambda_n}(x) - \varphi_{\lambda_n}(y)| \leq C(1+\lambda_n)^{\beta+1}\mathbf{d}(x,y), \tag{6}$$

where $\mathbf{d}(x,y)$ denotes the geodesic distance on M between x and y. Then the following holds true. If $f \in L^2(M)$, defined by (3) satisfies for some $\alpha > 1$,

$$\sum_{n\in\mathbb{N}} \lambda_n^{2\beta}(\log(1+\lambda_n))^{\alpha}|c_n|^2 < \infty$$

then F defined by (4) satisfies $F(\omega, x) \in C(M)$ a.s in ω.

Again, in the case when M is a flat d-dimensional torus with the φ_n the exponentials, the estimate (6) holds with $\beta = 0$. We can also show that (6) holds on any manifold M with $\beta = \frac{d-1}{2}$ and thus we obtain the following statement.

Corollary 6. If $f \in L^2(M)$, defined by (3) satisfies for some $\alpha > 1$,

$$\sum_{n\in\mathbb{N}} \lambda_n^{d-1}(\log(1+\lambda_n))^{\alpha}|c_n|^2 < \infty \tag{7}$$

then F defined by (4) satisfies $F(\omega, x) \in C(M)$ a.s in ω.

Again, we observe $\sim 1/2$ derivatives gain with respect to the Sobolev embedding restriction ensuring continuity which is essentially speaking (7) where $d-1$ is replaced by d.

If one considers the case of the standard sphere one may obtain that (7) is nearly optimal by considering zonal eigenfunctions. It should be however pointed out that at the present moment my understanding on the optimality of the restriction $\alpha > 1$ even in the case of the sphere is very poor. It seems to be an interesting problem.

1.3. **A link with a result of Marcus-Pisier.** Our approach can be applied to give some concrete criteria for the almost sure continuity of random series on compact Lie groups, the context being the one considered by Marcus and Pisier in [5] that we recall now.

Let G be a compact Lie group of dimension d. This means that G is a group having a structure of differentiable manifold so that the group operations are smooth. Denote by μ the Haar measure on G. Since G is compact this measure is bi-invariant. Moreover it is unique up to a multiplicative constant. We set $L^2(G) = L^2(G, d\mu)$ and we fix μ so that the volume of G is one. Since $L^2(G)$ is separable, the set of equivalence classes of irreducible unitary representations of G is countable. Denote by $(\pi_i)_{i=1}^{\infty}$ a sequence describing all irreductible, unitary non-equivalent representations of G. Suppose that $\pi_i : G \to \mathcal{H}_i$, where $\mathcal{H}_i$ a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$. Thanks to the Peter-Weyl theorem $\mathcal{H}_i$ is finite dimensional, say of

dimension d_i, and if $(e_1^i, \cdots, e_{d_i}^i)$ is an orthonormal bases of $\mathcal{H}_i$ then the family of functions on G

$$\varphi_{j,k}^i(g) = \sqrt{d_i}\langle \pi_i(g)e_j^i, e_k^i\rangle, \quad 1 \le j,k \le d_i,\ i = 1, 2, \cdots$$

is an orthonormal basis of $L^2(G)$.

We now repeat for a third time the discussion of the previous sections by replacing the corresponding objects. Let $f \in L^2(G)$. Then there exists $c_{j,k}^i$, $i \in \mathbb{N}$, $1 \le j,k \le d_i$ such that

$$f(g) = \sum_{i=1}^{\infty} \sum_{j,k=1}^{d_i} c_{j,k}^i \varphi_{j,k}^i(g) = \sum_{i=1}^{\infty} \mathrm{Tr}\left((\varphi_{j,k}^i(g))_{j,k=1}^{d_i} \big((c_{j,k}^i)_{j,k=1}^{d_i} \big)^t \right). \tag{8}$$

The result of the previous section applies to the randomisation

$$\sum_{i=1}^{\infty} \sum_{j,k=1}^{d_i} h_{j,k}^i(\omega) c_{j,k}^i \varphi_{j,k}^i(g), \tag{9}$$

where $h_{j,k}^i$ is a family of standard Bernoulli random variables. Let us now describe the randomisation used by Marcus-Pisier in [5] which is slightly different from (9). For $i = 1, 2, \cdots$, we define the random matrix $H_i(\omega)$ as $H_i(\omega) = (\frac{1}{\sqrt{d_i}} h_{j,k}^i(\omega))_{j,k=1}^{d_i}$, where the $h_{j,k}^i$ is again a family of standard Bernoulli random variables. We consider the left randomisation of f, defined by

$$F(g,\omega) = \sum_{i=1}^{\infty} \mathrm{Tr}\left(H_i(\omega) (\varphi_{j,k}^i(g))_{j,k=1}^{d_i} \big((c_{j,k}^i)_{j,k=1}^{d_i} \big)^t \right). \tag{10}$$

One can analyse similarly the right randomisation but we skip this considerations. Observe that $\|F\|_{L^2(G\times\Omega)} = \|f\|_{L^2(G)}$ which "explains" the presence of the factor $\frac{1}{\sqrt{d_i}}$ in the definition of the matrix H_i. We have the following statement.

Theorem 7 (Marcus-Pisier)**.** The random function $F(g,\omega)$ belongs to $C(G)$ a.s. in ω if and only if

$$\int_0^{\infty} \big(\log(N(\varepsilon))\big)^{1/2} d\varepsilon < \infty,$$

where $N(\varepsilon)$ is the entropy number associated to the pseudo-distance on G, defined by

$$d^2(g_1, g_2) = \sum_{i\ge 1} \mathrm{Tr}\left| (\varphi_{j,k}^i(g_1) - \varphi_{j,k}^i(g_2))_{j,k=1}^{d_i} \big((c_{j,k}^i)_{j,k=1}^{d_i} \big)^t \right|^2.$$

A very natural question is to find criteria in the spirit of the Paley-Zygmund result ensuring that $F(g,\omega) \in C(G)$ a.s. in ω. Here we propose one such criterion which will recover the result of Theorem 1 as a particular case.

Since G is compact there exists a riemannian bi-invariant metric on G. Let us denote by m one such metric which makes that (G,m) becomes a riemannian manifold. We denote by Δ_m the Laplace-Beltrami operator associated to the metric m. Let us define the left and right regular representations of G on $L^2(G)$ by

$$L(g)(u(x)) = u(g^{-1}x), \quad R(g)(u(x)) = u(xg), \quad g,x \in G, \quad u \in L^2(G)\,.$$

Since m is bi-invariant, the left and right multiplications on G are isometries for m. Using that the Laplace-Beltrami operator is invariant by isometries, we obtain that for every $g \in G$,

$$\Delta_m L(g) = L(g)\Delta_m, \quad \Delta_m R(g) = R(g)\Delta_m\,.$$

Therefore the restrictions of $L(g)$ and $R(g)$ to the eigenspaces of Δ_m are finite dimensional representations of G. By considering the decompositions of these representations into irreductible representations we obtain that for every $i = 1, 2, \cdots$ there exists $\nu_i \geq 0$ such that

$$-\Delta_m(\varphi^i_{j,k}) = \nu_i \varphi^i_{j,k}, \quad 1 \leq j,k \leq d_i\,.$$

Here is our riemannian Paley-Zygmund theorem in the context of the Marcus-Pisier analysis.

Theorem 8. If $f \in L^2(G)$, defined by (8) satisfies for some $\alpha > 1$,

$$\sum_{i=1}^{\infty}(1+\nu_i)^{d-1}(\log(1+\nu_i))^{\alpha} \sum_{j,k=1}^{d_i} |c^i_{j,k}|^2 < \infty \tag{11}$$

then F defined by (10) satisfies $F(g,\omega) \in C(G)$ a.s in ω.

One can prove a result in the spirit of Theorem 4 concerning the $L^p(G)$ properties of $F(g,\omega)$. Observe that the condition on $f(g)$ is the same as if we consider the randomisation of the previous section given by (9). One may show that the set of functions f satisfying conditions of type (11) is independent of the choice of the metric m (but the random function $F(\omega,x)$ constructed from f is dependent of the choice of the bases of $L^2(G)$ and thus of the metric).

2. Proof of the $L^p(M)$ theorem

In this section, we give the proof of Theorem 4 by invoking the argument of [1] in the considered setting. The starting point is the following classical $L^p - L^2$

inequality of Khinchin which contains the whole benefit if the considered randomisation.

Lemma 9. *There exists $C > 0$ such that for every $p \geq 2$, every sequence of complex numbers $(c_n) \in l^2$,*

$$(12) \quad \|\sum_n c_n h_n(\omega)\|_{L^p(\Omega)} \leq C\sqrt{p}\|\sum_n c_n h_n(\omega)\|_{L^2(\Omega)} = C\sqrt{p}\Big(\sum_n |c_n|^2\Big)^{\frac{1}{2}}.$$

One may wish to see the estimate of Lemma 9 is a consequence of a hypercontractivity property associated to a suitable semi-group. Such hypercontractivity bounds have the advantage to give estimates in the spirit of (12) for the L^p norms of the sum of products of independent Bernouli random variables.

Proof of Lemma 9. Since the claimed inequality is invariant under a multiplication of (c_n) by a constant, we can assume that $\sum_n |c_n|^2 = 1$. Estimate (12) is a consequence of the large deviation bound

$$(13) \qquad p(\omega : |\sum_n c_n h_n(\omega)| > \lambda) \leq C\exp(-c\lambda^2),$$

for some positive constants C and c independent of $\lambda \geq 0$ and (c_n) satisfying $\sum_n |c_n|^2 = 1$. Let us first show that (13) implies (12). Using (13) we get

$$\begin{aligned} \|\sum_n c_n h_n(\omega)\|_{L^p(\Omega)}^p &\leq C\int_0^\infty \lambda^{p-1} e^{-c\lambda^2} d\lambda \\ &= C\int_0^\infty e^{(p-1)\log\lambda - c\lambda^2} d\lambda \\ &\leq (C\sqrt{p})^p, \end{aligned}$$

where in the last inequality we applied (for instance) the Laplace asymptotics, the constant C being independent of p. Therefore (13) implies (12). Let us now prove (13). By separating the real and the imaginary part we may assume that c_n are real. By changing c_n with $-c_n$, we only need to evaluate the probability $p(\omega : \sum_n c_n h_n(\omega) > \lambda)$. By using the independence of h_n and using the inequality $\mathrm{ch}(x) \leq e^{x^2/2}$ for every $x \in \mathbb{R}$, we may write for $t > 0$,

$$\int_\Omega e^{t\sum_n c_n h_n(\omega)} dp(\omega) = \prod_n \mathrm{ch}(tc_n) \leq e^{t^2/2}.$$

Therefore, by the Chebishev inequality

$$p(\omega : \sum_n c_n h_n(\omega) > \lambda) \leq e^{t^2/2 - t\lambda}.$$

We take $t = \lambda$ in the above inequality which proves (13). This completes the proof of Lemma 9. □

Let us now give the proof of Theorem 4. Thanks to Lemma 9, we obtain that for a fixed $x \in M$,

$$\|F(\cdot, x)\|_{L^p(\Omega)} \leq C\sqrt{p}\Big(\sum_{n=0}^\infty |c_n|^2 |\varphi_n(x)|^2\Big)^{\frac{1}{2}}.$$

Therefore, using the triangle inequality and the Fubini theorem, we get

$$\begin{aligned}
\|F\|_{L^p(\Omega\times M)} &\leq C\sqrt{p}\Big\|\Big(\sum_{n=0}^{\infty}|c_n|^2|\varphi_n(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(M)} \\
&= C\sqrt{p}\Big\|\sum_{n=0}^{\infty}|c_n|^2|\varphi_n(x)|^2\Big\|^{\frac{1}{2}}_{L^{p/2}(M)} \\
&\leq C\sqrt{p}\Big(\sum_{n=0}^{\infty}|c_n|^2\|\varphi_n\|^2_{L^p(M)}\Big)^{\frac{1}{2}} \\
&\leq C\sqrt{p}\Big(\sum_{n=0}^{\infty}|c_n|^2\lambda_n^{2\delta(n)}\Big)^{\frac{1}{2}} \\
&\leq C\sqrt{p}\|f\|_{H^{\delta(p)}(M)} < \infty\,.
\end{aligned}$$

Therefore $F(\omega,x)\in L^p(M)$ a.s. in ω. This completes the proof of Theorem 4.

3. Proof of the $C(M)$ theorems

We shall use the following consequence of Lemma 9.

Lemma 10. *There exists a positive constant C such that the following holds true. If for $N\geq 2$, $X_1,\cdots X_N$ are random variables in $L^2(\Omega)$, defined by*

$$X_k(\omega)=\sum_n c_{n,k}h_n(\omega),\quad 1\leq k\leq N$$

then

$$\|\sup_{1\leq k\leq N}|X_k(\omega)|\|_{L^1(\Omega)}\leq C\sqrt{\log N}\sup_{1\leq k\leq N}\|X_k\|_{L^2(\Omega)}\,. \tag{14}$$

Let us remark that inequality (14) holds also for $N=1$ with $C\sqrt{\log N}$ replaced by 1.

Proof of Lemma 10. Write for $p\geq 1$

$$\begin{aligned}
\|\sup_{1\leq k\leq N}|X_k(\omega)|\|^p_{L^1(\Omega)} &\leq \|\sup_{1\leq k\leq N}|X_k(\omega)|\|^p_{L^p(\Omega)} \\
&= \int_\Omega \sup_{1\leq k\leq N}|X_k(\omega)|^p d\omega \\
&\leq \sum_{k=1}^{N}\|X_k\|^p_{L^p(\Omega)} \\
&\leq N\sup_{1\leq k\leq N}\|X_k\|^p_{L^p(\Omega)}\,.
\end{aligned}$$

By invoking Lemma 9, we further get

$$\|\sup_{1\leq k\leq N}|X_k(\omega)|\|_{L^1(\Omega)}\leq N^{\frac{1}{p}}\sup_{1\leq k\leq N}\|X_k\|_{L^p(\Omega)}\leq CN^{\frac{1}{p}}\sqrt{p}\sup_{1\leq k\leq N}\|X_k\|_{L^2(\Omega)}\,.$$

Optimizing a little in p, we get that for $p=2\log N$, $N\geq 2$,

$$\|\sup_{1\leq k\leq N}|X_k(\omega)|\|_{L^1(\Omega)}\leq C\sqrt{\log N}\sup_{1\leq k\leq N}\|X_k\|_{L^2(\Omega)}.$$

This completes the proof of Lemma 10. □

Next, we define a pseudo-distance δ (depending on f) on M by

$$\delta(x,y) \equiv \|F(\omega,x)-F(\omega,y)\|_{L^2(\Omega)} = \big(\sum_n |c_n|^2\,|\varphi_{\lambda_n}(x)-\varphi_{\lambda_n}(y)|^2\big)^{1/2}. \tag{15}$$

Denote by $\mathbf{d}$ the distance on M induced by the riemannian metric m. For $\alpha > 1$, let us define the function $\Phi_\alpha : (0,\infty) \to (0,\infty)$ by

$$\Phi_\alpha(t) = (-\log(t))^{\alpha/2}$$

for $t \in (0, 1/a]$ and $\Phi_\alpha(t) = \Phi_\alpha(1/a)$ for $t \geq 1/a$, where $a > 1$ is chosen in a way so that the function $t \longmapsto t\Phi_\alpha(t)$ is increasing on $(0,+\infty)$. Observe that Φ_α is non-increasing on $(0,+\infty)$. The next step toward the proof of Theorem 5 is the following lemma which relates the pseudo-distance δ to the distance $\mathbf{d}$.

Lemma 11. *Under the assumption of Theorem 5, there exists a positive constant C such that for every $x, y \in M$,*

$$\delta(x,y) \leq \frac{C}{\Phi_\alpha(\mathbf{d}(x,y))}\,, \tag{16}$$

where δ is defined by (15). In particular (16) implies that for a fixed $y \in M$, the function on M defined by $x \mapsto \delta(x,y)$ is continuous.

Proof. Write

$$(\delta(x,y))^2 = \sum_{n=0}^{\infty} |c_n|^2 |\varphi_{\lambda_n}(x) - \varphi_{\lambda_n}(y)|^2 \equiv I(x,y) + II(x,y)\,,$$

where

$$I(x,y) = \sum_{n\,:\,a(1+\lambda_n)\leq(\mathbf{d}(x,y))^{-1}} |c_n|^2 |\varphi_{\lambda_n}(x) - \varphi_{\lambda_n}(y)|^2$$

and

$$II(x,y) = \sum_{n\,:\,a(1+\lambda_n)>(\mathbf{d}(x,y))^{-1}} |c_n|^2 |\varphi_{\lambda_n}(x) - \varphi_{\lambda_n}(y)|^2\,.$$

We estimate separately $I(x,y)$ and $II(x,y)$. Using our assumption (6), we get

$$\begin{aligned} I(x,y) &\leq C \sum_{n\,:\,a(1+\lambda_n)\leq(\mathbf{d}(x,y))^{-1}} |c_n|^2 ((1+\lambda_n)^{1+\beta}\mathbf{d}(x,y))^2 \\ &= \frac{C}{\Phi_\alpha^2(\mathbf{d}(x,y))} \sum_{n\,:\,a(1+\lambda_n)\leq(\mathbf{d}(x,y))^{-1}} |c_n|^2 (1+\lambda_n)^{2+2\beta} \left(\mathbf{d}(x,y)\Phi_\alpha(\mathbf{d}(x,y))\right)^2. \end{aligned}$$

Since the function $t \longmapsto t\Phi_\alpha(t)$ is increasing, if n is such that $\mathbf{d}(x,y) \leq 1/(a(1+\lambda_n))$ we have

$$(\mathbf{d}(x,y)\Phi_\alpha(\mathbf{d}(x,y)))^2 \leq C \left(\frac{1}{a(1+\lambda_n)}\right)^2 (\log(1+\lambda_n))^\alpha\,.$$

This yields the following estimate for the term $I(x,y)$:

$$I(x,y) \leq \frac{C}{\Phi_\alpha^2(\mathbf{d}(x,y))} \sum_{n\,:\,a(1+\lambda_n)\leq(\mathbf{d}(x,y))^{-1}} |c_n|^2 (1+\lambda_n)^{2\beta} \big(\log(1+\lambda_n)\big)^\alpha.$$

Let us next analyse $II(x,y)$. Another use of our assumption (6) yields

$$II(x,y) \leq C \sum_{n\,:\,a(1+\lambda_n)>(\mathbf{d}(x,y))^{-1}} |c_n|^2(1+\lambda_n)^{2\beta}\,.$$

Since Φ_α is non-increasing, we infer that for n such that $a(1+\lambda_n) > (\mathbf{d}(x,y))^{-1}$,

$$\Phi_\alpha(\mathbf{d}(x,y)) \leq \Phi_\alpha\Big(\frac{1}{a(1+\lambda_n)}\Big) \leq C(\log(1+\lambda_n))^{\alpha/2}\,.$$

We thus obtain that

$$II(x,y) \leq \frac{C}{\Phi_\alpha^2(\mathbf{d}(x,y))} \sum_{n\,:\,a(1+\lambda_n)>(\mathbf{d}(x,y))^{-1}} |c_n|^2(1+\lambda_n)^{2\beta}(\log(1+\lambda_n))^{\alpha}\,.$$

Since by our assumption, the series

$$\sum_n |c_n|^2(\log(1+\lambda_n))^{\alpha}(1+\lambda_n)^{2\beta}$$

converges, putting together the estimates on $I(x,y)$ and $II(x,y)$ yields the required inequality. This completes the proof of Lemma 11. □

For every $\varepsilon>0$ we denote by $N_\delta(\varepsilon,M)$ the minimal number of open balls of radius ε for the pseudo-distance δ which are needed to cover M. Then the entropy integral is defined by

$$J(\delta,M)=\int_0^{+\infty}\sqrt{\log(N_\delta(\varepsilon,M))}d\varepsilon.$$

We will use the following consequence of Lemma 10.

Lemma 12 (Dudley criterion)**.** *If $J(\delta,M)$ is finite then $F(\omega,x)\in C(M)$ a.s. in ω.*

Proof. In this proof, we are inspired by [4]. For $j\geq 1$, we define N_j as $N_j=N_\delta(2^{-j},M)$. Let R_j be a family of points of M forming a 2^{-j}-net of (M,δ), i.e. the δ open balls of radius 2^{-j} centered at the points of R_j cover M (these balls are open sets of M thanks to Lemma 11). Let $(u_a^j)_{a\in R_j}$ be a partition of the unity such that

$$\mathrm{supp}(u_a^j)\subseteq(x\in M\,:\,\delta(a,x)<2^{-j}),\quad\forall\, a\in R_j\,.$$

Set

$$F_j(\omega,x)=\sum_{a\in R_j}u_a^j(x)F(\omega,a)\,.$$

The proof will be done by showing that the random variables F_j converge both in $L^2(\Omega;L^2(M))$ and $L^1(\Omega;C(M))$, and then identifying the limits.

We first show the convergence of F_j to F in $L^2(\Omega;L^2(M))$. We have that

$$\|F_j(\omega,x)-F(\omega,x)\|_{L^2(\Omega)}\leq\sum_{a\in R_j}u_a^j(x)\|F(\omega,a)-F(\omega,x)\|_{L^2(\Omega)}\,.$$

Moreover $u_a^j(x)=0$ for $\|F(\omega,a)-F(\omega,x)\|_{L^2(\Omega)}\geq 2^{-j}$. Therefore

$$\|F_j-F\|_{L^2(\Omega;L^2(M))}\leq(\mathrm{vol}(M))^{\frac12}\,2^{-j}$$

which proves that F_j converges to F in $L^2(\Omega; L^2(M))$.

We next prove that F_j converges to some limit in $L^1(\Omega; C(M))$. Define the random variable M_j by

$$M_j(\omega) = \sup_{(a,b)\in E_j} |F(\omega, a) - F(\omega, b)|,$$

where

$$E_j = \{(a,b) \in R_j \times R_{j+1} \,:\, \delta(a,b) < 2^{-(j-1)}\}.$$

We have

$$\begin{aligned} |F_j(\omega,x) - F_{j+1}(\omega,x)| &\leq \sum_{(a,b)\in R_j\times R_{j+1}} u_a^j(x) u_b^{j+1}(x) |F(\omega,a) - F(\omega,b)| \\ &\leq \sum_{(a,b)\in R_j\times R_{j+1}} u_a^j(x) u_b^{j+1}(x) M_j(\omega) \leq M_j(\omega), \end{aligned}$$

where we have made use of the fact that if $(a,b) \in R_j \times R_{j+1}$ and $u_a^j(x)u_b^{j+1}(x)$ is non zero then $\delta(a,b) < 2^{-j} + 2^{-(j+1)} \leq 2^{-(j-1)}$, i.e. $(a,b) \in E_j$. Hence

$$\sup_{x\in M} |F_j(\omega,x) - F_{j+1}(\omega,x)| \leq M_j(\omega)$$

a.s. in $\omega \in \Omega$. Thus, using Lemma 10 and the definition of δ, we get

$$\begin{aligned} \|F_j - F_{j+1}\|_{L^1(\Omega;C(M))} &\leq \|M_j\|_{L^1(\Omega)} \\ &\leq C\sqrt{\log(|E_j|+1)} \sup_{(a,b)\in E_j} \|F(\omega,a) - F(\omega,b)\|_{L^2(\Omega)} \\ &= C\sqrt{\log(|E_j|+1)} \sup_{(a,b)\in E_j} \delta(a,b)\,. \end{aligned}$$

This yields

$$\begin{aligned} \|F_j - F_{j+1}\|_{L^1(\Omega;C(M))} &\leq C\,2^{-j}\sqrt{\log(N_j N_{j+1} + 1)} \\ &\leq C\,2^{-(j+1)}\sqrt{\log(N_{j+1}+1)}\,. \end{aligned}$$

Summing over $j \geq 1$ yields

$$\sum_{j\geq 1} \|F_j - F_{j+1}\|_{L^1(\Omega,C(M))} \leq C_1 + C_2 \sum_{j\geq 1} 2^{-j}\sqrt{\log N_j}$$

for some positive constants C_1 and C_2. Coming back to the definition of the entropy we get

$$J(\delta, M) \geq C \sum_{j\geq 1} 2^{-j}\sqrt{\log N_j},$$

and thus the finiteness of the entropy integral implies that the series

$$\sum_{j\geq 1} \|F_j - F_{j+1}\|_{L^1(\Omega,C(M))}$$

converges. This shows that (F_j) converge to a certain limit in $L^1(\Omega, C(M))$. Using the $L^2(\Omega, L^2(M))$ convergence allows to identify this limit as F. This completes the proof of Lemma 12. $\square$

Recall that we denote by $\mathbf{d}$ the distance on M induced by the riemannian metric m. Denote by $N_{\mathbf{d}}(\varepsilon, M)$ the entropy number with respect to the distance $\mathbf{d}$. Then there exists a positive constant C such that for every $\varepsilon > 0$,

$$N_{\mathbf{d}}(\varepsilon, M) \leq C\varepsilon^{-d}\,. \tag{17}$$

Let us now complete the proof of Theorem 5. We plan to apply Lemma 12. Observe that we only need to study the convergence of the entropy integral for ε near zero. Indeed thanks to our assumption (6), we have that $\delta(x,y)$ remains bounded for $x, y \in M$. Therefore the integration in the expression defining the entropy integral is in fact on a compact set ($N_\delta(\varepsilon, M) = 1$ for $\varepsilon \gg 1$). Using the monotonicity of $N_\delta(\varepsilon, M)$ with respect to ε, we infer that indeed it suffices to study the convergence of the entropy integral for ε near zero. Thanks to Lemma 11

$$\Phi_\alpha(\mathbf{d}(x,y)) \geq \frac{C}{\varepsilon} \implies \delta(x,y) \leq \varepsilon. \tag{18}$$

Coming back to the definition of Φ_α, we obtain that for ε sufficiently small,

$$\mathbf{d}(x,y) \leq e^{-\left(\frac{C}{\varepsilon}\right)^{\frac{2}{\alpha}}} \implies \Phi_\alpha(\mathbf{d}(x,y)) \geq \frac{C}{\varepsilon}\,. \tag{19}$$

Combining (18) and (19), we get

$$N_\delta(\varepsilon, M) \leq N_{\mathbf{d}}\big(e^{-\left(\frac{C}{\varepsilon}\right)^{\frac{2}{\alpha}}}, M\big)\,.$$

Using (17), we get that if ε is sufficiently small,

$$\sqrt{\log N_\delta(\varepsilon, M)} \leq \Big(\frac{C}{\varepsilon}\Big)^{\frac{1}{\alpha}}$$

which is integrable near the point 0 thanks to the assumption $\alpha > 1$. Therefore Lemma 12 applies which completes the proof of Theorem 5. $\square$

Let us make a remark. The proof of Lemma 12 uses a decomposition in the physical space. This lemma then implies Theorem 5 and Corollary 6. The condition imposed on f in Corollary 6 involves only its spectral decomposition. It would be interesting to find a proof of Corollary 6 by decompositions in the frequency space only.

We now give the proof of Theorem 8 which is very similar to that of Theorem 5. Again, we shall use Lemma 12. Coming back to (10), we write

$$\begin{aligned} F(g,\omega) &= \sum_{i\geq 1} d_i^{-1/2} \sum_{j,k,l=1}^{d_i} \varphi^i_{j,k}(g) c^i_{l,k} h^i_{l,j}(\omega) \\ &= \sum_{i\geq 1} d_i^{-1/2} \sum_{j,l=1}^{d_i} \left(\sum_{k=1}^{d_i} \varphi^i_{j,k}(g) c^i_{l,k} \right) h^i_{l,j}(\omega). \end{aligned}$$

Recall that for almost every $\omega \in \Omega$, $F(g,\omega)$ belongs to $L^2(G)$. Indeed

$$\begin{aligned}
\|F\|^2_{L^2(\Omega\times G)} &= \sum_{i\geq 1} d_i^{-1} \sum_{j,l=1}^{d_i} \Big\| \sum_{k=1}^{d_i} \varphi^i_{j,k}(g) c^i_{l,k} \Big\|^2_{L^2(G)} \\
&= \sum_{i\geq 1} d_i^{-1} \sum_{j,l=1}^{d_i} \sum_{k=1}^{d_i} |c^i_{l,k}|^2 \\
&= \sum_{i\geq 1} \sum_{k,l=1}^{d_i} |c^i_{k,l}|^2
\end{aligned}$$

which is finite since $f \in L^2(G)$. The expression of $F(g,\omega)$ shows that we can apply the criterion of Lemma 12, and that $F(g,\omega)$ will coincide a.s. with a continuous function on G as soon as $J(\delta, G)$ is finite, where $J(\delta, G)$ is the entropy integral associated to the pseudo-distance (depending on f) δ on G defined by

$$\begin{aligned}
\delta(g,h) &= \|F(g,\omega) - F(h,\omega)\|_{L^2(\Omega)} \\
&= \Big(\sum_{i\geq 1} d_i^{-1} \sum_{j,l=1}^{d_i} \Big| \sum_{k=1}^{d_i} c^i_{l,k} \Big(\varphi^i_{j,k}(g) - \varphi^i_{j,k}(h) \Big) \Big|^2 \Big)^{\frac{1}{2}}.
\end{aligned}$$

As in the proof of Theorem 5, we introduce the real function $\Phi_\alpha(t)$ and we split $\delta^2(g,h) = I(g,h) + II(g,h)$, where

$$I(g,h) = \sum_{i\,:\,a(1+\nu_i)\leq(\mathbf{d}(g,h))^{-1}} d_i^{-1} \sum_{j,l=1}^{d_i} \Big| \sum_{k=1}^{d_i} c^i_{l,k} \Big(\varphi^i_{j,k}(g) - \varphi^i_{j,k}(h) \Big) \Big|^2$$

and

$$II(g,h) = \sum_{i\,:\,a(1+\nu_i)>(\mathbf{d}(g,h))^{-1}} d_i^{-1} \sum_{j,l=1}^{d_i} \Big| \sum_{k=1}^{d_i} c^i_{l,k} \Big(\varphi^i_{j,k}(g) - \varphi^i_{j,k}(h) \Big) \Big|^2,$$

where $\mathbf{d}(h,g)$ is the riemannian distance between g and h associated to the bi-invariant metric m. Let us set

$$\psi^i_{j,l}(g) = \sum_{k=1}^{d_i} c^i_{l,k} \varphi^i_{j,k}(g)\,.$$

These functions are eigenfunctions of the Laplace-Beltrami operator associated to the metric m with eigenvalue ν_i. We now use the Weyl bounds for the Laplace-Beltrami eigenfunctions (see e.g. [8]). We have that there exists a constant C such that for every i, every $1 \leq j,l \leq d_i$ and every $g,h \in G$,

$$|\psi^i_{j,l}(g)| \leq C\, \|\psi^i_{j,l}\|_{L^2(G)} (1+\nu_i)^{\frac{d-1}{2}}$$

and

$$|\psi^i_{j,l}(g) - \psi^i_{j,l}(h)| \leq C\, \|\psi^i_{j,l}\|_{L^2(G)} (1+\nu_i)^{\frac{d+1}{2}} \mathbf{d}(g,h)\,.$$

Then

$$I(g,h) \leq \; C \sum_{i\,:\,a(1+\nu_i)\leq(\mathbf{d}(g,h))^{-1}} d_i^{-1} \sum_{j,l=1}^{d_i} C\, \|\psi^i_{j,l}\|^2_{L^2(G)} ((1+\nu_i)^{d+1} \mathbf{d}(g,h))^2\,.$$

As in the proof of Lemma 11, we use that $t \longmapsto t\Phi_\alpha(t)$ is increasing on $(0,+\infty)$ to obtain that

$$I(g,h) \leq \frac{C}{\Phi_\alpha^2(\mathbf{d}(g,h))} \times \sum_{i\,:\,a(1+\nu_i)\leq(\mathbf{d}(g,h))^{-1}} d_i^{-1} \sum_{j,l=1}^{d_i} \|\psi_{j,l}^i\|^2_{L^2(G)}(1+\nu_i)^{d-1}\big(\log(1+\nu_i)\big)^\alpha.$$

For the second term, using that Φ_α is non-increasing on $(0,+\infty)$, we get

$$II(g,h) \leq \frac{C}{\Phi_\alpha^2(\mathbf{d}(g,h))} \times \sum_{i\,:\,a(1+\nu_i)>(\mathbf{d}(g,h))^{-1}} d_i^{-1} \sum_{j,l=1}^{d_i} \|\psi_{j,l}^i\|^2_{L^2(G)}(1+\nu_i)^{d-1}(\log(1+\nu_i))^\alpha\,.$$

Putting the estimates for I and II together, we obtain that

$$\begin{aligned}\delta^2(g,h) &\leq \frac{C}{\Phi_\alpha^2(\mathbf{d}(g,h))} \sum_{i\geq 1} d_i^{-1} \sum_{j,l=1}^{d_i} \|\psi_{j,l}^i\|^2_{L^2(G)}(1+\nu_i)^{d-1}(\log(1+\nu_i))^\alpha \\ &= \frac{C}{\Phi_\alpha^2(\mathbf{d}(g,h))} \sum_{i\geq 1} d_i^{-1} \sum_{j,l=1}^{d_i}\sum_{k=1}^{d_i} |c_{l,k}^i|^2(1+\nu_i)^{d-1}(\log(1+\nu_i))^\alpha \\ &= \frac{C}{\Phi_\alpha^2(\mathbf{d}(g,h))} \sum_{i\geq 1} \sum_{l,k=1}^{d_i} |c_{l,k}^i|^2(1+\nu_i)^{d-1}(\log(1+\nu_i))^\alpha\,.\end{aligned}$$

Therefore $\delta(g,h) \leq C/\Phi_\alpha(d(g,h))$ for some positive constant C. The rest of the proof of Theorem 8 goes on exactly as in the proof of Theorem 5.

References

[1] A. Ayache, N. Tzvetkov, *L^p properties for Gaussian random series*, Trans. AMS 360 (2008) 4425-4439.

[2] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations I, II*, Inventiones Math. 173 (2008) 449-496 .

[3] S. Grivaux, N. Tzvetkov, *A Riemannian Paley-Zygmund theorem*, in preparation.

[4] D. Li, H. Queffélec, *Introduction à l'étude des espaces de Banach*, Cours Spécialisés, 12. Société Mathématique de France, Paris, 2004.

[5] M. Marcus, G. Pisier, *Random Fourier series with applications to harmonic analysis*, Annals Math. Stud. 101 (1981).

[6] E. A. C. R. Paley, A. Zygmund, *On some series of functions (1) (2) (3)*, Proc. Camb. Philos. Soc. 26 (1930) 337-357, 458-474, 28 (1932) 190-205.

[7] C. Sogge, *Concerning the L^p norm of spectral clusters for second order elliptic operators on compact manifolds* , J. Funct. Anal. 77 (1988), 123-138.

[8] C. Sogge. *Fourier integrals in classical analysis*, *Cambridge tracts in Mathematics*, 1993.

U.M.R. CNRS 8524, U.F.R. de Mathmatiques, 59 655 Villeneuve d'Ascq Cedex
E-mail address: nikolay.tzvetkov@math.univ-lille1.fr