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Weyl asymptotics for non-self-adjoint operators with small multiplicative random perturbations

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Abstract

We study the Weyl asymptotics of the distribution of eigenvalues of non-self-adjoint (pseudo)differential operators with small random multiplicative perturbations in arbitrary dimension. We were led to quite essential improvements of many of the probabilistic aspects.

Résumé

Nous étudions l'asymptotique de Weyl de la distribution des valeurs propres d'opérateurs (pseudo-)différentiels avec des petites perturbations aléatoires multiplicatives en dimension quelconque. Nous avons été amenés à faire des améliorations essentielles des aspects probabilistes.

1 Introduction

Following works of E.B. Davies [2], M. Zworski [15] and others [3, 10] we know that quasimode constructions, going back to Hörmander (1960), can be used to show that non-self-adjoint pseudodifferential operators in general have wildly growing resolvents when the spectral parameter is in the interior of the range of the principal symbol. Correspondingly, the spectrum will in general be very unstable under small perturbations of the operator.

M. Hager [7] studied certain classes of semi-classical (pseudo)differential operators P on $\mathbf{R}$ with small random perturbations. She showed that with probability tending to 1 when $h \to 0$, the eigenvalues in the interior of the range of the leading symbol p distribute according to the Weyl law (well-known in the self-adjoint case, see [4] and further references there for the semi-classical case):

$$\#(\sigma(P_\delta) \cap W) = \frac{1}{2\pi h}(\mathrm{vol}\,(p^{-1}(W) + o(1)), \quad h \to 0, \tag{1.1}$$

for natural classes of domains W with smooth boundary, where $P_\delta = P + \delta Q_\omega$, and δQ_ω is the small random perturbation, $\sigma(P_\delta)$ is the spectrum of P_δ (here discrete) and $\#(\sigma(P_\delta) \cap W)$ is the number of eigenvalues in W, counted with their algebraic multiplicity. For the more concrete examples, she took $Q_\omega u(x) = q_\omega(x)u(x)$, where q_ω is a random linear combination of eigenfunctions of an auxiliary self-adjoint operator, with coefficients given by independent Gaussian random variables.

For example, if $P = \frac{1}{2}((hD_x)^2 + ix^2)$, we have $p(x,\xi) = \xi^2 + ix^2$, the range of p is equal to $[0,\infty[+i[0,\infty[$ while the spectrum of P is given by $\sigma(P) = \{e^{i\pi/4}(k+\frac{1}{2})h;\, k = 0,1,2,...\}$, so (1.1) does not hold as soon as the open set W intersects the range of p but not $e^{i\pi/4}[0,\infty[$. On the other hand the results of Hager give (1.1) with probability close to 1 for P_δ for domains $W \Subset]0,\infty[+i]0,\infty[$ with smooth boundary.

In the above example the symbol is even in ξ and some additional symmetry seems to be needed in the case of multiplicative perturbations, as can be seen from the simple example when $P = hD_x + g(x)$ on S^1 where g is smooth. Then the spectrum is contained in the line $\Im z = (2\pi)^{-1}\int_0^{2\pi} \Im g(x)dx$ while the range of $p(x,\xi) = \xi + g(x)$ is the band given by $\inf \Im g \le \Im z \le \sup \Im g$, so clearly we will not get the Weyl law for small multiplicative random perturbations of P, since such a perturbation will only displace slightly the line containing the spectrum.

W. Bordeaux Montrieux [1] adapted some of the results of Hager to the case of the compact manifold S^1 and extended them to the case of systems. He also considered elliptic differential operators on S^1 in the usual sense ($h = 1$) and showed with suitable symmetry assumptions on p and for a suitable class of random perturbations, that the Weyl law holds almost surely for the distribution of large eigenvalues in closed sectors, contained in the interior of the (union of the) range(s) of (the eigenvalues of) p.

In [8], we passed to higher dimensions and showed for quite general P on $\mathbf{R}^n$, that we still have (1.1) with probability tending to 1, when $h \to 0$. In this case it was conveneient to have random perturbations with Q_ω of the form

$$Q_\omega u(x) = S \sum_{j,k \in \mathbf{N}} \alpha_{j,k}(\omega)(Tu|f_k)e_j, \tag{1.2}$$

where S, T are elliptic h-pseudodifferential operators of Hilbert-Schmidt class, $\alpha_{j,k}(\omega)$ are independent $\mathcal{N}(0,1)$-laws and $e_0, e_1, ..., f_0, f_1, ...$ are orthonormal bases in $L^2(\mathbf{R}^n)$ (the choice of which does not affect the class of operators of the form (1.2)). Since δ is very small, the interpretation of this result is that we have Weyl asymptotics for "most" *h-pseudodifferential* operators. However, since our perturbations are not multiplicative, the same (rough) conclusion about *h-differential* operators with symmetry could not be made.

In this talk we shall describe the recent result from [13] that deals with multiplicative perturbations in any dimension. Several elements of [8] carry over to the multiplicative case, while the study of a certain effective Hamiltonian, here a finite random matrix, turned out to be more difficult. Because of that we were led to abandon the fairly explicit calculations with Gaussian random variables and instead resort to arguments from complex analysis. A basic difficulty was then to find at least one perturbation within the class of permissible ones, for which we have a lower bound on the determinant of the associated effective Hamiltonian. This is achieved via an iterative ("renormalization") procedure, with estimates on the singular values at each step. An advantage with the new approach is that we can treat much more general random perturbations.

2 The result

We first specify the assumptions about the unperturbed operator. Let $m \geq 1$ be an order function on $\mathbf{R}^{2n}$ in the sense that

$$m(\rho) \leq C_0 \langle \rho - \mu \rangle^{N_0} m(\mu),\ \ \rho, \mu \in \mathbf{R}^{2n}$$

for some fixed positive constants C_0, N_0, where we use the standard notation $\langle \rho \rangle = (1 + |\rho|^2)^{1/2}$.

Let

$$p \in S(m) := \{a \in C^\infty(\mathbf{R}^{2n});\ |\partial_\rho^\alpha a(\rho)| \leq C_\alpha m(\rho),\ \forall \rho \in \mathbf{R}^{2n}, \alpha \in \mathbf{N}^{2n}\}.$$

We assume that $p - z$ is elliptic (in the sense that $(p - z)^{-1} \in S(m^{-1})$) for at least one value $z \in \mathbf{C}$. Put $\Sigma = \overline{p(\mathbf{R}^{2n})} = p(\mathbf{R}^{2n}) \cup \Sigma_\infty$, where Σ_∞ is the set of accumulation values of $p(\rho)$ near $\rho = \infty$. Let $P(\rho) = P(\rho; h)$, $0 < h \leq h_0$ belong to $S(m)$ in the sense that $|\partial_\rho^\alpha P(\rho; h)| \leq C_\alpha m(\rho)$ as above, with contants that are independent of h. Assume that there exist $p_1, p_2, ... \in S(m)$ such that

$$P \sim p + hp_1 + ... \text{ in } S(m),\ \ h \to 0.$$

Let $\Omega \Subset \mathbf{C}$ be open, simply connected, with $\overline{\Omega} \cap \Sigma_\infty = \emptyset$, $\Omega \not\subset \Sigma$. Then for $h > 0$ small enough, the spectrum $\sigma(P)$ of P is discrete in Ω and constituted of eigenvalues of finite algebraic multiplicity ([7, 8]). We will also need the symmetry assumption,

$$P(x, -\xi; h) = P(x, \xi; h). \tag{2.1}$$

Let $V_z(t) := \mathrm{vol}\,(\{\rho \in \mathbf{R}^{2n};\ |p(\rho) - z|^2 \le t\})$. For $\kappa \in]0, 1]$, $z \in \Omega$, we consider the property that

$$V_z(t) = \mathcal{O}(t^\kappa),\ 0 \le t \ll 1. \tag{2.2}$$

Let $K \subset \mathbf{R}^n_x$ be a compact neighborhood of the x-space projection of $p^{-1}(\Omega)$. The random potential will be of the form

$$q_\omega(x) = \chi_0(x) \sum_{0<\mu_k\le L} \alpha_k(\omega)\epsilon_k(x),\ |\alpha|_{\mathbf{C}^D} \le R,\ R \asymp h^{-(sM-\kappa+3n/2)} \tag{2.3}$$

where ϵ_k is the orthonormal basis of eigenfunctions of $h^2\widetilde{R}$, and $\widetilde{R}$ is a positive elliptic h-independent 2nd order operator with smooth coefficients on a compact manifold of dimension n, containing an open set diffeomorphic to an open neighborhood of $\mathrm{supp}\,\chi_0$. Here $\chi_0 \in C_0^\infty(\mathbf{R}^n)$ is equal to 1 near K. μ_k^2 denote the corresponding eigenvalues, so that $h^2\widetilde{R}\epsilon_k = \mu_k^2\epsilon_k$. We choose $L = L(h)$ in the interval

$$h^{\frac{\kappa-3n}{s-\frac{n}{2}-\epsilon}} \ll L \le h^{-M}, \quad M = \mathrm{Const} \ge \frac{3n-\kappa}{s-\frac{n}{2}-\epsilon}, \tag{2.4}$$

for some $\epsilon \in]0, s - \frac{n}{2}[$, $s > \frac{n}{2}$, so by Weyl's law for the large eigenvalues of elliptic self-adjoint operators, the dimension D is of the order of magnitude $\mathcal{O}((L/h)^n)$. We introduce the small parameter $\delta = \tau_0 h^{N_1+n}$, $0 < \tau_0 \le \sqrt{h}$, where $N_1 := 2n - \kappa + (s + \frac{n}{2} + \epsilon)M$. The randomly perturbed operator is

$$P_\delta = P + \delta h^{N_1} q_\omega =: P + \delta Q_\omega. \tag{2.5}$$

The random variables $\alpha_j(\omega)$ will have a joint probability distribution

$$P(d\alpha) = C(h)e^{\Phi(\alpha;h)}L(d\alpha), \tag{2.6}$$

where for some $N_5 > 0$,

$$|\nabla_\alpha \Phi| = \mathcal{O}(h^{-N_5}), \tag{2.7}$$

and $L(d\alpha)$ is the Lebesgue measure. ($C(h)$ is the normalizing constant, assuring that the probability of $B_{\mathbf{C}^D}(0, R)$ is equal to 1.)

We also need the parameter

$$\epsilon_0(h) = (h^\kappa + h^n \ln \frac{1}{h})(\ln \frac{1}{\tau_0} + (\ln \frac{1}{h})^2) \tag{2.8}$$

and assume that $\tau_0 = \tau_0(h)$ is not too small, so that $\epsilon_0(h)$ is small. The main result of [13] is:

Theorem 2.1 *Under the assumptions above, let $\Gamma \Subset \Omega$ have smooth boundary, let $\kappa \in]0,1]$ be the parameter in (2.3), (2.4), (2.8) and assume that (2.2) holds uniformly for z in a neighborhood of $\partial\Gamma$. Then there exists a constant $C > 0$ such that for $C^{-1} \geq r > 0$, $\widetilde{\epsilon} \geq C\epsilon_0(h)$ we have with probability*

$$\geq 1 - \frac{C\epsilon_0(h)}{rh^{N_5+(n+s)M+\frac{7n}{2}-\kappa}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} \tag{2.9}$$

that:

$$\begin{aligned} &|\#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma))| \leq \\ &\frac{C}{h^n} \left(\frac{\widetilde{\epsilon}}{r} + C_{\widetilde{M}} (r^{\widetilde{M}} + \ln(\frac{1}{r}) \mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0,r)))) \right), \end{aligned} \tag{2.10}$$

for every $\widetilde{M} > 0$, where $C_{\widetilde{M}}$ depends on $\widetilde{M}$ but not on the other parameters. Here $\#(\sigma(P_\delta) \cap \Gamma)$ denotes the number of eigenvalues of P_δ in Γ, counted with their algebraic multiplicity.

When $\kappa > 1/2$ the second volume in (2.10) is $\mathcal{O}(r^{2\kappa-1})$ and choosing $\widetilde{M}$ sufficiently large, and r to be equal to a suitable power of $\widetilde{\epsilon}$, we see that the right hand side of (2.10) is $\mathcal{O}(\widetilde{\epsilon}^\alpha/h^n)$ for some $\alpha > 0$. We therefore have Weyl asymptotics in that case, provided that $\widetilde{\epsilon}$ is small. If we assume that $\tau_0 \geq \exp(-h^{-\kappa_0})$, $0 < \kappa_0 < \kappa$, then $\epsilon_0(h) = \mathcal{O}(h^{\kappa-\kappa_0} \ln \frac{1}{h})$ and it suffices to choose $\widetilde{\epsilon} = h^{\widetilde{\kappa}}$, $0 < \widetilde{\kappa} < \kappa - \kappa_0$. With these choices the lower bound (2.9) is $\geq 1 - Ch^{-N} \exp(-h^{\widetilde{\kappa}-(\kappa-\kappa_0)}(\ln \frac{1}{h})^{-1})$, which is very close to 1 in the limit of small h. When $0 < \kappa \leq 1/2$ it may still happen that the volume in (2.10) is r^β for some $\beta > 0$ and we get the same conclusion.

As in [8] we also have a result valid simultaneously for a family $\mathcal{C}$ of domains $\Gamma \subset \Omega$ satisfying the assumptions of Theorem 2.1 uniformly in the natural sense: With a probability

$$\geq 1 - \frac{\mathcal{O}(1)\epsilon_0(h)}{r^2 h^{N_5+(n+s)M+\frac{7n}{2}-\kappa}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}}, \tag{2.11}$$

the estimate (2.10) holds simultaneously for all $\Gamma \in \mathcal{C}$.

Remark 2.2 When $\widetilde{R}$ has real coefficients, we may assume that the eigenfunctions ϵ_j are real. Then Theorem 2.1 remains valid if we restrict q_ω to be real.

Example 2.3 Let $1 \le m_0(x)$ be an order function on $\mathbf{R}^n$ and let $V \in S(m_0)$ be a smooth potential which is elliptic in the sense that $|V(x)| \ge m_0(x)/C$ and assume that $-\pi + \epsilon_0 \le \arg(V(x)) \le \pi - \epsilon_0$ for some fixed $\epsilon_0 > 0$. Then it is easy to see that $p(x,\xi) := \xi^2 + V(x)$ is an elliptic element of $S(m)$, where $m(x,\xi)$ is the order function $m_0(x) + \xi^2$. Let $\Sigma_\infty(V)$ be the set of accumulation points of $V(x)$ at infinity and define $\Sigma(V) = \overline{V(\mathbf{R}^n)} = V(\mathbf{R}^n) \cup \Sigma_\infty(V)$. Then with Σ and Σ_∞ defined for p as above, we get $\Sigma = \Sigma(V) + [0, +\infty[$, $\Sigma_\infty = \Sigma_\infty(V) + [0, +\infty[$. Using the fact that $\partial_{\xi_1}^2 \Re p \ge 1/C$, we further see that if $\widetilde{K} \subset \mathbf{C}$ is compact and disjoint from Σ_∞, then (2.2) holds uniformly for $z \in \widetilde{K}$ with $\kappa = 1/4$. The non-self-adjoint Schrödinger operator $P := -h^2\Delta + V(x)$ has $P(x,\xi) = p(x,\xi)$ as its symbol and (2.1) is fulfilled. This means that Theorem 2.1 is applicable.

In the remainder of this text, we shall describe the ideas of the proof. The strategy is the same as in [8], but there are also some essential differences.

3 Some H^s properties

Let $H^s = \langle hD\rangle^{-s} L^2(\mathbf{R}^n)$ be the semi-classical Sobolev space of order s. For $s > \frac{n}{2}$, we have $\|u\|_{L^\infty} \le Ch^{-n/2}\|u\|_{H^s}$, $\|uv\|_{H^s} \le Ch^{-n/2}\|u\|_{H^s}\|v\|_{H^s}$, for all $u, v \in H^s$.

We can also define $H^s(\widetilde{\Omega})$ when $\widetilde{\Omega}$ is a compact n-dimensional manifold by using local coordinates and a partition of unity. Using essentially the functional calculus of R. Seeley [11], we get that $H^s(\widetilde{\Omega}) = (1+h^2\widetilde{R})^{-s/2}L^2(\widetilde{\Omega})$ where $\widetilde{R}$ is as in Section 2. It follows that for q_ω as in (2.3), we have for $s_1 \in \mathbf{R}$,

$$\|q_\omega\|_{H^{s_1}}^2 \asymp \sum_{0<\mu_k\le L} |\alpha_k|^2 \langle \mu_k\rangle^{2s_1}.$$

If $s_1 \ge 0$, we get, using that $|\alpha| \le R$: $\|q_\omega\|_{H^{s_1}} \le CRL^{s_1}$. In particular with our choices of parameters, we have for $s_1 = s > n/2$:

$$\|\delta h^{N_1} q_\omega\|_{H^s} \le C\tau_0 h^{\frac{7n}{2} - \kappa + (n+2\epsilon)M} \ll 1. \tag{3.1}$$

4 Grushin problems and strategy of [7, 8]

We construct $\widetilde{p} \in S(m)$, equal to p outside a compact set, such that if $\widetilde{P} = P + (\widetilde{p} - p)$, then $\widetilde{P} - z$ has a bounded inverse in $\mathrm{Op}\,(S(\frac{1}{m}))$ for every

$z \in \Omega$. The eigenvalues of P in Ω coincide with the zeros of the holomorphic function,

$$\Omega \ni z \mapsto \det(\widetilde{P}-z)^{-1}(P-z) = \det(1-(\widetilde{P}-z)^{-1}(\widetilde{P}-P)).$$

If we introduce a more general perturbed operator $P_\delta = P + \delta Q$, $Qu(x) = q(x)u(x)$, $\|q\|_{H^s} \le 1$, then for $\delta \ll h^{n/2}$, $h \ll 1$, $\widetilde{P}_\delta := \widetilde{P} + \delta Q$ has no spectrum in Ω and $(\widetilde{P}_\delta - z)^{-1}$ is bounded. The eigenvalues of P_δ in Ω are the zeros of

$$\Omega \ni z \mapsto \det(P_{\delta,z}),$$

where

$$P_{\delta,z} = (\widetilde{P}_\delta - z)^{-1}(P_\delta - z) = 1 - (\widetilde{P}_\delta - z)^{-1}(\widetilde{P}-P).$$

Put $P_z = P_{0,z}$.

If $\mathcal{H}$ is a complex separable Hilbert space and $A : \mathcal{H} \to \mathcal{H}$ a bounded operator, we let $s_1(A), s_2(A), \dots$ denote the singular values of A, i.e. the decreasing sequence of eigenvalues of $(A^*A)^{1/2}$ starting with $s_1(A) = \|A\|$ (and possibly continued by an infinite repetition of $\sup \sigma_{\mathrm{ess}}((A^*A)^{1/2})$ when there are only finitely many discrete eigenvalues above $\sigma_{\mathrm{ess}}((A^*A)^{1/2})$). When A is a Fredholm operator of index 0 we let $0 \le t_1(A) \le t_2(A) \le \dots$ denote the increasing sequence of discrete eigenvalues of $(A^*A)^{1/2}$ (possibly continued as an infinite repetition of $\inf \sigma_{\mathrm{ess}}((A^*A)^{1/2})$, when there are only finitely many such eigenvalues below the essential spectrum).

For $0 < \alpha \ll 1$, let $0 \le t_1(P_z) \le t_2(P_z) \le \dots \le t_N(P_z)$ be the singular values of P_z in $[0, \sqrt{\alpha}[$ and let $e_1, \dots, e_N$ be a corresponding orthonormal family of eigenfunctions of $P_z^*P_z$. The t_j^2 are also the eigenvalues of $P_zP_z^*$ in $[0, \alpha[$, and we can choose a corresponding orthonormal family of eigenfunctions $f_1, \dots, f_N$ so that $P_z e_j = t_j f_j$, $P_z^* f_j = t_j e_j$. Define $R_+ : L^2 \to \mathbf{C}^N$, $R_- : \mathbf{C}^N \to L^2$ by $R_+u(j) = (u|e_j)$, $R_-u_- = \sum_1^N u_-(j) f_j$. Then

$$\mathcal{P}_z = \begin{pmatrix} P_z & R_- \\ R_+ & 0 \end{pmatrix} : L^2 \times \mathbf{C}^N \to L^2 \times \mathbf{C}^N$$

is bijective and the inverse

$$\mathcal{E}^0(z) = \begin{pmatrix} E^0 & E_+^0 \\ E_-^0 & E_{-+}^0 \end{pmatrix}$$

is quite explicit: E^0 can be identified with $P_z^{-1} : (f_1, \dots, f_N)^\perp \to (e_1, \dots, e_N)^\perp$, $E_+^0 v_+ = \sum v_+(j) e_j$, $E_-^0 v(j) = (v|f_j)$, $E_{-+}^0 = \mathrm{diag}\,(t_j)$. In particular, $\|E^0\| \le 1/t_{N+1} \le 1/\sqrt{\alpha}$, $\|E_\pm^0\| \le 1$, $\|E_{-+}^0\| \le t_N$.

Now $P_{\delta,z} = P_z + \mathcal{O}(\|\delta Q\|) = P_z + \mathcal{O}(\delta h^{-n/2})$ and if the norm of the perturbation is $\ll \sqrt{\alpha}$, the perturbed Grushin matrix

$$\mathcal{P}_\delta = \begin{pmatrix} P_{z,\delta} & R_- \\ R_+ & 0 \end{pmatrix}$$

has a bounded inverse

$$\mathcal{E}^\delta(z) = \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix}$$

and we have Neumann series expansions for the different entries, in particular,

$$E_{-+}^\delta = E_{-+}^0 + \delta E_-^0 \widetilde{Q} E_+^0 + \delta^2 E_-^0 \widetilde{Q} E^0 \widetilde{Q} E_+^0 + ..., \tag{4.1}$$

where $\widetilde{Q} := ((\widetilde{P}_\delta - z)^{-1} - (\widetilde{P} - z)^{-1})(\widetilde{P} - P)$.

The strategy in [8] (close to the one of [7]) was the following:

- Step 1. Show that with probability close to 1, we have for all z in a neighborhood of Γ that

$$\ln |\det P_{\delta,z}| \le \frac{1}{(2\pi h)^n} (\int \ln |p_z(\rho)| d\rho + o(1)), \tag{4.2}$$

  where $p_z(\rho) = (\widetilde{p}(\rho) - z)^{-1}(p(\rho) - z)$ is the principal symbol of P_z.

- Step 2. Show that for each z in a neighborhood of Γ we have with probability close to one that

$$\ln |\det P_{\delta,z}| \ge \frac{1}{(2\pi h)^n} (\int \ln |p_z(\rho)| d\rho + o(1)). \tag{4.3}$$

  Here the probability is so close to one that we can take a finite set of z, of cardinal not growing too fast when $h \to 0$, and have (4.3) similtaneously for all z in that set, still with probability close to 1 for small h.

- Step 3. Apply results from [6, 7, 8] about counting zeros of holomorphoic functions with exponential growth near the boundary of Γ. Very roughly, these results say that if $u(z) = u(z; \widetilde{h})$ is holomorphic with respect to z in a neighborhood of $\overline{\Gamma}$ such that $|u(z)| \le \exp(\phi(z)/h)$ near the boundary of Γ and such that we have a reverse estimate $|u(z_j)| \ge \exp((\phi(z_j) - \text{"small"})/h)$ for a finite set of points, distributed "densely" along the boundary, then the number of zeros of u in Γ is equal to $(2\pi\widetilde{h})^{-1}(\iint_\Gamma \Delta\phi(z) d\Re z d\Im z + \text{"small"})$. This is applied with $\widetilde{h} = (2\pi h)^n$, $\phi(z) = \int \ln |p_z(\rho)| d\rho$.

The first step could be carried out along the lines of [9] with some sharpening in order to improve the remainder estimates.

The second step was more delicate. Using some calculations from [14] we first established under more general assumptions, that

$$\det P_{\delta,z} = \det \mathcal{P}_{\delta,z} \det E_{-+}^{\delta} \tag{4.4}$$

and showed by using some functional calculus that with $\alpha = h$:

$$\ln|\det \mathcal{P}_{\delta,z}| = \frac{1}{(2\pi h)^n}\Big(\int \ln|p_z(\rho)|d\rho + \text{“small”}\Big). \tag{4.5}$$

Hence the problem is reduced to showing that

$$\ln|\det E_{-+}^{\delta}| \geq -\frac{\text{“small”}}{(2\pi h)^n} \tag{4.6}$$

with a probability very close to 1. In [8] we did so by using the Gaussian nature of the random variables. Using (4.1) and especially the second term there, we showed that E_{-+}^{δ} is close to a random matrix with independent Gaussian entries, for which the probability of smallness of the determinant could be estimated. An essential feature is that the size N of E_{-+}^{δ} satisfies

$$N = \mathcal{O}(h^{\kappa-n}) \ll h^{-n}, \tag{4.7}$$

which is heuristically clear, since the volume of the set of ρ for which $|p_z(\rho)| \leq \sqrt{\alpha} = \sqrt{h}$, is $\mathcal{O}(h^{\kappa})$

In the case of multiplicative perturbations we follow the same strategy. The steps 1 and 3 work quite the same way and again we have (4.4), (4.5) so the problem is to get (4.6) with probability close to 1. Here an essential difference seems to appear since we were not able to approximate E_{-+}^{δ} by an easily understandable random matrix, even when assuming the α_j to be independent and Gaussian. We chose a different and less explicit method. The first step is to construct at least one admissible perturbation of the form (2.3) for which $|\det E_{-+}^{\delta}|$ is not too small, and the second step is to use arguments of complex analysis in the variables $\alpha = \alpha(\omega)$, to show that $|\det E_{-+}^{\delta}|$ is not too small for “most” α.

5 Construction of a special admissible perturbation

We can show that $t_j(P_{\delta,z}) \asymp t_j(E_{-+}^{\delta})$ for $1 \leq j \leq N(\alpha)$, so the problem is to construct a perturbation q_ω of the form (2.3) for which we have a nice lower bound on $\prod_1^N t_j(P_{\delta,z})$. Using simple inequalities between the singular values, the problem can further be reduced to that of finding a perturbation for which $\prod_1^N t_j(P_\delta - z)$ is not too small and this can be done using a Grushin problem for $P_\delta - z$ associated to the singular values $t_j(P-z)$ in $[0,\sqrt{\alpha}]$ whose number is roughly equal to $N(\alpha)$. We now have operator matrices,

$$\mathcal{P} = \begin{pmatrix} P-z & R_- \\ R_+ & 0 \end{pmatrix}, \quad \mathcal{P}_\delta = \begin{pmatrix} P_\delta - z & R_- \\ R_+ & 0 \end{pmatrix},$$

constructed in the same way as for P_z and $P_{\delta,z}$, and the problem is then to find P_δ so that the singular values of the new matrix E_{-+}^δ, appearing in the inverse of the second matrix above, is not too small. Again, we have the expansion (4.1), now with $\widetilde{Q}$ replaced by $-Q = -Q_\omega$, given by $Q_\omega u(x) = h^{N_1} q_\omega(x) u(x)$. As can be expected, the main problem is then to construct q so that we get a nice lower bound on the singular values of $M_q := E_-^0 Q E_+^0$, where $E_\pm^0$ now denote the operators that appear at the appropriate places in the inverse of $\mathcal{P}$. The matrix of M_q is given by

$$M_{q;j,k} = \int q(x) e_k(x) \overline{f_j(x)} dx,$$

where $e_1, ..., e_N$ and $f_1, ..., f_N$ are orthonormal systems of eigenfunctions of $(P-z)^*(P-z)$ and $(P-z)(P-z)^*$ respectively. The symmetry assumption (2.1) is equivalent to the statement that $P^* = \Gamma P \Gamma$ on the operator level, where $\Gamma u = \overline{u}$, and (up to the use of a unitary map) we may assume that $f_j = \overline{e_j}$, so

$$M_{q;j,k} = \int q(x) e_j(x) e_k(x) dx.$$

We start by looking for q as a sum of N Dirac masses. We have the following general result:

Lemma 5.1 *Let $\Omega \Subset \mathbf{R}^n$ be open and let $e_1, ..., e_N \in C(\overline{\Omega}) \cap L^2(\Omega)$. Let $L \subset \mathbf{C}^N$ be a linear subspace of dimension $M-1$, for some $M \in \{1, 2, ..., N\}$. Then there exists $x \in \overline{\Omega}$ such that*

$$\mathrm{dist}\,(\overrightarrow{e}(x), L)^2 \ge \frac{1}{\mathrm{vol}\,(\Omega)} \mathrm{tr}\,((1-\pi_L)\mathcal{E}_\Omega),$$

where $\mathcal{E}_\Omega = ((e_k|e_k)_{L^2(\Omega)})$ and π_L is the orthogonal projection $\mathbf{C}^N \to L$. Here, $\overrightarrow{e}(x) = (e_1(x), ..., e_N(x))^t$.

Proof A straight forward calculation gives,

$$\int_\Omega \mathrm{dist}\,(\overrightarrow{e}(x), L)^2 dx = \mathrm{tr}\,((1-\pi_L)\mathcal{E}_\Omega).$$

□

Continuing the general discussion, let $0 \le \epsilon_1 \le \epsilon_2 \le ... \le \epsilon_N$ denote the eigenvalues of $\mathcal{E}_\Omega$. Using the mini-max principle, we can show that

$$\inf_{\dim L=M-1} \operatorname{tr}((1-\pi_L)\mathcal{E}_\Omega) = \epsilon_1 + ... + \epsilon_{N-M+1} =: E_M.$$

Using this and the lemma we can choose successively $a_1, a_2, ..., e_N \in \overline{\Omega}$ such that

$$\begin{aligned} \|\overrightarrow{e}(a_1)\|^2 &\ge \frac{E_1}{\operatorname{vol}(\Omega)}, \\ \operatorname{dist}(\overrightarrow{e}(a_2), \mathbf{C}\overrightarrow{e}(a_1))^2 &\ge \frac{E_2}{\operatorname{vol}(\Omega)}, \\ &\ldots \\ \operatorname{dist}(\overrightarrow{e}(a_M), \mathbf{C}\overrightarrow{e}(a_1)\oplus ... \oplus \mathbf{C}\overrightarrow{e}(a_{M-1}))^2 &\ge \frac{E_M}{\operatorname{vol}(\Omega)}, \\ &\ldots \end{aligned}$$

Consider the $N \times N$ matrix E given by the columns $\overrightarrow{e}(a_1), ..., \overrightarrow{e}(a_N)$. Expressing these columns in the Gram-Schmidt orthonormalization of the basis $\overrightarrow{e}(a_1), ..., \overrightarrow{e}(a_N)$, we see that

$$|\det E| = |c_1 \cdot ... \cdot c_N|, \quad c_j = \operatorname{dist}(\overrightarrow{e}(a_j), \mathbf{C}\overrightarrow{e}(a_1)\oplus ... \oplus \mathbf{C}\overrightarrow{e}(a_{j-1})),$$

so

$$|\det E| \ge \frac{(E_1E_2...E_N)^{\frac{1}{2}}}{(\operatorname{vol}(\Omega))^{\frac{N}{2}}}.$$

Now, with $q = \sum_1^N \delta(x-a_j)$ we get

$$M_q = E^t \circ E,$$

so

$$\det M_q = (\det E)^2, \quad |\det M_q| \ge \frac{E_1 \cdot ... \cdot E_N}{\operatorname{vol}(\Omega)^N}.$$

When the e_j form an orthonormal system in $L^2(\Omega)$, we have $\mathcal{E}_\Omega = 1$ and $E_j = j$, $E_1 \cdot ... \cdot E_N = N!$. If $s_j = s_j(M_q)$, then using that $|\det M_q| = \prod_1^N s_j$, we get

$$s_k \ge s_1 \Big(\prod_1^N \big(\frac{E_j}{s_1 \operatorname{vol}(\Omega)}\big)\Big)^{\frac{1}{N-k+1}}.$$

Returning to P and more generally to P_δ we can now

- approximate δ-measures in H^{-s} with admissible potentials as in (2.3),

- establish nice H^s-properties for the e_j,
- show that $\mathcal{E}_\Omega = 1 + \mathcal{O}(h^\infty)$ if Ω is a neighborhood of the x-space projection of supp $(\widetilde{p} - p)$,

and obtain nice estimates on M_q:

Proposition 5.2 *We can find an admissible potential q as in (2.3) such that the matrix M_q satisfies,*

$$\|M_q\| \leq CNh^{-n}, \tag{5.1}$$

$$s_k(M_q) \geq \frac{(1+\mathcal{O}(h^\infty))}{C^{\frac{k-1}{N-k+1}}(\mathrm{vol}\,(\Omega))^{\frac{N}{N-k+1}}}\left(\frac{h^n}{N}\right)^{\frac{k-1}{N-k+1}}(N!)^{\frac{1}{N-k+1}} - C_\epsilon L^{-(s-\frac{n}{2}-\epsilon)}h^{-n}N. \tag{5.2}$$

Here the negative term in (5.2) corresponds to losses appearing when we approximate our delta potential by an admissible one. Using Stirling's formula, we get for $k \leq N/2$ that

$$s_k(M_q) \geq \frac{h^n}{C} - C_\epsilon L^{-(s-\frac{n}{2}-\epsilon)}\frac{N}{h^n}.$$

Here $N = \mathcal{O}(h^{\kappa-n})$ and by the choice of L, we get

$$s_k(M_q) \geq \frac{h^n}{C},\ 1 \leq k \leq \frac{N}{2}, \tag{5.3}$$

for a new constant $C > 0$.

Under the assumptions of Theorem 2.1, we fix $\theta \in]0, \frac{1}{4}[$. Consider the Grushin problem for P with $\sqrt{\alpha}$ replaced by a parameter $\tau_0 = \mathcal{O}(\sqrt{h})$, so that $t_N(P) < \tau_0 \leq t_{N+1}(P)$. If $N_2 = 2(N_1 + n) + \epsilon_0$ with $\epsilon_0 > 0$ fixed, we consider two cases:

- Case 1. $s_j(E_{-+}) \geq \tau_0 h^{N_2}$ for $1 \leq j \leq N - [(1-\theta)N]$. Then we keep P unchanged so that the "perturbation" is $P_\delta = P$, and replace N by $\widetilde{N} = [(1-\theta)N]$. Then, we get a new $\widetilde{\tau}_0 := t_{\widetilde{N}+1}(P) \geq \tau_0 h^{N_2}$.
- Case 2. $s_j(E_{-+}) < \tau_0 h^{N_2}$ for some $j \leq N - [(1-\theta)N]$. Then we replace P by $P_\delta = P + \frac{\delta h^{N_1}}{\widetilde{C}} q = P + \delta Q$ with q as above. Using the development (4.1), or rather its analogue with $\widetilde{Q}$ replaced by $-Q$, and well-known inequalities for the singular values of sums of operators ([5]), we get

$$\begin{aligned} s_\nu(E^\delta_{-+}) &\geq 8\tau_0 h^{N_2},\ 1 \leq \nu \leq N - [(1-\theta)N], \\ t_j(P_\delta) &\geq \tau_0 h^{N_2},\ [(1-\theta)N] + 1 \leq j \leq N. \end{aligned}$$

  Moreover, the perturbation is so small that it will not modify very much the $t_j(P)$ for $j > N$, since these values are already $\geq \tau_0$.

We repeat the procedure with (P, N, τ_0) replaced by $(P_\delta, [(1-\theta)N], \tau_0 h^{N_2})$, so in the next step we pose a Grushin problem for $P_\delta - z$ instead of $P - z$. Again, we consider two cases and add a new (smaller) perturbation in the second case. (Once N reaches a fixed bounded value, we decrease N by one unit during the last steps of the iteration.) We end up with a perturbation P_δ for which

$$t_\nu(P_\delta - z) \geq (1 - \mathcal{O}(h^{N_1+n}))\tau_0^{(k)}, \text{ for } N^{(k)} < \nu \leq N^{(k-1)},$$

where $\tau_0^{(k)} = \tau_0 h^{kN_2}$, and $N^{(k)} = [(1-\theta)N^{(k-1)}]$ as long as $N^{(k-1)} \gg 1$ and $N^{(k)} = N^{(k-1)} - 1$ at the end of the iteration until we reach 0. This can be used to prove:

Proposition 5.3 *There exists an admissible perturbation P_δ such that*

$$\ln|\det P_{\delta,z}| \geq \frac{1}{(2\pi h)^n}\left(\int \ln|p_z(\rho)|d\rho - \mathcal{O}((h^\kappa + h^n \ln\frac{1}{h})(\ln\frac{1}{\tau_0} + (\ln\frac{1}{h})^2))\right).$$

6 End of step 2

Still with z fixed in a small neighborhood of the boundary of Γ, we consider the holomorphic function

$$F(\alpha) = (\det P_{\delta,z})\exp\left(-\frac{1}{(2\pi h)^n}\iint \ln|p_z|dxd\xi\right), \tag{6.1}$$

Then we can establish the upper bound,

$$\ln|F(\alpha)| \leq \epsilon_0(h)h^{-n}, \ |\alpha| < 2R, \tag{6.2}$$

and for one particular value $\alpha = \alpha_0$ with $|\alpha_0| \leq \frac{1}{2}R$, we also have the lower bound,

$$\ln|F(\alpha_0)| \geq -\epsilon_0(h)h^{-n}, \tag{6.3}$$

where we put

$$\epsilon_0(h) = C\left((h^\kappa + h^n \ln\frac{1}{h})(\ln\frac{1}{\tau_0} + (\ln\frac{1}{h})^2)\right). \tag{6.4}$$

Let $\alpha^1 \in \mathbf{C}^D$ with $|\alpha^1| = R$ and consider the holomorphic function of one complex variable

$$f(w) = F(\alpha^0 + w\alpha^1). \tag{6.5}$$

We will mainly consider this function for w in the disc determined by the condition $|\alpha^0 + w\alpha^1| < R$:

$$D_{\alpha^0,\alpha^1} : \left| w + \left(\frac{\alpha^0}{R} | \frac{\alpha^1}{R}\right) \right|^2 < 1 - \left|\frac{\alpha^0}{R}\right|^2 + \left|\left(\frac{\alpha^0}{R} | \frac{\alpha^1}{R}\right)\right|^2 =: r_0^2, \tag{6.6}$$

whose radius is between $\frac{\sqrt{3}}{2}$ and 1.

From (6.2), (6.3) we get

$$\ln |f(0)| \geq -\epsilon_0(h)h^{-n}, \ \ln |f(w)| \leq \epsilon_0(h)h^{-n}. \tag{6.7}$$

By (6.2), we may assume that the last estimate holds in a larger disc, say $D(-(\frac{\alpha^0}{R}|\frac{\alpha^1}{R}), 2r_0)$. Let $w_1, ..., w_M$ be the zeros of f in $D(-(\frac{\alpha^0}{R}|\frac{\alpha^1}{R}), 3r_0/2)$. Then it is standard to get the factorization

$$f(w) = e^{g(w)} \prod_1^M (w - w_j), \ w \in D(-(\frac{\alpha^0}{R}|\frac{\alpha^1}{R}), 4r_0/3), \tag{6.8}$$

together with the bounds

$$|g(w)| \leq \mathcal{O}(\epsilon_0(h)h^{-n}), \ M = \mathcal{O}(\epsilon_0(h)h^{-n}). \tag{6.9}$$

See for instance Section 5 in [12] where further references are also given.

For $0 < \epsilon \ll 1$, put

$$\Omega(\epsilon) = \{r \in [0, 1[; \exists w \in D_{\alpha^0,\alpha^1} \text{ such that } |w| = r \text{ and } |f(w)| < \epsilon\}. \tag{6.10}$$

If $r \in \Omega(\epsilon)$ and w is a corresponding point in D_{α^0,α^1}, we have with $r_j = |w_j|$,

$$\prod_1^M |r - r_j| \leq \prod_1^M |w - w_j| \leq \epsilon \exp(\mathcal{O}(\epsilon_0(h)h^{-n})). \tag{6.11}$$

Then at least one of the factors $|r-r_j|$ is bounded by $(\epsilon e^{\mathcal{O}(\epsilon_0(h)h^{-n})})^{1/M}$. In particular, the Lebesgue measure $\lambda(\Omega(\epsilon))$ of $\Omega(\epsilon)$ is bounded by $2M(\epsilon e^{\mathcal{O}(\epsilon_0(h)h^{-n})})^{1/M}$. Noticing that the last bound increases with M when the last member of (6.11) is ≤ 1, we get

Proposition 6.1 *Let $\alpha^1 \in \mathbf{C}^D$ with $|\alpha^1| = R$ and assume that $\epsilon > 0$ is small enough so that the last member of (6.11) is ≤ 1. Then*

$$\lambda(\{r \in [0, 1]; \ |\alpha^0 + r\alpha^1| < R, \ |F(\alpha^0 + r\alpha^1)| < \epsilon\}) \leq \frac{\epsilon_0(h)}{h^n} \exp(\mathcal{O}(1) + \frac{h^n}{\mathcal{O}(1)\epsilon_0(h)} \ln \epsilon). \tag{6.12}$$

Here, the symbol $\mathcal{O}(1)$ in the denominator indicates a bounded positive quantity.

Writing $\alpha = \alpha^0 + Rr\alpha^1$, $0 \le r < r(\alpha^1)$, $\alpha^1 \in S^{2D-1}$, we get

$$P(d\alpha) = \widetilde{C}(h)e^{\phi(r)}r^{2D-1}drS(d\alpha^1), \tag{6.13}$$

where $\phi(r) = \phi_{\alpha^0,\alpha^1}(r) = \Phi(\alpha^0 + rR\alpha^1)$ so that $\phi'(r) = \mathcal{O}(h^{-N_6})$, $N_6 = N_5 + N_4$. Here $\frac{\sqrt{3}}{2} \le r(\alpha^1) \le 1$, $R = \mathcal{O}(h^{-N_4})$, $N_4 = sM - \kappa + \frac{3n}{2}$, where M is the constant in (2.4). $S(d\alpha^1)$ denotes the Lebesgue measure on S^{2D-1}.

Comparing this measure with the Lebesgue measure in r, we get an estimate similar to (6.12) for the normalized radial part of the measure (6.13) and after integration with respect to α^1, we get

Proposition 6.2 *Let $\epsilon > 0$ be small enough for the right hand side of (6.11) to be ≤ 1. Then*

$$P(|F(\alpha)| < \epsilon) \le \mathcal{O}(1)h^{-N_7}\frac{\epsilon_0(h)}{h^n}\exp(\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)}\ln\epsilon), \tag{6.14}$$

where $N_7 = N_6 + n(M+1)$.

This concludes the step 2 in the procedure outlined in Section 4, and we can perform the step 3 as in [8].

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