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Taoufik Hmidi

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U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

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ON THE GLOBAL WELL-POSEDNESS OF THE BOUSSINESQ SYSTEM WITH ZERO VISCOSITY

TAOUFIK HMIDI
(JOINT WORK WITH S. KERAANI)

ABSTRACT. In this paper we prove the global well-posedness of the two-dimensional Boussinesq system with zero viscosity for rough initial data.

1. INTRODUCTION

This paper deals with the global well-posedness for the two-dimensional Boussinesq system,

$$(B_{\nu,\kappa})\begin{cases}\partial_t v + v\cdot\nabla v - \nu\Delta v + \nabla\pi = \theta e_2,\\ \partial_t\theta + v\cdot\nabla\theta - \kappa\Delta\theta = 0,\\ \operatorname{div} v = 0,\\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0.\end{cases}$$

Here, e_2 denotes the vector $(0,1)$, $v = (v_1, v_2)$ is the velocity field, π the scalar pressure and θ the temperature. The coefficients ν and κ are assumed to be positive; ν is called the kinematic viscosity and κ the molecular conductivity.

In the case of strictly positive coefficients ν and κ both velocity and temperature have sufficiently smoothing effects leading to the global well-posedness of smooth solutions. This was proved by numerous authors in various function spaces (see [4, 9, 15] and the references therein).

For $\nu > 0$ and $\kappa = 0$ the problem of global well-posedness is well understood. In [5], Chae proved global well-posedness for initial data (v^0, θ^0) lying in Sobolev spaces $H^s \times H^s$, with $s > 2$ (see also [14]). This result has been recently improved by Keraani and the author [12] to initial data in $H^s \times H^s$, with $s > 0$. However we give only a global existence result without uniqueness in the energy space $L^2 \times L^2$. In a joint work with Abidi [1], we prove the uniqueness for data belonging to $L^2 \cap \mathscr{B}^{-1}_{\infty,1} \times \mathscr{B}^0_{2,1}$. More recently Danchin and Paicu [8] have proved the uniqueness in the energy space.

Our goal here is to study the global well-posedness of the system $(B_{0,\kappa})$, with $\kappa > 0$. First of all, let us recall that the two-dimensional incompressible Euler system, corresponding to $\theta^0 = 0$, is globally well-posed in the Sobolev space H^s, with $s > 2$. This is due to the advection of the vorticity by the flow: there is no accumulation of the vorticity and thus there is no finite-time singularities according to B-K-M criterion [3]. In critical spaces like $\mathscr{B}^{\frac{2}{p}+1}_{p,1}$ the situation is more complicate because we do not know if the B-K-M criterion works or not. In [16], Vishik proved that Euler system is globally well-posed in these critical Besov spaces. He used for the proof a new logarithmic estimate taking advantage of the particular structure of the vorticity equation in dimension two. For the Boussinesq system $(B_{0,\kappa})$, Chae proved in [5] the global well-posedness for initial data v^0, θ^0 lying in Soboloev space H^s, with $s > 2$. We intend here to improve this result for rough initial data. Our results reads as follows.

Theorem 1.1. *Let $p \in [1, \infty[$, $v^0 \in \mathscr{B}_{p,1}^{1+\frac{2}{p}}$ be a divergence-free vector field of $\mathbb{R}^2$ and $\theta^0 \in \mathscr{B}_{p,1}^{-1+\frac{2}{p}} \cap L^r$, with $2 < r < \infty$. There exists a unique global solution (v, θ) to the Boussinesq system $(B_{0,\kappa}), \kappa > 0$ such that*

$$v \in \mathscr{C}(\mathbb{R}_+; \mathscr{B}_{p,1}^{1+\frac{2}{p}}) \quad \text{and} \quad \theta \in L^\infty_{\text{loc}}(\mathbb{R}_+; \mathscr{B}_{p,1}^{-1+\frac{2}{p}} \cap L^r) \cap \widetilde{L}^1_{\text{loc}}(\mathbb{R}_+; \mathscr{B}_{p,1}^{1+\frac{2}{p}} \cap \mathscr{B}^2_{r,\infty}).$$

The situation in the case $p = +\infty$ is more subtle since Leray's projector is not continuous on L^∞ and we overcome this by working in homogeneous Besov spaces leading to more technical difficulties. Before stating our result we introduce the following sub-space of L^∞ :

$$u \in \mathscr{B}^\infty \Leftrightarrow \|u\|_{\mathscr{B}^\infty} := \|u\|_{L^\infty} + \|\Delta_{-1} u\|_{\dot{\mathscr{B}}^0_{\infty,1}} < \infty.$$

We notice that $\mathscr{B}^\infty$ is a Banach space and independent of the choice of the dyadic partition of unity. For the definition of Besov spaces and the frequency localization operator Δ_{-1} we can see next section. Our second main result is the following:

Theorem 1.2. *Let $v^0 \in \mathscr{B}^1_{\infty,1}$, with zero divergence and $\theta^0 \in \mathscr{B}^\infty$. Then there exists a unique global solution (v, θ) to the Boussinesq system $(B_{0,\kappa}), \kappa > 0$ such that*

$$v \in \mathscr{C}(\mathbb{R}_+; \mathscr{B}^1_{\infty,1}) \quad \text{and} \quad \theta \in L^\infty_{\text{loc}}(\mathbb{R}_+; \mathscr{B}^\infty) \cap \widetilde{L}^1_{\text{loc}}(\mathbb{R}_+; \mathscr{B}^2_{\infty,\infty}).$$

For the proofs it suffices to estimate the quantities $\|\nabla v(t)\|_{L^\infty}$ and $\|\nabla \theta\|_{L^1_t L^\infty}$. This will be done by using some logarithmic estimates combined with some smoothing effects. Theorem 3.1 is very crucial and it describes new smoothing effects for the transport-diffusion equation governed by a vector field which is not necessary Lipschitz but only quasi-lipschitz.
The rest of this paper is organized as follows. In section 2, we recall some preliminary results on Besov spaces. Section 3 is devoted to the proof of smoothing effects. In section 4 and 5 we give respectively the proof of Theorem 1.1 and 1.2. We give in the appendix a commutator lemma.

2. NOTATION AND PRELIMINARIES

Throughout this paper we shall denote by C some real positive constant which may be different in each occurrence and by C_0 a real positive constant depending on the norms of the initial data.
Let us introduce the so-called Littlewood-Paley decomposition and the corresponding cut-off operators. There exist two radial positive functions $\chi \in \mathscr{D}(\mathbb{R}^d)$ and $\varphi \in \mathscr{D}(\mathbb{R}^d \backslash \{0\})$ such that

i) $\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1; \quad \forall\, q \geq 1, \text{supp}\,\chi \cap \text{supp}\,\varphi(2^{-q}) = \varnothing$

ii) $\text{supp}\,\varphi(2^{-p}\cdot) \cap \text{supp}\,\varphi(2^{-q}\cdot) = \varnothing$, if $|p - q| \geq 2$.

For every $v \in \mathscr{S}'(\mathbb{R}^d)$ we set

$$\Delta_{-1} v = \chi(\mathrm{D}) v\,;\ \forall q \in \mathbb{N},\ \Delta_q v = \varphi(2^{-q}\mathrm{D}) v \quad \text{and} \quad S_q = \sum_{-1 \le p \le q-1} \Delta_p.$$

The homogeneous operators are defined by

$$\dot\Delta_q v = \varphi(2^{-q}\mathrm{D}) v, \quad \dot S_q v = \sum_{j \le q-1} \dot\Delta_j v, \quad \forall q \in \mathbb{Z}.$$

¿From [2] we split the product uv into three parts:

$$uv = T_u v + T_v u + R(u,v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v \quad \text{and} \quad R(u,v) = \sum_{|q'-q| \le 1} \Delta_q u \Delta_{q'} v.$$

Let us now define inhomogeneous and homogeneous Besov spaces. For $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$ we define the inhomogeneous Besov space $\mathscr{B}^s_{p,r}$ as the set of tempered distributions u such that

$$\|u\|_{\mathscr{B}^s_{p,r}} := \Big(2^{qs} \|\Delta_q u\|_{L^p}\Big)_{\ell^r} < +\infty.$$

The homogeneous Besov space $\dot{\mathscr{B}}^s_{p,r}$ is defined as the set of $u \in \mathscr{S}'(\mathbb{R}^d)$ up to polynomials such that

$$\|u\|_{\dot{\mathscr{B}}^s_{p,r}} := \Big(2^{qs} \|\dot\Delta_q u\|_{L^p}\Big)_{\ell^r(\mathbb{Z})} < +\infty.$$

Let $T > 0$ and $\rho \ge 1$, we denote by $L^\rho_T \mathscr{B}^s_{p,r}$ the space of distributions u such that

$$\|u\|_{L^\rho_T \mathscr{B}^s_{p,r}} := \Big\| \Big(2^{qs} \|\Delta_q u\|_{L^p}\Big)_{\ell^r} \Big\|_{L^\rho_T} < +\infty.$$

We say that u belongs to the space $\widetilde{L}^\rho_T \mathscr{B}^s_{p,r}$ if

$$\|u\|_{\widetilde{L}^\rho_T \mathscr{B}^s_{p,r}} := \Big(2^{qs} \|\Delta_q u\|_{L^\rho_T L^p}\Big)_{\ell^r} < +\infty.$$

The following result is due to Vishik [16].

Lemma 2.1. *Let $d \ge 2$, there exists a positive constant C such that for any smooth function f and for any diffeomorphism ψ of $\mathbb{R}^d$ preserving Lebesgue measure, we have for all $p \in [1,+\infty]$ and for all $j, q \in \mathbb{Z}$,*

$$\|\dot\Delta_j(\dot\Delta_q f \circ \psi)\|_{L^p} \le C 2^{-|j-q|} \|\nabla \psi^{\eta(j,q)}\|_{L^\infty} \|\dot\Delta_q f\|_{L^p},$$

with

$$\eta(j,q) = sign(j-q).$$

Let us now recall the following result proven in [8, 10].

Proposition 2.2. *Let $\nu \ge 0$, $(p,r) \in [1,\infty]^2$, $s \in]-1,1[$, $v \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{Lip}(\mathbb{R}^d))$ with zero divergence and f be a smooth function. Let a be any smooth solution of the transport-diffusion equation*

$$\partial_t a + v \cdot \nabla a - \nu \Delta a = f.$$

Then there is a constant $C := C(s,d)$ such that for every $t \in \mathbb{R}_+$

$$\|a\|_{\widetilde{L}^\infty_t \mathscr{B}^s_{p,r}} + \nu^{\frac{1}{m}} \|a - \Delta_{-1} a\|_{\widetilde{L}^m_t B^{s+\frac{2}{m}}_{p,r}} \le Ce^{CV(t)} \Big(\|a^0\|_{\mathscr{B}^s_{p,r}} + \int_0^t \|f(\tau)\|_{\mathscr{B}^s_{p,r}} d\tau \Big),$$

where $V(t) := \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau.$

We shall now give a logarithmic estimate which is an extension of Vishik's one [16]. For the proof we refer to [13].

Proposition 2.3. *Let p, $r \in [1, +\infty]$, v be a divergence-free vector field belonging to the space $L^1_{loc}(\mathbb{R}_+; \mathrm{Lip}(\mathbb{R}^d))$ and let a be a smooth solution of the following transport-diffusion equation (with $\nu \ge 0$),*

$$\begin{cases} \partial_t a + v \cdot \nabla a - \nu \Delta a = f \\ a_{|t=0} = a^0. \end{cases}$$

If the initial data $a^0 \in B^0_{p,r}$, then we have for all $t \in \mathbb{R}_+$

$$\|a\|_{\widetilde{L}^\infty_t B^0_{p,r}} \le C\big(\|a^0\|_{B^0_{p,r}} + \|f\|_{\widetilde{L}^1_t B^0_{p,r}}\big) \Big(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\Big),$$

where C depends only on the dimension d but not on the viscosity ν.

3. SMOOTHING EFFECTS

This section is devoted to the proof of a smoothing effect for a transport-diffusion equation with respect to a vector field which is not necessary Lipschitz. This problem was studied by the author [11] in the case of singular vortex patches for two dimensional Navier-Stokes equations. The estimate given below is more precise than [11].

Theorem 3.1. *Let v be a smooth divergence-free vector field of $\mathbb{R}^d$ with vorticity $\omega := \mathrm{curl}\, v$. Let a be a smooth solution of the transport-diffusion equation*

$$\partial_t a + v \cdot \nabla a - \Delta a = 0; \quad a_{|t=0} = a^0.$$

Then we have for $q \in \mathbb{N} \cup \{-1\}$ and $t \ge 0$

$$2^{2q} \int_0^t \|\Delta_q a(\tau)\|_{L^\infty} d\tau \quad \lesssim \quad \|a^0\|_{L^\infty} \Big(1 + t + (q+2)\|\omega\|_{L^1_t L^\infty} + \|\nabla \Delta_{-1} v\|_{L^1_t L^\infty}\Big).$$

Remark 1. In [10], the author proved in the case of Lipschitz velocity the following estimate

$$2^{2q} \int_0^t \|\Delta_q a(\tau)\|_{L^\infty} d\tau \lesssim \|a^0\|_{L^\infty} \Big(1 + t + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\Big). \tag{1}$$

But this is not useful in our case. We emphasize that the above theorem is also true when we change L^∞ by L^p, with $p \in [1, \infty]$.

Proof. The idea of the proof is the same as in [10]. We use Lagrangian formulation combined with intensive use of paradifferential calculus.
Let $q \in \mathbb{N}^*$, then the Fourier localized function $a_q := \Delta_q a$ satisfies

$$\partial_t a_q + S_{q-1}v\cdot\nabla a_q - \Delta a_q = (S_{q-1}-\mathrm{Id})v\cdot\nabla a_q - [\Delta_q, v\cdot\nabla]a := g_q. \tag{2}$$

Let ψ_q denote the flow of the regularized velocity $S_{q-1}v$:

$$\psi_q(t,x) = x + \int_0^t S_{q-1}v\big(\tau,\psi_q(\tau,x)\big)d\tau.$$

We set

$$\bar{a}_q(t,x) = a_q(t,\psi_q(t,x)) \quad \text{and} \quad \bar{g}_q(t,x) = g_q(t,\psi_q(t,x)).$$

¿From Leibniz formula we deduce the following identity

$$\Delta\bar{a}_q(t,x) = \sum_{i=1}^{d}\Big\langle H_q\cdot(\partial^i\psi_q)(t,x),(\partial^i\psi_q)(t,x)\Big\rangle + (\nabla a_q)(t,\psi_q(t,x))\cdot\Delta\psi_q(t,x),$$

where $H_q(t,x) := (\nabla^2 a_q)(t,\psi_q(t,x))$ is the Hessian matrix.
Straightforward computations based on the definition of the flow and Gronwall's inequality yield

$$\partial^i\psi_q(t,x) = e_i + h_q^i(t,x),$$

where $(e_i)_{i=1}^d$ is the canonical basis of $\mathbb{R}^d$ and the function h_q^i is estimated as follows

$$\|h_q^i(t)\|_{L^\infty} \lesssim V_q(t)e^{CV_q(t)}, \quad \text{with} \quad V_q(t) := \int_0^t \|\nabla S_{q-1}v(\tau)\|_{L^\infty}d\tau. \tag{3}$$

Applying Leibniz formula and Bernstein inequality we find

$$\|\Delta\psi_q(t)\|_{L^\infty} \lesssim 2^q V_q(t)e^{CV_q(t)}. \tag{4}$$

The outcome is

$$\Delta\bar{a}_q(t,x) = (\Delta a_q)(t,\psi_q(t,x)) - \mathcal{R}_q(t,x), \tag{5}$$

with

$$\begin{aligned}\|\mathcal{R}_q(t)\|_{L^\infty} &\lesssim \|\nabla a_q(t)\|_{L^\infty}\|\Delta\psi_q(t)\|_{L^\infty}\\ &+ \|\nabla^2 a_q(t)\|_{L^\infty}\sup_i\big(\|h_q^i(t)\|_{L^\infty} + \|h_q^i(t)\|_{L^\infty}^2\big)\\ &\lesssim 2^{2q}V_q(t)e^{CV_q(t)}\|a_q(t)\|_{L^\infty}. \qquad (6)\end{aligned}$$

In the last line we have used Bernstein inequality.
¿From (2) and (5) we see that $\bar{a}_q$ satisfies

$$(\partial_t - \Delta)\bar{a}_q(t,x) = \mathcal{R}_q(t,x) + \bar{g}_q(t,x).$$

Now, we will again localize in frequency this equation through the operator Δ_j. So we write from Duhamel formula,

$$\begin{aligned}\Delta_j\bar{a}_q(t,x) &= e^{t\Delta}\Delta_j a_q(0) + \int_0^t e^{(t-\tau)\Delta}\Delta_j\mathcal{R}_q(\tau,x)d\tau\\ &+ \int_0^t e^{(t-\tau)\Delta}\Delta_j\bar{g}_q(\tau,x)d\tau. \qquad (7)\end{aligned}$$

At this stage we need the following lemma (see for instance [7]).

Lemma 3.2. *For $u \in L^\infty$ and $j \in \mathbb{N}$,*

$$\|e^{t\Delta}\Delta_j u\|_{L^\infty} \leq Ce^{-ct2^{2j}}\|\Delta_j u\|_{L^\infty}, \tag{8}$$

where the constants C and c depend only on the dimension d.

Combined with (6) this lemma yields, for every $j \in \mathbb{N}$,

$$\|e^{(t-\tau)\Delta}\Delta_j\mathcal{R}_q(\tau)\|_{L^\infty} \lesssim 2^{2q}V_q(\tau)e^{CV_q(\tau)}e^{-c(t-\tau)2^{2j}}\|a_q(\tau)\|_{L^\infty}. \tag{9}$$

Since the flow is an homeomorphism then we get again in view of Lemma 8

$$\begin{aligned}\|e^{(t-\tau)\Delta}\Delta_j\bar{g}_q(\tau)\|_{L^\infty} &\lesssim e^{-c(t-\tau)2^{2j}}\Big(\|[\Delta_q, v.\nabla]a(\tau)\|_{L^\infty}\\ &+ \|(S_{q-1}v - v)\cdot\nabla a_q\|_{L^\infty}\Big).\end{aligned} \tag{10}$$

From Proposition 6.1 we have

$$\begin{aligned}\|[\Delta_q, v\cdot\nabla]a(t)\|_{L^\infty} &\lesssim \|a(t)\|_{L^\infty}\Big(\|\nabla\Delta_{-1}v(t)\|_{L^\infty} + (q+2)\|\omega(t)\|_{L^\infty}\Big)\\ &\lesssim \|a^0\|_{L^\infty}\Big(\|\nabla\Delta_{-1}v(t)\|_{L^\infty} + (q+2)\|\omega(t)\|_{L^\infty}\Big).\end{aligned} \tag{11}$$

We have used in the last line the maximum principle: $\|a(t)\|_{L^\infty} \leq \|a^0\|_{L^\infty}$.
On the other hand since $q \in \mathbb{N}^*$, we can easily obtain

$$\begin{aligned}\|(S_{q-1}v - v)\cdot\nabla a_q\|_{L^\infty} &\lesssim \|a_q\|_{L^\infty}2^q\sum_{j\geq q-1}2^{-j}\|\Delta_j\omega\|_{L^\infty}\\ &\lesssim \|a^0\|_{L^\infty}\|\omega\|_{L^\infty}.\end{aligned} \tag{12}$$

Putting together (7), (9), (10), (11) and (12) we find

$$\begin{aligned}\|\Delta_j\bar{a}_q(t)\|_{L^\infty} &\lesssim e^{-ct2^{2j}}\|\Delta_j a_q^0\|_{L^\infty}\\ &+ V_q(t)e^{CV_q(t)}2^{2q}\int_0^t e^{-c(t-\tau)2^{2j}}\|a_q(\tau)\|_{L^\infty}d\tau\\ &+ (q+2)\|a^0\|_{L^\infty}\int_0^t e^{-c(t-\tau)2^{2j}}\|\omega(\tau)\|_{L^\infty}d\tau\\ &+ \|a^0\|_{L^\infty}\int_0^t e^{-c(t-\tau)2^{2j}}\|\nabla\Delta_{-1}v(\tau)\|_{L^\infty}d\tau.\end{aligned}$$

Integrating in time and using Young inequalities, we obtain for all $j \in \mathbb{N}$

$$\begin{aligned}\|\Delta_j\bar{a}_q\|_{L^1_tL^\infty} &\lesssim (2^{2j})^{-1}\Big(\|\Delta_j a_q^0\|_{L^\infty} + (q+2)\|a^0\|_{L^\infty}\|\omega\|_{L^1_tL^\infty} +\\ &\|a^0\|_{L^\infty}\|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big) + V_q(t)e^{CV_q(t)}2^{2(q-j)}\|a_q\|_{L^1_tL^\infty}.\end{aligned}$$

Let N be a large integer that will be chosen later. Since the flow is an homeomorphism, then we can write

$$\begin{aligned}2^{2q}\|a_q\|_{L^1_tL^\infty} &= 2^{2q}\|\bar{a}_q\|_{L^1_tL^\infty}\\ &\leq 2^{2q}\Big(\sum_{|j-q|<N}\|\Delta_j\bar{a}_q\|_{L^1_tL^\infty} + \sum_{|j-q|\geq N}\|\Delta_j\bar{a}_q\|_{L^1_tL^\infty}\Big).\end{aligned}$$

Hence, for all $q > N$, one has

$$\begin{aligned} 2^{2q}\|a_q\|_{L^1_tL^\infty} &\lesssim \|a^0\|_{L^\infty} + 2^{2N}\|a^0\|_{L^\infty}\Big((q+2)\|\omega\|_{L^1_tL^\infty} + \|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big) \\ &+ V_q(t)e^{CV_q(t)}2^{2N}2^{2q}\|a_q\|_{L^1_tL^\infty} + 2^{2q}\sum_{|j-q|\geq N}\|\Delta_j\bar{a}_q\|_{L^1_tL^\infty}. \end{aligned}$$

According to Lemma 2.1, we have

$$\|\Delta_j\bar{a}_q(t)\|_{L^\infty} \lesssim 2^{-|q-j|}e^{CV_q(t)}\|a_q(t)\|_{L^\infty}.$$

Thus, we infer

$$\begin{aligned} 2^{2q}\|a_q\|_{L^1_tL^\infty} &\lesssim \|a^0\|_{L^\infty} + 2^{2N}\|a^0\|_{L^\infty}\Big((q+2)\|\omega\|_{L^1_tL^\infty} + \|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big) \\ &+ V_q(t)e^{CV_q(t)}2^{2N}2^{2q}\|a_q\|_{L^1_tL^\infty} + 2^{-N}e^{CV_q(t)}2^{2q}\|a_q\|_{L^1_tL^\infty}. \end{aligned}$$

For low frequencies, $q \leq N$, we write

$$2^{2q}\|a_q\|_{L^1_tL^\infty} \lesssim 2^{2N}\|a\|_{L^1_tL^\infty}.$$

Therefore we get for $q \in \mathbb{N} \cup \{-1\}$,

$$\begin{aligned} 2^{2q}\|a_q\|_{L^1_tL^\infty} &\lesssim \|a^0\|_{L^\infty} + 2^{2N}\|a\|_{L^1_tL^\infty} \\ &+ 2^{2N}\|a^0\|_{L^\infty}\Big((q+2)\|\omega\|_{L^1_tL^\infty} + \|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big) \\ &+ \Big(V_q(t)e^{CV_q(t)}2^{2N} + 2^{-N}e^{CV_q(t)}\Big)2^{2q}\|a_q\|_{L^1_tL^\infty}. \end{aligned}$$

Choosing N and t such that

$$V_q(t)e^{CV_q(t)}2^{2N} + e^{CV_q(t)}2^{-N} \lesssim \epsilon,$$

where $\epsilon << 1$. This is possible for small time t such that

$$V_q(t) \leq C_1,$$

where C_1 is a small absolute constant.
Under this assumption, one obtains for $q \geq -1$

$$2^{2q}\|a_q\|_{L^1_tL^\infty} \lesssim \|a\|_{L^1_tL^\infty} + \|a^0\|_{L^\infty}\Big(1 + (q+2)\|\omega\|_{L^1_tL^\infty} + \|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big).$$

Let us now see how to extend this for arbitrarily large time T. We take a partition $(T_i)_{i=1}^M$ of $[0,T]$ such that

$$\int_{T_i}^{T_{i+1}}\|\nabla S_{q-1}v(t)\|_{L^\infty}dt \simeq C_1.$$

Reproducing the same arguments as above we find in view of $\|a(T_i)\|_{L^\infty} \leq \|a^0\|_{L^\infty}$,

$$
\begin{aligned}
2^{2q}\int_{T_i}^{T_{i+1}}\|a_q(t)\|_{L^\infty}dt &\lesssim \int_{T_i}^{T_{i+1}}\|a(t)\|_{L^\infty}dt+\|a^0\|_{L^\infty}\\
&+ \|a^0\|_{L^\infty}\Big((q+2)\int_{T_i}^{T_{i+1}}\|\omega(t)\|_{L^\infty}dt\\
&+ \int_{T_i}^{T_{i+1}}\|\nabla\Delta_{-1}v(t)\|_{L^\infty}dt\Big).
\end{aligned}
$$

Summing these estimates we get

$$
\begin{aligned}
2^{2q}\|a_q\|_{L^1_TL^\infty} &\lesssim \|a\|_{L^1_TL^\infty}+(M+1)\|a^0\|_{L^\infty}+\\
&+ \|a^0\|_{L^\infty}\Big((q+2)\|\omega\|_{L^1_TL^\infty}+\|\nabla\Delta_{-1}v\|_{L^1_TL^\infty}\Big).
\end{aligned}
$$

As $M\approx V_q(T)$, then

$$
\begin{aligned}
2^{2q}\|a_q\|_{L^1_TL^\infty} &\lesssim \|a\|_{L^1_TL^\infty}+(V_q(T)+1)\|a^0\|_{L^\infty}+\\
&+ \|a^0\|_{L^\infty}\Big((q+2)\|\omega\|_{L^1_TL^\infty}+\|\nabla\Delta_{-1}v\|_{L^1_TL^\infty}\Big).
\end{aligned}
$$

Since

$$
\|\nabla S_{q-1}v\|_{L^\infty}\le\|\nabla\Delta_{-1}v\|_{L^\infty}+(q+2)\|\omega\|_{L^\infty},
$$

then inserting this estimate into the previous one

$$
2^{2q}\|a_q\|_{L^1_TL^\infty} \lesssim \|a^0\|_{L^\infty}\Big((1+T)+(q+2)\|\omega\|_{L^1_TL^\infty}+\|\nabla\Delta_{-1}v\|_{L^1_TL^\infty}\Big).
$$

This is the desired result. □

4. PROOF OF THEOREM 1.1

We restrict ourselves to the *a priori* estimates. The existence and uniqueness parts can be done in classical manner.

Proposition 4.1. *For $v^0\in\mathscr{B}_{p,1}^{1+\frac{2}{p}}$ and $\theta^0\in L^r$, with $2<r<\infty$, we have for $t\in\mathbb{R}_+$*

1)
$$
\|\theta(t)\|_{L^r}\le\|\theta^0\|_{L^r}.
$$

2)
$$
\|\theta\|_{L^1_t\mathscr{B}_{r,1}^{1+\frac{2}{r}}}+\|\omega(t)\|_{L^\infty}+\|\omega\|_{\widetilde{L}^\infty_t\mathscr{B}^0_{\infty,1}}\le C_0e^{e^{C_0t}}.
$$

3)
$$
\|\theta\|_{\widetilde{L}^1_t\mathscr{B}^2_{r,\infty}}+\|\theta\|_{L^1_t\mathscr{B}_{p,1}^{1+\frac{2}{p}}}+\|v\|_{\widetilde{L}^\infty_t\mathscr{B}_{p,1}^{1+\frac{2}{p}}}\le C_0e^{e^{e^{C_0t}}},
$$

where the constant C_0 depends on the initial data.

Proof. The first estimate can be easily obtained from L^r energy estimate. For the second one, we recall that the vorticity $\omega = \partial_1 v^2 - \partial_2 v^1$ satisifies the equation

$$\partial_t\omega + v\cdot\nabla\omega = \partial_1\theta. \tag{13}$$

Taking the L^∞ norm we get

$$\|\omega(t)\|_{L^\infty} \leq \|\omega^0\|_{L^\infty} + \|\nabla\theta\|_{L^1_tL^\infty}. \tag{14}$$

Using the embedding $\mathscr{B}^{1+\frac{2}{r}}_{r,1} \hookrightarrow \mathrm{Lip}(\mathbb{R}^2)$ we obtain

$$\|\omega(t)\|_{L^\infty} \lesssim \|\omega^0\|_{L^\infty} + \|\theta\|_{L^1_t\mathscr{B}^{1+\frac{2}{r}}_{r,1}}. \tag{15}$$

¿From Theorem 3.1 applied to the temperature equation and by Bernstein inequalities we deduce for $\epsilon > 0$,

$$\begin{aligned}\|\theta\|_{L^1_t\mathscr{B}^{2-\epsilon}_{r,\infty}} &\lesssim \|\theta^0\|_{L^r}\Big(1 + t + \|\omega\|_{L^1_tL^\infty} + \|\Delta_{-1}\nabla v\|_{L^1_tL^\infty}\Big)\\ &\lesssim \|\theta^0\|_{L^r}\Big(1 + t + \|\omega\|_{L^1_tL^\infty} + \|\nabla v\|_{L^1_tL^{\bar p}}\Big),\end{aligned}$$

with $\bar p := \max\{p, r\}$. This leads for $r > 2$ to the inequality

$$\|\theta\|_{L^1_t\mathscr{B}^{1+\frac{2}{r}}_{r,\infty}} \lesssim \|\theta^0\|_{L^r}\Big(1 + t + \|\omega\|_{L^1_tL^\infty} + \|\nabla v\|_{L^1_tL^{\bar p}}\Big).$$

On the other hand we have the classical result $\|\nabla v\|_{L^{\bar p}} \approx \|\omega\|_{L^{\bar p}}$. Thus we get

$$\|\theta\|_{L^1_t\mathscr{B}^{1+\frac{2}{r}}_{r,1}} \lesssim \|\theta^0\|_{L^r}\Big(1 + t + \|\omega\|_{L^1_tL^\infty} + \|\omega\|_{L^1_tL^{\bar p}}\Big). \tag{16}$$

The estimate of the $L^{\bar p}$ norm of the vorticity can be done as its L^∞ norm

$$\begin{aligned}\|\omega(t)\|_{L^{\bar p}} &\leq \|\omega^0\|_{L^{\bar p}} + \|\nabla\theta\|_{L^1_tL^{\bar p}}.\\ &\lesssim \|\omega^0\|_{L^{\bar p}} + \|\theta\|_{L^1_t\mathscr{B}^{1+\frac{2}{r}}_{r,1}}.\end{aligned} \tag{17}$$

Set $f(t) := \|\theta\|_{L^1_t\mathscr{B}^{1+\frac{2}{r}}_{r,1}}$, then combining (15), (16) and (17) leads

$$f(t) \lesssim \|\theta^0\|_{L^r}(1 + t + t\|\omega^0\|_{L^\infty\cap L^{\bar p}}) + \|\theta^0\|_{L^r}\int_0^t f(\tau)d\tau.$$

According to Gronwall's inequality, one has

$$\|\theta\|_{L^1_t\mathscr{B}^{1+\frac{2}{r}}_{r,1}} \leq C_0e^{C_0t}, \tag{18}$$

where C_0 is a constant depending on the initial data. From (15) and (16) we deduce

$$\|\omega^0\|_{L^\infty\cap L^{\bar p}} \leq C_0e^{C_0t}.$$

Let us now turn to the estimate of $\|\omega(t)\|_{\mathscr{B}^0_{\infty,1}}$. Applying Proposition 2.3 to the vorticity equation and using Besov embeddings,

$$\begin{aligned}\|\omega\|_{\widetilde{L}^\infty_t\mathscr{B}^0_{\infty,1}} &\lesssim \big(\|\omega^0\|_{L^\infty} + \|\theta\|_{L^1_t\mathscr{B}^1_{\infty,1}}\big)\big(1 + \|\nabla v\|_{L^1_tL^\infty}\big)\\ &\lesssim \big(\|\omega^0\|_{L^\infty} + \|\theta\|_{L^1_t\mathscr{B}^{1+\frac{2}{r}}_{r,1}}\big)\big(1 + \|\nabla v\|_{L^1_tL^\infty}\big).\end{aligned} \tag{19}$$

On the other hand we have

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty} &\leq \|\nabla\Delta_{-1}v(t)\|_{L^\infty}+\sum_{q\in\mathbb{N}}\|\Delta_q\nabla v(t)\|_{L^\infty}\\ &\lesssim \|\nabla\Delta_{-1}v(t)\|_{L^{\bar p}}+\|\omega(t)\|_{\mathscr{B}^0_{\infty,1}}\\ &\lesssim \|\omega(t)\|_{L^{\bar p}}+\|\omega\|_{\widetilde L^\infty_t\mathscr{B}^0_{\infty,1}}. \qquad (20)\end{aligned}$$

Putting together (18), (19) and (20) and using Gronwall's inequality gives

$$\|\nabla v\|_{L^\infty}+\|\omega\|_{\widetilde L^\infty_t\mathscr{B}^0_{\infty,1}}\leq C_0e^{e^{C_0t}}. \qquad (21)$$

It remains to prove the third point of the proposition. The first smoothing effect on θ is a direct consequence of (1) and the above inequality,

$$\|\theta\|_{\widetilde L^1_t\mathscr{B}^2_{r,\infty}}\leq C_0e^{e^{C_0t}}.$$

For the second one we apply Proposition 2.2 to the temperature equation .
To establish the velocity estimate we write

$$\|v\|_{\widetilde L^\infty_t\mathscr{B}^{1+\frac{2}{p}}_{p,1}}\lesssim\|v\|_{L^\infty_tL^p}+\|\omega\|_{\widetilde L^\infty_t\mathscr{B}^{\frac{2}{p}}_{p,1}}.$$

Using the velocity equation, we obatin

$$\|v(t)\|_{L^p}\lesssim\|v^0\|_{L^p}+t\|\theta^0\|_{L^p}+\int_0^t\|\mathcal{P}(v\cdot\nabla v)(\tau)\|_{L^p}d\tau.$$

where $\mathcal{P}$ denotes Leray projector. It is well-known that Riesz transforms act continuously on L^p, with $1<p<\infty$, which yields

$$\|\mathcal{P}(v\cdot\nabla v)\|_{L^p}\lesssim\|v\cdot\nabla v\|_{L^p}\lesssim\|v\|_{L^p}\|\nabla v\|_{L^\infty}.$$

Thus we get in view of Gronwall's inequality and (21)

$$\|v\|_{L^\infty_tL^p}\leq C_0e^{e^{e^{C_0t}}}. \qquad (22)$$

It remains to estimate $\|\omega(t)\|_{\mathscr{B}^{\frac{2}{p}}_{p,1}}$. We apply Proposition 2.2 to the vorticity equation combined with the temperature smoothing, (18) and (21),

$$\|\omega\|_{\widetilde L^\infty_t\mathscr{B}^{\frac{2}{p}}_{p,1}}\lesssim e^{CV(t)}(\|\omega^0\|_{\mathscr{B}^{\frac{2}{p}}_{p,1}}+\|\theta\|_{L^1_t\mathscr{B}^{1+\frac{2}{p}}_{p,1}}).$$

□

5. PROOF OF THEOREM 1.2

The case $p=+\infty$ is more subtle and the difficulty comes from the term $\|\nabla\Delta_{-1}v\|_{L^\infty}$, since Riesz transforms do not map L^∞ to itself. To avoid this problem we use a frequency interpolation method. The main result is the following:

Proposition 5.1. *There exists a constant C_0 depending on $\|v^0\|_{\mathscr{B}^0_{\infty,1}}$ and $\|\theta^0\|_{\mathscr{B}^\infty}$ such that for $t \in [0,\infty[$*

$$\|\theta(t)\|_{L^\infty} \leq \|\theta^0\|_{L^\infty}; \quad \|\theta\|_{L^1_t\mathscr{B}^1_{\infty,1}} \leq C_0 e^{C_0 t^3} \quad \textit{and}$$
$$\|v\|_{\widetilde{L}^\infty_t\mathscr{B}^1_{\infty,1}} + \|\theta\|_{\widetilde{L}^1_t\mathscr{B}^2_{\infty,\infty}} + \|\theta(t)\|_{\mathscr{B}^\infty} \leq C_0 e^{e^{C_0 t^3}}.$$

Proof. The L^∞-bound of the temperature can be easily obtained from the maximum principle. To give the other bounds we start with the following estimate for the vorticity, which is again a direct consequence of the maximum principle,

$$\|\omega(t)\|_{L^\infty} \leq \|\omega^0\|_{L^\infty} + \|\nabla\theta\|_{L^1_tL^\infty} \leq \|\omega^0\|_{L^\infty} + \|\theta\|_{L^1_t\mathscr{B}^1_{\infty,1}}. \tag{23}$$

Let $N \in \mathbb{N}^*$, then we get by definition of Besov spaces and the maximum principle

$$\begin{aligned}\|\theta\|_{L^1_t\mathscr{B}^1_{\infty,1}} &= \sum_{q\leq N-1} 2^q\|\Delta_q\theta\|_{L^1_tL^\infty} + \sum_{q\geq N} 2^q\|\Delta_q\theta\|_{L^1_tL^\infty}\\ &\lesssim 2^N t\|\theta^0\|_{L^\infty} + \sum_{q\geq N} 2^q\|\Delta_q\theta\|_{L^1_tL^\infty}.\end{aligned}$$

By virtue of Theorem 3.1 one has

$$\begin{aligned}\|\theta\|_{L^1_t\mathscr{B}^1_{\infty,1}} &\lesssim 2^N t\|\theta^0\|_{L^\infty} + 2^{-N}\|\theta^0\|_{L^\infty}\Big(1 + t + \|\nabla\Delta_{-1}v\|_{L^1_tL^\infty} + N\|\omega\|_{L^1_tL^\infty}\Big)\\ &\lesssim 2^N t\|\theta^0\|_{L^\infty} + 2^{-N}\|\theta^0\|_{L^\infty}\Big(1 + t + \|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big) + \|\omega\|_{L^1_tL^\infty}.\end{aligned}$$

Choosing judiciously N we get

$$\|\theta\|_{L^1_t\mathscr{B}^1_{\infty,1}} \lesssim \|\omega\|_{L^1_tL^\infty} + t^{\frac{1}{2}}\|\theta^0\|_{L^\infty}\Big(1 + t + \|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big)^{\frac{1}{2}} \tag{24}$$

The following lemma gives an estimate of the low frequency of the velocity.

Lemma 5.2. *For all $t \geq 0$, wed have*

$$\|\nabla\Delta_{-1}v(t)\|_{L^\infty} \lesssim 1 + C_0\big(1+t\big)\|\omega\|_{L^\infty_tL^\infty} + t\|\omega\|^2_{L^\infty_tL^\infty},$$

with C_0 a constant depending on the norms of the initial data.

Proof. Fix $N \in \mathbb{N}^*$. Since $\Delta_{-1} = \Delta_{-1}(\dot{S}_{-N} + \sum_{q=-N}^0 \dot{\Delta}_q)$ then we have

$$\begin{aligned}\|\nabla\Delta_{-1}v\|_{L^\infty} &\lesssim \|\nabla\dot{S}_{-N}v\|_{L^\infty} + \sum_{q=-N}^{0}\|\nabla\dot{\Delta}_q v\|_{L^\infty}\\ &\lesssim 2^{-N}\|v\|_{L^\infty} + \sum_{-N}^{0}\|\dot{\Delta}_q\omega\|_{L^\infty}\\ &\lesssim 2^{-N}\|v\|_{L^\infty} + N\|\omega\|_{L^\infty}.\end{aligned}$$

Taking $N \approx \log(e + \|v\|_{L^\infty})$ we get

$$\|\nabla\Delta_{-1}v\|_{L^\infty} \lesssim 1 + \|\omega\|_{L^\infty}\log(e + \|v\|_{L^\infty}). \tag{25}$$

It remains to estimate $\|v\|_{L^\infty}$. Let $M \in \mathbb{N}$ then we have

$$\|v\|_{L^\infty} \lesssim \|\dot{S}_{-M}v\|_{L^\infty} + 2^M\|\omega\|_{L^\infty}.$$

Now using the equation of the velocity we get

$$\begin{aligned}
\|\dot{S}_{-M}v(t)\|_{L^\infty} &\leq \|\dot{S}_{-M}v^0\|_{L^\infty} + \|\mathcal{P}\dot{S}_{-M}\theta\|_{L^1_tL^\infty}\\
&+ \int_0^t \|\dot{S}_{-M}\mathrm{div}\,\mathcal{P}(v\otimes v)(\tau)\|_{L^\infty}d\tau\\
&\lesssim \|v^0\|_{L^\infty} + \|\Delta_{-1}\theta\|_{L^1_t\dot{\mathscr{B}}^0_{\infty,1}} + 2^{-M}\int_0^t \|v(\tau)\|^2_{L^\infty}d\tau.
\end{aligned}$$

We have used the following inequality based on the uniform continuity with respect to q of the operator $\dot{\Delta}_q\mathcal{P}: L^\infty \to L^\infty$:

$$\begin{aligned}
\|\dot{S}_{-M}\mathrm{div}\,\mathcal{P}(v\otimes v)\|_{L^\infty} &\leq \sum_{q\leq -M-1}\|\dot{\Delta}_q\mathrm{div}\,\mathcal{P}(v\otimes v)\|_{L^\infty}\\
&\lesssim \sum_{q\leq -M-1} 2^q\|v\otimes v\|_{L^\infty}.
\end{aligned}$$

Thus we obtain

$$\|v\|_{L^\infty} \lesssim \|v^0\|_{L^\infty} + \|\Delta_{-1}\theta\|_{L^1_t\dot{\mathscr{B}}^0_{\infty,1}} + 2^{-M}\int_0^t \|v(\tau)\|^2_{L^\infty}d\tau + 2^M\|\omega(t)\|_{L^\infty}.$$

To estimate $\|\Delta_{-1}\theta\|_{L^1_t\dot{\mathscr{B}}^0_{\infty,1}}$ we use the temperature equation,

$$\begin{aligned}
\|\dot{\Delta}_q\theta(t)\|_{L^\infty} &\leq \|\dot{\Delta}_q\theta^0\|_{L^\infty} + \|\dot{\Delta}_q(v\cdot\nabla\theta)\|_{L^1_tL^\infty} + \|\dot{\Delta}_q\Delta\theta\|_{L^1_tL^\infty}\\
&\lesssim \|\dot{\Delta}_q\theta^0\|_{L^\infty} + 2^q\|v\,\theta\|_{L^1_tL^\infty} + 2^{2q}\|\theta\|_{L^1_tL^\infty}\\
&\lesssim \|\dot{\Delta}_q\theta^0\|_{L^\infty} + 2^q\|\theta^0\|_{L^\infty}\|v\|_{L^1_tL^\infty} + 2^{2q}t\|\theta^0\|_{L^\infty}.
\end{aligned}$$

Therefore we get

$$\sum_{q\leq 0}\|\dot{\Delta}_q\theta(t)\|_{L^\infty} \lesssim \sum_{q\leq 0}\|\dot{\Delta}_q\theta^0\|_{L^\infty} + t\|\theta^0\|_{L^\infty} + \|\theta^0\|_{L^\infty}\int_0^t \|v(\tau)\|_{L^\infty}d\tau.$$

Taking M such that

$$2^{2M} \approx \frac{\int_0^t \|v\|^2_{L^\infty}d\tau}{\|\omega\|_{L^\infty}},$$

we find

$$\|v\|_{L^\infty} \lesssim C_0(1+t) + \|\theta^0\|_{L^\infty}\int_0^t \|v(\tau)\|_{L^\infty}d\tau + \|\omega(t)\|^{\frac{1}{2}}_{L^\infty}\Big(\int_0^t \|v(\tau)\|^2_{L^\infty}d\tau\Big)^{\frac{1}{2}}.$$

According to Gronwall's inequality we get

$$\|v\|_{L^\infty} \leq C_0 e^{C_0 t}e^{Ct\|\omega\|_{L^\infty_tL^\infty}}. \tag{26}$$

Inserting this estimate into (25) we find the desired inequality. □

Lemma 5.2 and (24) yield

$$\begin{aligned}\|\theta\|^2_{L^1_t\mathscr{B}^1_{\infty,1}} &\leq C_0(1+t^2)+\|\omega\|^2_{L^1_tL^\infty}+C_0(1+t^2)\int_0^t\|\omega\|^2_{L^\infty_\tau L^\infty}d\tau\\ &\leq C_0(1+t^2)\Big(1+\int_0^t\|\omega\|^2_{L^\infty_\tau L^\infty}d\tau\Big).\end{aligned}$$

Combining this estimate with (23) yields

$$\|\omega\|^2_{L^\infty_tL^\infty}\leq C_0(1+t^2)\Big(1+\int_0^t\|\omega\|^2_{L^\infty_\tau L^\infty}d\tau\Big).$$

Applying Gronwall's inequality we get

$$\|\omega(t)\|_{L^\infty}\leq C_0e^{C_0t^3}. \tag{27}$$

This gives

$$\|\theta\|_{L^1_t\mathscr{B}^1_{\infty,1}}\leq C_0e^{C_0t^3}. \tag{28}$$

¿From Lemma 5.2 we have

$$\|\nabla\Delta_{-1}v(t)\|_{L^\infty}\leq C_0e^{C_0t^3}. \tag{29}$$

Let us now turn to the estimate of the vorticity in $\mathscr{B}^0_{\infty,1}$ space. For this purpose we apply Proposition 2.3 to the vorticity equation, with $p=+\infty$ and $r=1$

$$\|\omega\|_{\widetilde{L}^\infty_t\mathscr{B}^0_{\infty,1}}\lesssim\big(\|\omega^0\|_{L^\infty}+\|\nabla\theta\|_{L^1_t\mathscr{B}^0_{\infty,1}}\big)\Big(1+\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\Big). \tag{30}$$

On the other hand we have by definition and from (29) and (30)

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty}\lesssim\|v\|_{\widetilde{L}^\infty_t\mathscr{B}^1_{\infty,1}} &\lesssim \|\nabla\Delta_{-1}v\|_{L^\infty_tL^\infty}+\sum_{q\in\mathbb{N}}\|\Delta_q\omega\|_{L^\infty_tL^\infty}\\ &\lesssim C_0e^{C_0t^3}+\|\omega\|_{\widetilde{L}^\infty_t\mathscr{B}^0_{\infty,1}}\\ &\lesssim C_0e^{C_0t^3}\Big(1+\int_0^t\|v(\tau)\|_{\mathscr{B}^1_{\infty,1}}d\tau\Big).\end{aligned}$$

It suffices now to use Gronwall's inequality.

To estimate $\|\theta\|_{L^1_tB^2_{\infty,\infty}}$ it suffices to combine (1) with Lipschitz estimate of the velocity. This concludes the proof of the proposition. □

6. Appendix: Commutator estimate

Our task now is to prove the following commutator result.

Proposition 6.1. *Let u be a smooth function and v be a divergence-free vector field of $\mathbb{R}^d$ such that its vorticity $\omega:=$ curl v belongs to L^∞. Then we have for all $q\geq-1$,*

$$\big\|[\Delta_q,v\cdot\nabla]u\big\|_{L^\infty}\lesssim\|u\|_{L^\infty}\Big(\|\nabla\Delta_{-1}v\|_{L^\infty}+(q+2)\|\omega\|_{L^\infty}\Big).$$

Proof. The main tool is Bony's decomposition [2]:

$$[\Delta_q, v\cdot\nabla]u = [\Delta_q, T_v\cdot\nabla]u + [\Delta_q, T_{\nabla\cdot}\cdot v]u + [\Delta_q, R(v\cdot\nabla,.)]u, \tag{31}$$

where

$$\begin{aligned}
[\Delta_q, T_v\cdot\nabla]u &= \Delta_q(T_v\cdot\nabla u) - T_v\cdot\nabla\Delta_q u\\
[\Delta_q, T_{\nabla\cdot}\cdot v]u &= \Delta_q(T_{\nabla u}\cdot v) - T_{\nabla\Delta_q u}\cdot v\\
[\Delta_q, R(v\cdot\nabla,.)]u &= \Delta_q(R(v\cdot\nabla,u)) - R(v\cdot\nabla,\Delta_q u).
\end{aligned}$$

¿From the definition of the paraproduct and according to Bernstein inequalities

$$\begin{aligned}
\|[\Delta_q, T_{\nabla\cdot}\cdot v]u\|_{L^\infty} &\lesssim \sum_{|j-q|\le 4}\|S_{j-1}\nabla u\|_{L^\infty}\|\Delta_j v\|_{L^\infty}\\
&\lesssim \|u\|_{L^\infty}\|\omega\|_{L^\infty}, \qquad (32)
\end{aligned}$$

where we have used here the following equivalence: $\forall j\in\mathbb{N}$,

$$\|\Delta_j v\|_{L^\infty} \approx 2^{-j}\|\Delta_j\omega\|_{L^\infty}.$$

For the second term of the right-hand side of (31), we have

$$\begin{aligned}
[\Delta_q, T_v\cdot\nabla]u &= \sum_{j\ge 1}[\Delta_q, S_{j-1}v\cdot\nabla\Delta_j]u,\\
&= \sum_{|j-q|\le 4}[\Delta_q, S_{j-1}v\cdot\nabla]\Delta_j u.
\end{aligned}$$

To estimate each commutator, we write Δ_q as a convolution

$$[\Delta_q, S_{j-1}v\cdot\nabla]\Delta_j u(x) = 2^{qd}\int h(2^q(x-y))\big(S_{j-1}v(y) - S_{j-1}v(x)\big)\cdot\nabla\Delta_j u(y)dy.$$

Thus, Young and Bernstein inequalities yield, for $|j-q|\le 4$,

$$\begin{aligned}
\big\|[\Delta_q, S_{j-1}v\cdot\nabla]\Delta_j u\big\|_{L^\infty} &\lesssim 2^{-q}\|\nabla S_{j-1}v\|_{L^\infty}\|\Delta_j\nabla u\|_{L^\infty} \qquad (33)\\
&\lesssim \|\nabla S_{j-1}v\|_{L^\infty}\|u\|_{L^\infty}\\
&\lesssim \Big(\|\nabla\Delta_{-1}v\|_{L^\infty} + (q+2)\|\omega\|_{L^\infty}\Big)\|u\|_{L^\infty}.
\end{aligned}$$

Let us move to the remainder term. It can be written, in view of the definition, as

$$\mathrm{J}_q := [\Delta_q, R(v\cdot\nabla,.)]u = \sum_{\substack{j\ge q-4, j\ge 0\\ i\in\{\mp 1,0\}}}[\Delta_q, \Delta_j v]\cdot\nabla\Delta_{j+i}u + \sum_{i\in\{0,1\}}[\Delta_q, \Delta_{-1}v]\cdot\nabla\Delta_{-1+i}u.$$

It follows from the zero divergence condition that

$$\mathrm{J}_q = \sum_{i\in\{0,1\}}[\Delta_q, \Delta_{-1}v]\cdot\nabla\Delta_{-1+i}u + \sum_{\substack{j\ge q-4, j\ge 0\\ i\in\{\mp 1,0\}}}\mathrm{div}\,\big([\Delta_q, \Delta_j v]\otimes\Delta_{j+i}u\big) = \mathrm{J}_q^1 + \mathrm{J}_q^2.$$

By the same way as (33) one has

$$\begin{aligned}
\|\mathrm{J}_q^1\|_{L^2} &\lesssim 2^{-q}\|\nabla\Delta_{-1}v\|_{L^\infty}\sum_{i=0}^{1}\|\nabla\Delta_{-1+i}u\|_{L^\infty}\\
&\lesssim \|\nabla\Delta_{-1}v\|_{L^\infty}\|u\|_{L^\infty}.
\end{aligned}$$

To estimate the second term we use Bernstein inequality

$$\begin{aligned}\|\mathrm{J}_q^2\|_{L^2} &\lesssim \sum_{\substack{j\geq q-4, j\geq 0\\ i\in\{\mp 1,0\}}} 2^q \|\Delta_j v\|_{L^\infty}\|\Delta_{j+i}u\|_{L^\infty}\\ &\lesssim \|u\|_{L^\infty}\sum_{j\geq q-4} 2^{q-j}\|\Delta_j\omega\|_{L^\infty}\\ &\lesssim \|\omega\|_{L^\infty}\|u\|_{L^\infty},\end{aligned}$$

This completes the proof of Proposition 6.1. □

References

[1] H. Abidi, T. Hmidi, *On the global well-posedness for Boussinesq system*, J. Diff. Equa.,**233**, 1 (2007) 199-220.

[2] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. de l' Ecole Norm. Sup., **14** (1981) 209-246.

[3] J. T. Beale, T. Kato, A. Majda, *Remarks on the breakdown of smooth solutions for* 3*-D Euler equations*, Comm. Math. Phys **94** (1984) 61-66.

[4] J. R. Cannon, E. Dibenedetto, *The initial value problem for the Boussinesq equations with data in L^p*, in Approximation Methods for Navier-Stokes Problems, Lecture Notes in Math. **771**, Springer, Berlin 1980, 129-144.

[5] D. Chae, *Global regularity for the* 2*-D Boussinesq equations with partial viscous terms,* Advances in Math., **203**, 2 (2006) 497-513.

[6] J.-Y. Chemin, *Perfect incompressible Fluids*, Oxford University Press.

[7] J.-Y. Chemin, *Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel*, J. Anal. Math. **77** (1999) 27-50.

[8] R. Danchin, M. Paicu, *Le théorème de Leray et le théorème de Fujita-Kato pour le système de Boussinesq partiellement visqueux,* to appear in Bull. S. M. F.

[9] B. Guo, *Spectral method for solving two-dimensional Newoton-Boussinesq equation*, Acta Math. Appl. Sinica, **5** (1989) 201-218.

[10] T. Hmidi, *Régularité höldérienne des poches de tourbillon visqueuses,* J. Math. Pures Appl. (9) **84**, 11 (2005) 1455-1495.

[11] T. Hmidi, Poches de tourbillon singulières dans un fluide faiblement visqueux. Rev. Mat. Iberoamericana, **22**, 2 (2006) 489-543.

[12] T. Hmidi, S. Keraani, *On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity,* Adv. Diff. Equations, **12**, 4 (2007) 461-480.

[13] T. Hmidi, S. Keraani, *Incompressible viscous flows in borderline Besov spaces,* to appear in Arch. Ratio. Mech. Ana.

[14] T. Y. Hou, C. Li, *Global well-posedness of the viscous Boussinesq equations*, Discrete and Continuous Dynamical Systems, **12**, 1 (2005) 1-12.

[15] O. Sawada, Y. Taniuchi, *On the Boussinesq flow with nondecaying initial data,* Funkcial. Ekvac. **47**, 2 (2004) 225-250.

[16] M. Vishik, *Hydrodynamics in Besov Spaces*, Arch. Rational Mech. Anal **145**(1998) 197-214.