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ENTROPY AND LOCALIZATION OF EIGENFUNCTIONS

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1. Motivations

The theory of quantum chaos tries to understand how the chaotic behaviour of a classical Hamiltonian system is reflected in its quantum counterpart. For instance, let M be a compact Riemannian C^∞ manifold, with negative sectional curvatures. The geodesic flow has the Anosov property, which is considered as the ideal chaotic behaviour in the theory of dynamical systems. The corresponding quantum dynamics is the unitary flow generated by the Laplace-Beltrami operator on $L^2(M)$. One expects that the chaotic properties of the geodesic flow influence the spectral theory of the Laplacian, for large eigenvalues. The Random Matrix conjecture [5] asserts that the large eigenvalues should, after proper renormalization, statistically resemble those of a large random matrix, at least for a generic Anosov metric. The Quantum Unique Ergodicity conjecture [21] (see also [4, 24]) deals with the corresponding eigenfunctions ψ: it claims that the probability density $|\psi(x)|^2dx$ should approach (in a weak sense) the Riemannian volume, when the eigenvalue tends to infinity. In fact a stronger property should hold for the *Wigner transform* W_ψ, a distribution on the cotangent bundle T^*M which describes the distribution of the wave function ψ on the classical phase space T^*M (position and momentum).

To describe the problem in a more precise way, we will adopt a semiclassical point of view, that is, consider the eigenstates of eigenvalue unity of the semiclassical Laplacian $-\hbar^2\triangle$, in the semiclassical limit $\hbar \to 0$. We denote by $(\psi_k)_{k\in\mathbb{N}}$ an orthonormal basis of $L^2(M)$ made of eigenfunctions of the Laplacian, and by $(-\frac{1}{\hbar_k^2})_{k\in\mathbb{N}}$ the corresponding eigenvalues:

$$-\hbar_k^2\,\triangle\psi_k = \psi_k, \quad \text{with} \quad \hbar_{k+1} \le \hbar_k\,. \tag{1.1}$$

We are interested in the high-energy eigenfunctions of $-\triangle$, in other words the semiclassical limit $\hbar_k \to 0$.

The Wigner distribution associated to an eigenfunction ψ_k is defined by

$$W_k(a) = \langle \mathrm{Op}_{\hbar_k}(a)\psi_k, \psi_k\rangle_{L^2(M)}, \qquad a \in C_c^\infty(T^*M)\,.$$

Here $\mathrm{Op}_{\hbar_k}$ is a quantization procedure, set at the scale $\hbar_k$, which associates a bounded operator on $L^2(M)$ to any smooth phase space function a with nice behaviour at infinity. See for instance [10] or [11] for various definitions of Op on $\mathbb{R}^d$. On a manifold, we just use local coordinates to define Op in a finite system of charts, then glue the objects defined locally thanks to a smooth partition of unity. If a is a function on the manifold M, $\mathrm{Op}_\hbar(a)$ is the multiplication by a, and thus we have $W_k(a) = \int_M a(x)|\psi_k(x)|^2dx$: the distribution W_k is a *microlocal lift* of the probability measure $|\psi_k(x)|^2dx$ into a phase space distribution. Although the definition of W_k depends on a certain number of choices, like the choice of local coordinates, or of the quantization procedure (Weyl, anti-Wick, "right" or "left" quantization...), its asymptotic behaviour when $\hbar_k \longrightarrow 0$ does not. Accordingly, we call *semiclassical measures* the limit points of the sequence $(W_k)_{k\in\mathbb{N}}$, in the distribution topology.

In the semiclassical limit $\hbar \longrightarrow 0$, "quantum mechanics converges to classical mechanics". We will denote $|\cdot|_x$ the norm on T_x^*M given by the metric. The geodesic flow $(g^t)_{t\in\mathbb{R}}$ is the Hamiltonian flow on T^*M generated by the Hamiltonian $H(x,\xi) = \frac{|\xi|_x^2}{2}$. The corresponding quantum operator is $-\frac{\hbar^2\triangle}{2}$, which generates the unitary flow $(U_\hbar^t) = (\exp(it\hbar\frac{\triangle}{2}))$ acting on $L^2(M)$. The convergence of $(U_\hbar^t)$ to (g^t) when $\hbar \longrightarrow 0$ is expressed in the Egorov Theorem :

Theorem 1.1. *Let $a \in C_c^\infty(T^*M)$. Then, for any given t in $\mathbb{R}$,*

$$\|U_\hbar^{-t}\,\mathrm{Op}_\hbar(a)U_\hbar^t - \mathrm{Op}_\hbar(a\circ g^t)\|_{L^2(M)} = \mathcal{O}(\hbar). \tag{1.2}$$

The remainder term depends on t, and this is a notorious source of problems when one wants to use semiclassical methods to study the large time behaviour of $(U_\hbar^t)$.

Using (1.2) and other standard semiclassical arguments, one shows the following :

Proposition 1.2. *Any semiclassical measure is a probability measure carried on the energy layer $\mathcal{E} = H^{-1}(\frac{1}{2})$ (which coincides with the unit cotangent bundle $\mathcal{E} = S^*M$). This measure is invariant under the geodesic flow.*

If the geodesic flow has the Anosov property — for instance if M has negative sectional curvature — then there exist many invariant probability measures on $\mathcal{E}$, in addition to the Liouville measure. The geodesic flow has countably many periodic orbits, each of them carrying an invariant probability measure. There are still many others, like the equilibrium states obtained by variational principles [14].

For manifolds with an ergodic geodesic flow (with respect to the Liouville measure), it has been known for some time that *almost all* eigenfunctions become uniformly distributed over $\mathcal{E}$, in the semiclassical limit. This property is dubbed as Quantum Ergodicity :

Theorem 1.3. [22, 26, 8] *Let M be a compact Riemannian manifold, assume that the action of the geodesic flow on $\mathcal{E} = S^*M$ is ergodic with respect to the Liouville measure. Let $(\psi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of the Laplacian* (1.1)*, and let (W_k) be the associated Wigner distributions on T^*M.*

Then, there exists a subset $\mathcal{S} \subset \mathbb{N}$ of density 1, such that

$$W_k \underset{k\longrightarrow\infty, k\in\mathcal{S}}{\longrightarrow} \text{Liouville}.$$

The question of knowing, in particular cases, if there can exist "exceptional" subsequences with a different behaviour is widely open. On a negatively curved manifold, the geodesic flow satisfies the ergodicity assumption, and in fact much stronger properties : mixing, K–property,... In this case, the Quantum Unique Ergodicity conjecture [21] expresses the belief that there exists a unique semiclassical measure, namely the Liouville measure on $\mathcal{E}$: $W_k \underset{k\longrightarrow\infty}{\longrightarrow}$ Liouville. In other words, in the semiclassical régime all eigenfunctions should become uniformly distributed over $\mathcal{E}$.

So far the most precise results on this question were obtained for manifolds M with constant negative curvature and *arithmetic* properties: see Rudnick–Sarnak [21], Wolpert [25]. In that very particular situation, there exists a countable commutative family of self–adjoint operators commuting with the Laplacian : the Hecke operators. One may thus decide to restrict the attention to common bases of eigenfunctions, often called "arithmetic" eigenstates, or Hecke eigenstates. A few years ago, Lindenstrauss [19] proved that the arithmetic eigenstates become asymptotically equidistributed (Arithmetic Quantum Unique Ergodicity). If there is some degeneracy in the spectrum of the Laplacian, note that it could be possible that the Quantum Unique Ergodicity conjectured by Rudnick and Sarnak holds for one orthonormal basis but not for another. In the arithmetic case, it is believed that the spectrum of the Laplacian has bounded multiplicity, in which case it would be a harmless assumption to consider only Hecke eigenstates.

Nevertheless, one may be less optimistic about the general conjecture. Faure–Nonnenmacher–De Bièvre exhibited in [12] a simple example of a symplectic Anosov dynamical system, namely the action of the linear hyperbolic automorphism $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on the 2-torus, the Weyl–quantization of which does not satisfy the Quantum Unique Ergodicity conjecture. In this model, it is known [17] that there is one orthonormal family of eigenfunctions satisfying Quantum Unique Ergodicity, but, due to high degeneracies in the spectrum, one can also construct eigenfunctions with a different behaviour. Precisely, they construct a family of eigenstates for which the semiclassical measure consists in two ergodic components: half of it is the Liouville measure, while the other half is a Dirac peak on a single unstable periodic orbit. It was also shown that this half-localization on a periodic orbit is *maximal* for this model [13] : a semiclassical measure cannot have more than half the mass carried by a finite union of closed orbits. Another type of semiclassical measure was recently obtained by Kelmer for a quantized automorphism on a higher-dimensional torus [15]: it consists in the Lebesgue measure on some invariant co-isotropic subspace of the torus. For these torus automorphisms, the existence of exceptional eigenstates is due to some nongeneric algebraic properties of the classical and quantized systems.

2. Main result.

We wish to study certain aspects of the problem by considering the *Kolmogorov–Sinai* entropy of semiclassical measures. We work on a compact manifold M of arbitrary dimension, and only assume that the geodesic flow has the Anosov property. In fact, our method is very general, and can without doubt be adapted to more general Anosov Hamiltonian systems.

The Kolmogorov–Sinai entropy, also called metric entropy, of a (g^t)-invariant probability measure μ is a nonnegative number $h_{KS}(\mu)$ that describes, in some sense, the complexity of a μ-typical orbit of the flow. The precise definition will be given later, but for the moment let us just give a few facts. A measure carried on a closed geodesic has zero entropy. In constant curvature, the entropy is known to be maximal for the Liouville measure. More generally, an upper bound on the entropy is given by the Ruelle inequality:

since the geodesic flow has the Anosov property, the energy layer $\mathcal{E}$ is foliated into unstable manifolds of the flow, and for any invariant probability measure μ one has

$$h_{KS}(\mu) \leq \left| \int_{\mathcal{E}} \log J^u(\rho) d\mu(\rho) \right| . \tag{2.1}$$

In this inequality, $J^u(\rho)$ is the *unstable Jacobian* of the flow at the point $\rho \in \mathcal{E}$, defined as the Jacobian of the map g^{-1} restricted to the unstable manifold at the point $g^1\rho$ (the average of $\log J^u$ over any invariant measure is negative). If M has dimension d and has constant sectional curvature -1, this inequality just reads $h_{KS}(\mu) \leq d-1$. The equality holds in (2.1) if and only if μ is the Liouville measure on $\mathcal{E}$ [18].

Let μ be a (g^t)–invariant probability measure on $\mathcal{E}$. The Birkhoff ergodic theorem says that, for μ–almost every $\rho \in \mathcal{E}$, the weak limit

$$\mu^\rho = \lim_{|t| \longrightarrow \infty} \frac{1}{t} \int_0^t \delta_{g^s\rho} ds$$

exists, and is an ergodic probability measure. We can then write

$$\mu = \int_{\mathcal{E}} \mu^\rho d\mu(\rho),$$

which is called the ergodic decomposition of μ. Note that the ergodic probability measures are the extremal points of the compact convex set of (g^t)–invariant probability measures.

To understand the connection of our results with the previous discussion, it is important to know that the entropy if an *affine* functional on the convex set of (g^t)–invariant probability measures. In fact, we have

$$h_{KS}(\mu) = \int_{\mathcal{E}} h_{KS}(\mu^\rho) d\mu(\rho).$$

Introduce the positive real number

$$\Lambda = -\sup_\mu \int \log J^u(\rho) d\mu(\rho) = \inf_\gamma \sum_{i=1}^{d-1} \lambda_i^+(\gamma).$$

The first sup runs over the set of (g^t)–invariant probability measures μ on $\mathcal{E}$. The second inf runs over the set of closed geodesics, and the λ_i^+ denote the positive Lyapunov exponents, which are, for a closed orbit, the logarithms of the eigenvalues of the Poincaré map. The identity between these two expressions of Λ comes from

— the density of invariant measures carried by closed orbits, in the set of invariant probability measures;

— the simple remark that, on a closed orbit, integrating $-\log J^u$ is the same as evaluating $\sum \lambda_i^+$.

For instance, for a d-dimensional manifold of constant sectional curvature -1, we find $\Lambda = d-1$.

In the whole article, we consider a certain subsequence of eigenstates $(\psi_{k_j})_{j\in\mathbb{N}}$ of the Laplacian, such that the corresponding sequence of Wigner functions (W_{k_j}) converges to a certain semiclassical measure μ (see the discussion preceding Proposition 1.2). The subsequence (ψ_{k_j}) will simply be denoted by $(\psi_\hbar)_{\hbar\to 0}$, using the slightly abusive notation $\psi_\hbar = \psi_{\hbar_{k_j}}$ for the eigenstate ψ_{k_j}. Each state $\psi_\hbar$ satisfies

$$(-\hbar^2 \triangle -1)\psi_\hbar = 0\,. \tag{2.2}$$

Theorem 2.1. [1] *We find a number $\kappa > 0$, and two continuous decreasing functions $\tau : [0,1] \longrightarrow [0,1]$, $\vartheta : (0,1] \longrightarrow \mathbb{R}_+$ with $\tau(0) = 1$, $\vartheta(0) = +\infty$, such that:*
If μ is a semi-classical invariant measure, and

$$\mu = \int_{S^1M} \mu^\rho d\mu(\rho)$$

is its decomposition in ergodic components, then, for all $\delta > 0$,

$$\mu\left(\{\rho, h_{KS}(\mu^\rho) \geq \frac{\Lambda}{2}(1-\delta)\}\right) \geq \left(\frac{\kappa}{\vartheta(\delta)}\right)^2 (1-\tau(\delta)).$$

Corollary 1. This implies that $h_{KS}(\mu) > 0$, and gives a lower bound for the topological entropy of the support, $h_{top}(supp\ \mu) \geq \frac{\Lambda}{2}$.

In the case of constant sectional curvature -1, this last statement can be rephrased in terms of the Hausdorff dimension : $\dim(supp\ \mu) \geq d$.

Theorem 2.1 is compatible with the kind of counter–examples found by Faure–Nonnenmacher–De Bièvre [12]. It allows certain ergodic components of μ to be carried by closed geodesics, but says there also have to be components of positive entropy. This stands in contrast with the much stronger result obtained in the arithmetic case by Bourgain and Lindenstrauss :

Theorem 2.2. [6] *Let M be a congruence arithmetic surface, and (ψ_j) an orthonormal basis of eigenfunctions for the laplacian and the Hecke operators.*

Let μ be a corresponding semiclassical measure, with ergodic decomposition $\mu = \int_{S^*X} \mu^\rho d\mu(\rho)$, then for almost all ergodic components we have $h_{KS}(\mu^\rho) \geq \frac{1}{9}$.

What we prove in [1] is in fact a more general result about quasi-modes of order $\hbar|\log\hbar|^{-1}$:

Theorem 2.3. [1] *We find a number $\kappa > 0$, and two continuous decreasing functions $\tau : [0,1] \longrightarrow [0,1]$, $\vartheta : (0,1] \longrightarrow \mathbb{R}_+$ with $\tau(0) = 1$, $\vartheta(0) = +\infty$, such that:*
If $(\psi_\hbar)$ is a sequence of normalized L^2 functions with

$$\|(-\hbar^2 \triangle -1)\psi_\hbar\|_{L^2(M)} \leq c\hbar|\log\hbar|^{-1},$$

then for any semi-classical invariant measure μ associated to $(\psi_\hbar)$, for any $\delta > 0$,

$$\mu\left(\{\rho, h_{KS}(\mu^\rho) \geq \frac{\Lambda}{2}(1-\delta)\}\right) \geq (1-\tau(\delta))\left(\frac{\kappa}{\vartheta(\delta)} - c\vartheta(\delta)\right)_+^2 - c\kappa.$$

If c is small enough, this implies that μ has positive entropy.

If c is too large, it is rather easy to construct quasimodes of order $c\hbar|\log\hbar|^{-1}$ whose semiclassical measures are carried on a closed geodesic.

In the theorem, all quantities κ, τ, ϑ can, in principle, be expressed explicitly in terms of certain dynamical quantities (Lyapunov exponents,...), but not in a satisfactory way. This is why the following theorem, proved in [3], is in many respects more pleasant.

Theorem 2.4. [3] *Let μ be a semiclassical measure associated to the eigenfunctions of the Laplacian on M. Then its metric entropy satisfies*

$$h_{KS}(\mu) \geq \frac{3}{2}\left|\int_{\mathcal{E}} \log J^u(\rho) d\mu(\rho)\right| - (d-1)\lambda_{\max}\,, \tag{2.3}$$

where $d = \dim M$ and $\lambda_{\max} = \lim_{t\to\pm\infty} \frac{1}{t} \log \sup_{\rho\in\mathcal{E}} |dg^t_\rho|$ is the maximal expansion rate of the geodesic flow on $\mathcal{E}$.

In particular, if M has constant sectional curvature -1, this means that

$$h_{KS}(\mu) \geq \frac{d-1}{2}. \tag{2.4}$$

The bound (2.4) in the above theorem is much sharper than Theorem 2.1 in the case of constant curvature. On the other hand, if the curvature varies a lot (still being negative everywhere), the right hand side of (2.3) may actually be negative, in which case the above bound is trivial. We believe this to be but a technical shortcoming of our method, and would actually expect the following bound to hold:

$$h_{KS}(\mu) \geq \frac{1}{2}\left|\int_{\mathcal{E}} \log J^u(\rho) d\mu(\rho)\right|\,. \tag{2.5}$$

Quantum Unique Ergodicity would mean that $h_{KS}(\mu) = \left|\int_{\mathcal{E}} \log J^u(\rho)\, d\mu(\rho)\right|$ [18]. We believe however that (2.5) is the optimal result that can be obtained without using more precise information, like for instance upper bounds on the multiplicities of eigenvalues. Indeed, in the above mentioned examples of Anosov systems where the Quantum Unique Ergodicity conjecture is wrong, the bound (2.5) is actually *sharp* [12, 15, 2]. In those examples, the spectrum has very high degeneracies, which allows for much freedom to select the eigenstates, and could be responsible for the failure of Quantum Unique Ergodicity. Such high degeneracies are not expected to happen in the case of the Laplacian on a negatively curved manifold. For the moment, however, there is no clear understanding of the precise relation between spectral degeneracies and failure of Quantum Unique Ergodicity.

3. Outline of the proof

3.1. **Definition of entropy, and main idea of the proof.** Let μ be a probability measure on $\mathcal{E}$. Let $(P_1, \ldots, P_K)$ be a finite measurable partition of the unit tangent bundle : $\mathcal{E} = P_1 \sqcup \ldots \sqcup P_K$. The entropy of μ with respect to the partition P is

$$h_P(\mu) = -\sum_{k=1}^{K} \mu(P_k) \log \mu(P_k).$$

Assume now that μ is (g^t)–invariant. For any integer n, denote $P^{\vee n}$ the partition formed of the sets $P_{\alpha_0} \cap g^{-1}P_{\alpha_1} ... \cap g^{-n}P_{\alpha_n}$. Denote

$$h_n(\mu, P) = h_{P^{\vee n}}(\mu) = -\sum_{(\alpha_j)\in\{1,\ldots,K\}^{\{0,\ldots,n\}}} \mu(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} ... \cap g^{-n}P_{\alpha_n}) \log \mu(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} ... \cap g^{-n}P_{\alpha_n}).$$

If μ is (g^t)–invariant, it follows from the convexity of the logarithm that

$$h_{n+m}(\mu, P) \leq h_n(\mu, P) + h_{m-1}(\mu, P), \tag{3.1}$$

in other words the sequence $(h_{n+1}(\mu, P))_{n\in\mathbb{N}}$ is subadditive. The entropy of μ with respect to the action of geodesic flow and to the partition P is defined by

$$h_{KS}(\mu, P) = \lim_{n\longrightarrow+\infty} -\frac{1}{n}h_n(\mu, P) = \inf_{n\in\mathbb{N}} -\frac{1}{n}h_n(\mu, P). \tag{3.2}$$

Note that $\mu(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} ... \cap g^{-n}P_{\alpha_n})$ measures the μ–probability to visit successively P_{α_0}, P_{α_1},..., P_{α_n} at times 1, 2,..., n of the geodesic flow. The entropy tries to measure the exponential decay of these probabilities when n gets large : it is easy to see that $h_{KS}(\mu, P) \geq \beta$ if there exists C such that $\mu(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} ... \cap g^{-n}P_{\alpha_n}) \leq Ce^{-\beta n}$, for all n and all $\alpha_0, \ldots, \alpha_n$.

The entropy of μ with respect to the action of the geodesic flow is defined as

$$h_{KS}(\mu) = \sup_P h_{KS}(\mu, P),$$

the supremum running over all finite measurable partitions P. For Anosov systems, this supremum is reached for a well-chosen partition P : in fact, as soon as the diameter of the P_is is small enough. Besides, we may restrict our attention to partitions P which are actually partitions of the manifold M (lifted to $\mathcal{E}$). This will simplify certain aspects of the analysis.

The existence of the limit in (3.2), and the fact that it coincides with the inf follow from a standard subadditivity argument. It has a crucial consequence : if (μ_k) is a sequence of (g^t)–invariant probability measures converging weakly to μ, then

$$h_{KS}(\mu, P) \geq \limsup_k h_{KS}(\mu_k, P)$$

(provided μ does not charge the boundary of P, which is a harmless assumption on P once μ is fixed). In particular : if (μ_k) converges weakly to μ, and if we have an estimate

$$\mu_k(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} ... \cap g^{-n}P_{\alpha_n}) \leq C_k e^{-\beta n}$$

where C_k may depend on k, but not β, we have $h_{KS}(\mu_k) \geq \beta$ for all k, and this estimate goes to the limit to yield $h_{KS}(\mu) \geq \beta$.

Since our semiclassical measure μ is defined as a limit of Wigner measures W_k, a naive idea would be to estimate from below the entropy of W_k and then pass to the limit. This idea cannot work so easily, since the W_k are not (g^t)–invariant probability measures. They are not positive measures, and they are not (g^t)–invariant. One can adjust the definition of Op to have one of these properties, but never both. However, we proved in [1] the following estimate :

Theorem 3.1. (The main estimate, [1]) *For every $\mathcal{K} > 0$, there exists $\hbar_{\mathcal{K}} > 0$ such that, uniformly for all $\hbar < \hbar_{\mathcal{K}}$, for all $n \leq \mathcal{K}|\log \hbar|$, for all $\alpha_0, ..., \alpha_n$,*

$$\left| \langle \mathbb{1}^{sm}_{P_{\alpha_n}}(n) \mathbb{1}^{sm}_{P_{\alpha_{n-1}}}(n-1) ... \mathbb{1}^{sm}_{P_{\alpha_0}} \psi_\hbar, \psi_\hbar \rangle \right| \leq (2\pi\hbar)^{-d/2} e^{-\frac{\Lambda}{2}n} (1 + \mathcal{O}(\mathrm{diam}(P)))^n.$$

In the theorem,

— $\mathbb{1}_{P_\alpha}$ is the characteristic function of P_α (recall that we restricted our attention to partitions of M); $\mathbb{1}^{sm}_\alpha$ means that we smooth this function by convolution to get a C^∞–function of M, keeping the property $\sum_\alpha \mathbb{1}^{sm2}_{P_\alpha} \equiv 1$.

— we see $\mathbb{1}^{sm}_\alpha$ as a multiplication operator on $L^2(M)$, and denote $\mathbb{1}^{sm}_\alpha(t)$ its evolution under the Schrödinger flow. That is, $\mathbb{1}^{sm}_\alpha(t) = \exp(-it\hbar\frac{\triangle}{2})\mathbb{1}^{sm}_\alpha \exp(it\hbar\frac{\triangle}{2})$.

— $\mathrm{diam}(P)$ denotes an upper bound on the diameter of the P_k, it can be made arbitrarily small.

The estimate holds for any family of wave functions $(\psi_\hbar)$ which are $\hbar$–microlocalized near the energy layer $\mathcal{E}$, and, in particular, for the solutions of $(-\hbar^2 \triangle -1)\psi_\hbar = 0$. It is actually a result about the kernel of the operator $\mathbb{1}^{sm}_{\alpha_n}(n)\mathbb{1}^{sm}_{\alpha_{n-1}}(n-1)...\mathbb{1}^{sm}_{\alpha_0}$, and not about eigenfunctions. Theorem 3.1 is proved by writing

$$\mathbb{1}^{sm}_{\alpha_n}(n)\mathbb{1}^{sm}_{\alpha_{n-1}}(n-1)...\mathbb{1}^{sm}_{\alpha_0} = \exp\left(-in\hbar\frac{\triangle}{2}\right)\mathbb{1}^{sm}_{\alpha_n} \exp\left(i\hbar\frac{\triangle}{2}\right)\mathbb{1}^{sm}_{\alpha_{n-1}} \exp\left(i\hbar\frac{\triangle}{2}\right)\ldots \exp\left(i\hbar\frac{\triangle}{2}\right)\mathbb{1}^{sm}_{\alpha_0},$$

by writing the asymptotic expansion of the kernel of $\exp(i\hbar\frac{\triangle}{2})\mathbb{1}^{sm}_{\alpha_k}$, when $\hbar \longrightarrow 0$, as a Fourier integral operator, and by iterating a large number of times this family of Fourier integral operators. For these kinds of techniques, $\mathbb{1}^{sm}_{\alpha_k}$ needs to be a smooth function, which is why we had to apply a convolution to $\mathbb{1}_{\alpha_k}$.

In quantum mechanics, the quantity $\langle \mathbb{1}^{sm}_{P_{\alpha_n}}(n)\mathbb{1}^{sm}_{P_{\alpha_{n-1}}}(n-1)...\mathbb{1}^{sm}_{\alpha_0}\psi_\hbar, \psi_\hbar\rangle$ is the probability amplitude, in the state $\psi_\hbar$, to visit successively P_{α_0}, P_{α_1},..., P_{α_n} at times 1, 2,..., n of the Schrödinger flow. Theorem 3.1 says that this probability decays exponentially fast[1] with rate $\frac{\Lambda}{2}$ with n, with a leading prefactor $(2\pi\hbar)^{-d/2}$ that explodes polynomially in $\hbar^{-1}$.

We would like to deduce from Theorem 3.1 that the entropy of the state $\psi_\hbar$ is larger than $\frac{\Lambda}{2}$ for any $\hbar$, before passing to the limit $\hbar \longrightarrow 0$, to conclude that $h_{KS}(\mu) \geq \frac{\Lambda}{2}$ for any semiclassical measure μ.

The difficulty is that the probability amplitude $\langle \mathbb{1}^{sm}_{\alpha_n}(n)\mathbb{1}^{sm}_{\alpha_{n-1}}(n-1)...\mathbb{1}^{sm}_{\alpha_0}\psi_\hbar, \psi_\hbar\rangle$ is a complex number. We are in the situation of a family of non–commutative dynamical systems converging to a classical dynamical system when $\hbar \longrightarrow 0$. We have to find a convenient notion of "entropy" for the state $\psi_\hbar$ under the action of the Schrödinger flow. Unfortunately, there seems to be no adequate notion of non–commutative entropy that would converge to the usual Kolmogorov–Sinai entropy when $\hbar \longrightarrow 0$ (for instance, the non–commutative entropy of [9] vanishes as soon as the quantum evolution is unitary with discrete spectrum, which is the case for the Schrödinger flow).

[1]Strictly speaking, the result does not hold for all n, but only for n of order $\mathcal{K}|\log \hbar|$, with $\mathcal{K}$ arbitrarily large. This is enough for our purposes. Note that, because of the prefactor $(2\pi\hbar)^{-d/2}$, the estimate is only non trivial when n is a large multiple of $|\log \hbar|$.

This difficulty explains the complicated formulation of Theorems 2.1, 2.3. We will describe the proof of Theorem 2.4, which was made much easier by the use of the *entropic uncertainty principle.*

3.2. **Weighted entropic uncertainty principle.** Our main tool is an adaptation of the entropic uncertainty principle first introduced in [16, 20]. This principle states that if a unitary matrix has "small" entries, then any of its eigenvectors must have a "large" Shannon entropy.

Let $(\mathcal{H}, \langle ., . \rangle)$ be a complex Hilbert space, and denote $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$ the associated norm. Let $(\pi_k)_{k=1}^{\mathcal{N}}$ be an quantum partition of unity, that is, a family of operators on $\mathcal{H}$ such that

$$\sum_{k=1}^{\mathcal{N}} \pi_k \pi_k^* = Id. \tag{3.3}$$

In other words, for all $\psi \in \mathcal{H}$ we have

$$\|\psi\|^2 = \sum_{k=1}^{\mathcal{N}} \|\psi_k\|^2$$

if we denote $\psi_k = \pi_k^* \psi$. If $\|\psi\| = 1$, we define the entropy of ψ with respect to the partition π as

$$h_\pi(\psi) = -\sum_{k=1}^{\mathcal{N}} \|\psi_k\|^2 \log \|\psi_k\|^2.$$

We extend this definition by introducing the notion of pressure, associated to a family $(\alpha_k)_{k=1,\dots,\mathcal{N}}$ of positive real numbers: it is defined by

$$p_{\pi,\alpha}(\psi) = -\sum_{k=1}^{\mathcal{N}} \|\psi_k\|^2 \log \|\psi_k\|^2 - \sum_{k=1}^{\mathcal{N}} \|\psi_k\|^2 \log \alpha_k^2.$$

In Theorem 3.2, we have two families of weights $(\alpha_k)_{k=1,\dots,\mathcal{N}}$, $(\beta_j)_{j=1,\dots,\mathcal{N}}$, and consider the corresponding pressures $p_{\pi,\alpha}(\psi), p_{\pi,\beta}(\psi)$. Besides the appearance of the weights α, β, we bring another modification to the statement in [20] by introducing an auxiliary operator $\mathcal{O}$ — for purely technical reasons.

Theorem 3.2. *Let $\mathcal{O}$ be a bounded operator on $\mathcal{H}$. Let $\mathcal{U}$ be an isometry on $\mathcal{H}$. Define $c_{\mathcal{O}}^{(\alpha,\beta)}(\mathcal{U}) \stackrel{\text{def}}{=} \sup_{j,k} \alpha_k \beta_j \|\pi_j^* \, \mathcal{U} \, \pi_k \, \mathcal{O}\|_{\mathcal{L}(\mathcal{H})}$.*

Then, for any $\epsilon \geq 0$, for any normalized $\psi \in \mathcal{H}$ satisfying

$$\forall k = 1, \dots, \mathcal{N}, \qquad \|(Id - \mathcal{O})\pi_k^* \psi\| \leq \epsilon\,,$$

the pressures $p_{\pi,\beta}\big(\mathcal{U}\psi\big)$, $p_{\pi,\alpha}\big(\psi\big)$ satisfy

$$p_{\pi,\beta}\big(\mathcal{U}\psi\big) + p_{\pi,\alpha}\big(\psi\big) \geq -2 \log \left(c_{\mathcal{O}}^{(\alpha,\beta)}(\mathcal{U}) + \mathcal{N}\, A\, B\, \epsilon \right)$$

where $A = \max \alpha_k$, $B = \max \beta_j$.

Example 1. The original result of [20] corresponds to the case $\mathcal{O} = Id$, $\epsilon = 0$, $\alpha_k = \beta_j = 1$, and the operators π_k are orthogonal projectors on an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$ (take $\mathcal{N} = \infty$ if $\mathcal{H}$ is infinite dimensional). In this case, the theorem says that

$$h_\pi(\mathcal{U}\psi) + h_\pi(\psi) \geq -2\log c(\mathcal{U})$$

where $c(\mathcal{U}) = \sup_{j,k} |\langle e_k, \mathcal{U} e_j\rangle|$ is the supremum of all matrix elements of $\mathcal{U}$ in the orthonormal basis associated to π. As a special case, we get $h_\pi(\psi) \geq -\log c(\mathcal{U})$ if ψ is an eigenfunction of $\mathcal{U}$.

3.3. **Applying the entropic uncertainty principle to the Laplacian eigenstates.** In this section we define the data to input in Theorem 3.2, in order to obtain informations on the eigenstates $\psi_\hbar$ and the measure μ. Only the Hilbert space is fixed, $\mathcal{H} \stackrel{\text{def}}{=} L^2(M)$. All other data depend on the semiclassical parameter $\hbar$: the quantum partition π, the operator $\mathcal{O}$, the positive real number ϵ, the weights (α_j), (β_k) and the unitary operator $\mathcal{U}$.

3.3.1. *Smooth partition of unity.* As usual when computing the Kolmogorov–Sinai entropy, we start by decomposing the manifold M into small cells of diameter $\varepsilon > 0$. More precisely, let $(\Omega_k)_{k=1,\ldots,K}$ be an open cover of M such that all Ω_k have diameters $\leq \varepsilon$, and let $(P_k)_{k=1,\ldots,K}$ be a family of smooth real functions on M, with $\operatorname{supp} P_k \Subset \Omega_k$, such that

$$\forall x \in M, \qquad \sum_{k=1}^{K} P_k^2(x) = 1\,. \tag{3.4}$$

Most of the time, the notation P_k will actually denote the operator of multiplication by $P_k(x)$ on the Hilbert space $L^2(M) = \mathcal{H}$: the above equation shows that they form a quantum partition of unity (3.3), which we will call $\mathcal{P}^{(0)}$.

3.3.2. *Refinement of the partition under the Schrödinger flow.* We denote by $U^t = \exp(it\hbar\triangle\,/\,2)$ the quantum propagator. With no loss of generality, we will assume that the injectivity radius of M is greater than 2, and work with the propagator at time unity, $U = U^1$. This propagator quantizes the flow at time one, g^1. The $\hbar$-dependence of U will be implicit in our notations.

As one does to compute the Kolmogorov–Sinai entropy of an invariant measure, we define a new quantum partition of unity by evolving and refining the initial partition $\mathcal{P}^{(0)}$ under the quantum evolution. For each time $n \in \mathbb{N}$ and any sequence of symbols $\boldsymbol{\epsilon} = (\epsilon_0 \cdots \epsilon_n)$, $\epsilon_i \in [1, K]$ (we say that the sequence $\boldsymbol{\epsilon}$ is of *length* $|\boldsymbol{\epsilon}| = n$), we define the operators

$$P_{\boldsymbol{\epsilon}} = P_{\epsilon_n}(n) P_{\epsilon_{n-1}}(n-1) \ldots P_{\epsilon_0}\,. \tag{3.5}$$

Throughout the paper we will use the notation $A(t) = U^{-t} A U^t$ for the quantum evolution of an operator A. From (3.4) and the unitarity of U, the family of operators $\{P_{\boldsymbol{\epsilon}}\}_{|\boldsymbol{\epsilon}|=n}$ obviously satisfies the resolution of identity $\sum_{|\boldsymbol{\epsilon}|=n} P_{\boldsymbol{\epsilon}} P_{\boldsymbol{\epsilon}}^* = Id_{L^2}$, and therefore forms a quantum partition which we call $\mathcal{P}^{(n)}$.

3.3.3. *Energy localization.* In the semiclassically setting, the eigenstate $\psi_\hbar$ of (2.2) is associated with the energy layer $\mathcal{E} = \mathcal{E}(1/2) = \{\rho \in T^*M,\ H(\rho) = 1/2\}$. Starting from the cotangent bundle T^*M, we restrict ourselves to a compact phase space by introducing an energy cutoff (actually, several cutoffs) near $\mathcal{E}$. To optimize our estimates, we will need this cutoff to depend on $\hbar$ in a sharp way. For some fixed $\delta \in (0,1)$, we consider a smooth function $\chi_\delta \in C^\infty(\mathbb{R}; [0,1])$, with $\chi_\delta(t) = 1$ for $|t| \leq \mathrm{e}^{-\delta/2}$ and $\chi_\delta(t) = 0$ for $|t| \geq 1$. Then, we rescale that function to obtain a family of $\hbar$-dependent cutoffs near $\mathcal{E}$:

$$(3.6) \qquad \forall \hbar \in (0,1),\ \forall n \in \mathbb{N},\ \forall \rho \in T^*M, \qquad \chi^{(n)}(\rho;\hbar) \stackrel{\text{def}}{=} \chi_\delta\big(\mathrm{e}^{-n\delta}\,\hbar^{-1+\delta}(H(\rho) - 1/2)\big)\,.$$

The cutoff $\chi^{(0)}$ is localized in a tubular neighbourhood of $\mathcal{E}$ of width $2\hbar^{1-\delta}$.

These cutoffs can be quantized into pseudodifferential operators $\mathrm{Op}(\chi^{(n)}) = \mathrm{Op}_{\mathcal{E},\hbar}(\chi^{(n)})$ described in [3] (the quantization uses a nonstandard pseudodifferential calculus drawn from [23]). The eigenstate $\psi_\hbar$ satisfies

$$(3.7) \qquad \|\big(\mathrm{Op}(\chi^{(0)}) - 1\big)\psi_\hbar\| = \mathcal{O}(\hbar^\infty)\,\|\psi_\hbar\|$$

(here and below, the norm $\|\cdot\|$ will either denote the Hilbert norm on $\mathcal{H} = L^2(M)$, or the corresponding operator norm).

In the whole paper, we fix a small $\delta' > 2\theta$, and call "Ehrenfest time" the $\hbar$-dependent integer

$$(3.8) \qquad n_E(\hbar) \stackrel{\text{def}}{=} \Big\lfloor \frac{(1-\delta')|\log\hbar|}{\lambda_{\max}} \Big\rfloor\,.$$

Unless indicated otherwise, the integer n will always be taken equal to this Ehrenfest time. The significance of this time scale will be discussed in §3.3.7.

The following proposition says that the operator $P^*_{\boldsymbol{\epsilon}}$ almost preserves the energy localization (3.7) of $\psi_\hbar$:

Proposition 3.3. *For any $L > 0$, there exists $\hbar_L$ such that, for any $\hbar \leq \hbar_L$, the Laplacian eigenstate satisfies*

$$(3.9) \qquad \forall \boldsymbol{\epsilon}, |\boldsymbol{\epsilon}| = n,\ \ \|\big(\mathrm{Op}(\chi^{(n)}) - Id\big)P^*_{\boldsymbol{\epsilon}}\psi_\hbar\| \leq \hbar^L \|\psi_\hbar\|\,.$$

3.3.4. We now precise some of the data we will use in the entropic uncertainty principle, Theorem 3.2:

- the quantum partition π is given by the family of operators $\{P_{\boldsymbol{\epsilon}},\ |\boldsymbol{\epsilon}| = n = n_E(\hbar)\}$. In the semiclassical limit, this partition has cardinality $\mathcal{N} = K^n \asymp \hbar^{-K_0}$ for some fixed $K_0 > 0$.
- the operator $\mathcal{O}$ is $\mathcal{O} = \mathrm{Op}(\chi^{(n)})$, and $\epsilon = \hbar^L$, where L will be chosen very large (see §3.3.6).
- the isometry will be $\mathcal{U} = U^n = U^{n_E(\hbar)}$.
- the weights $\alpha_{\boldsymbol{\epsilon}}$, $\beta_{\boldsymbol{\epsilon}}$ will be selected in §3.3.6. They will be semiclassically tempered, meaning that there exists $K_1 > 0$ such that, for $\hbar$ small enough, all $\alpha_{\boldsymbol{\epsilon}}$, $\beta_{\boldsymbol{\epsilon}}$ are contained in the interval $[1, \hbar^{-K_1}]$.

As in Theorem 3.2, the entropy and pressures associated with a state $\psi \in \mathcal{H}$ are given by

$$h_n(\psi) = h_{\mathcal{P}^{(n)}}(\psi) = -\sum_{|\boldsymbol{\epsilon}|=n} \|P_{\boldsymbol{\epsilon}}^* \psi\|^2 \log\left(\|P_{\boldsymbol{\epsilon}}^* \psi\|^2\right), \tag{3.10}$$

$$p_{n,\alpha}(\psi) = h_n(\psi) - 2\sum_{|\boldsymbol{\epsilon}|=n} \|P_{\boldsymbol{\epsilon}}^* \psi\|^2 \log \alpha_{\boldsymbol{\epsilon}}. \tag{3.11}$$

We may apply Theorem 3.2 to any sequence of states satisfying (3.9), in particular the eigenstates $\psi_\hbar$.

Corollary 3.4. *Define*

$$c^{\alpha,\beta}_{\mathrm{Op}\,\chi^{(n)}}(U^n) \stackrel{\mathrm{def}}{=} \max_{|\boldsymbol{\epsilon}|=|\boldsymbol{\epsilon}'|=n} \left(\alpha_{\boldsymbol{\epsilon}}\, \beta_{\boldsymbol{\epsilon}'} \|P_{\boldsymbol{\epsilon}'}^*\, U^n\, P_{\boldsymbol{\epsilon}}\, \mathrm{Op}(\chi^{(n)})\|\right). \tag{3.12}$$

Then for any normalized state ϕ satisfying (3.9),

$$p_{n,\beta}(U^n\,\phi) + p_{n,\alpha}(\phi) \geq -2\log\left(c^{\alpha,\beta}_{\mathrm{Op}\,\chi^{(n)}}(U^n) + h^{L-K_0-2K_1}\right).$$

Most of [3] is devoted to obtaining a good upper bound for the norms $\|P_{\boldsymbol{\epsilon}'}^*\, U^n\, P_{\boldsymbol{\epsilon}}\, \mathrm{Op}(\chi^{(n)})\|$ involved in the above quantity. The bound is given in Theorem 3.5 below. Our choice for the weights $\alpha_{\boldsymbol{\epsilon}}$, $\beta_{\boldsymbol{\epsilon}}$ will then be guided by these upper bounds.

3.3.5. *Unstable Jacobian for the geodesic flow.* We need to recall a few definitions pertaining to Anosov flows. For any $\lambda > 0$, the geodesic flow g^t is Anosov on the energy layer $\mathcal{E}(\lambda) = H^{-1}(\lambda) \subset T^*M$. This implies that for each $\rho \in \mathcal{E}(\lambda)$, the tangent space $T_\rho\mathcal{E}(\lambda)$ splits into

$$T_\rho\mathcal{E}(\lambda) = E^u(\rho) \oplus E^s(\rho) \oplus \mathbb{R}\, X_H(\rho)$$

where E^u is the unstable subspace and E^s the stable subspace. The unstable Jacobian $J^u(\rho)$ at the point ρ is defined as the Jacobian of the map g^{-1}, restricted to the unstable subspace at the point $g^1\rho$: $J^u(\rho) = \det\left(dg^{-1}_{|E^u(g^1\rho)}\right)$ (the unstable spaces at ρ and $g^1\rho$ are equipped with the induced Riemannian metric). This Jacobian can be "discretized" as follows in a neighbourhood $\mathcal{E}^\varepsilon \stackrel{\mathrm{def}}{=} \mathcal{E}([1/2-\varepsilon, 1/2+\varepsilon])$ of $\mathcal{E}$. For any pair $(\epsilon_0, \epsilon_1) \in [1, K]^2$, we define

$$J_1^u(\epsilon_0, \epsilon_1) \stackrel{\mathrm{def}}{=} \sup\left\{J^u(\rho)\ :\ \rho \in T^*\Omega_{\epsilon_0} \cap \mathcal{E}^\varepsilon,\ g^1\rho \in T^*\Omega_{\epsilon_1}\right\} \tag{3.13}$$

if the set on the right hand side is not empty, and $J_1^u(\epsilon_0, \epsilon_1) = e^{-R}$ otherwise, where $R > 0$ is a fixed large number. For any sequence of symbols $\boldsymbol{\epsilon}$ of length n, we define

$$J_n^u(\boldsymbol{\epsilon}) \stackrel{\mathrm{def}}{=} J_1^u(\epsilon_0, \epsilon_1) \ldots J_1^u(\epsilon_{n-1}, \epsilon_n). \tag{3.14}$$

Although J^u and $J_1^u(\epsilon_0, \epsilon_1)$ are not necessarily everywhere smaller than unity, there exists $C, \lambda_+,\ \lambda_- > 0$ such that, for any $n > 0$, for any $\boldsymbol{\epsilon}$ with $|\boldsymbol{\epsilon}| = n$,

$$C^{-1}\, \mathrm{e}^{-n(d-1)\,\lambda_+} \leq J_n^u(\boldsymbol{\epsilon}) \leq C\, \mathrm{e}^{-n(d-1)\,\lambda_-}. \tag{3.15}$$

One can take $\lambda_+ = \lambda_{\max}(1+\varepsilon)$. We can now give our central estimate, proven in [3].

Theorem 3.5. *Given a partition $\mathcal{P}^{(0)}$ and $\delta, \delta' > 0$ small enough, there exists $\hbar_{\mathcal{P}^{(0)},\delta,\delta'}$ such that, for any $\hbar \leq \hbar_{\mathcal{P}^{(0)},\delta,\delta'}$, for any positive integer $n \leq n_E(\hbar)$, and any pair of sequences $\boldsymbol{\epsilon}$, $\boldsymbol{\epsilon}'$ of length n,*

$$\|P^*_{\boldsymbol{\epsilon}'}\, U^n\, P_{\boldsymbol{\epsilon}}\, \mathrm{Op}(\chi^{(n)})\| \leq C\, \hbar^{-(d-1+c\delta)}\, J^u_n(\boldsymbol{\epsilon})^{1/2}\, J^u_n(\boldsymbol{\epsilon}')\,. \tag{3.16}$$

The constants c, C only depend on the Riemannian manifold (M, g).

The theorem bounds from above the norm of the operator $P^*_{\boldsymbol{\epsilon}'}\, U^n\, P_{\boldsymbol{\epsilon}}\, \mathrm{Op}(\chi^{(n)})$. This norm can be obtained as follows:

$$\|P^*_{\boldsymbol{\epsilon}'}\, U^n\, P_{\boldsymbol{\epsilon}}\, \mathrm{Op}(\chi^{(n)})\| = \sup\left\{|\langle P_{\boldsymbol{\epsilon}'}\Phi,\, U^n\, P_{\boldsymbol{\epsilon}}\, \mathrm{Op}(\chi^{(n)})\Psi\rangle| \;:\; \Psi,\, \Phi \in \mathcal{H},\; \|\Psi\| = \|\Phi\| = 1\right\}\,.$$

The theorem is a corollary of the following estimate :

Proposition 3.6. *For $\hbar$ small enough, any time $n \leq \frac{(1-\delta')|\log\hbar|}{\lambda_{\max}}$, any sequences $\boldsymbol{\epsilon}$, $\boldsymbol{\epsilon}'$ of length n and any normalized states Ψ, $\Phi \in L^2(M)$, one has*

$$|\langle P_{\boldsymbol{\epsilon}'}\, \mathrm{Op}(\chi^{(4n)})\,\Phi,\, U^n\, P_{\boldsymbol{\epsilon}}\, \mathrm{Op}(\chi^{(n)})\Psi\rangle| \leq C\, \hbar^{-(d-1)-c\delta}\, J^u_n(\boldsymbol{\epsilon})^{1/2} J^u_n(\boldsymbol{\epsilon}')\,. \tag{3.17}$$

The constants C and $c = 2 + 5/\lambda_{\max}$ only depend on the Riemannian manifold M.

The idea in Proposition 3.6 is rather simple, although the technical implementation becomes cumbersome. We first show that any state of the form $\mathrm{Op}(\chi)\Psi$, as those appearing in the scalar product (3.17), can be decomposed as a superposition of essentially $\hbar^{-\frac{(d-1)}{2}}$ normalized lagrangian states, supported on lagrangian manifolds transverse to the stable leaves of the flow. In fact the lagrangian states we work with are truncated δ–functions, supported on spheres $S^*_z M$. The action of the operator $U^n\, P_{\boldsymbol{\epsilon}} = P_{\epsilon_n} U P_{\epsilon_{n-1}} U \cdots U P_{\epsilon_0}$ on such lagrangian states is described by the theory of Fourier integral operators (WKB methods), and is intuitively simple to understand : each application of U stretches the lagrangian in the unstable direction (the rate of elongation being described by the unstable Jacobian) whereas each multiplication by P_ϵ cuts a small piece of lagrangian. This iteration of stretching and cutting accounts for the exponential decay.

3.3.6. *Applying the entropic uncertainty principle.* There remains to choose the weights $(\alpha_{\boldsymbol{\epsilon}}, \beta_{\boldsymbol{\epsilon}})$ to use in Theorem 3.2. Our choice is guided by the following idea: in the quantity (3.12), the weights should balance the variations (with respect to $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}'$) in the norms, such as to make all terms in (3.12) of the same order. Using the upper bounds (3.16), we end up with the following choice for all $\boldsymbol{\epsilon}$ of length n:

$$\alpha_{\boldsymbol{\epsilon}} \stackrel{\mathrm{def}}{=} J^u_n(\boldsymbol{\epsilon})^{-1/2} \quad \text{and} \quad \beta_{\boldsymbol{\epsilon}} \stackrel{\mathrm{def}}{=} J^u_n(\boldsymbol{\epsilon})^{-1}\,. \tag{3.18}$$

All these quantities are defined using the Ehrenfest time $n = n_E(\hbar)$. From (3.15), there exists $K_1 > 0$ such that, for $\hbar$ small enough, all the weights are bounded by

$$1 \leq |\alpha_{\boldsymbol{\epsilon}}| \leq \hbar^{-K_1}, \qquad 1 \leq |\beta_{\boldsymbol{\epsilon}}| \leq \hbar^{-K_1}\,, \tag{3.19}$$

as announced in §3.3.4.

The estimate (3.16) can then be rewritten as

$$c^{\alpha,\beta}_{\mathrm{Op}\,\chi^{(n)}}(U^n) \leq C\, \hbar^{-(d-1+c\delta)}\,.$$

We now apply Corollary 3.4 to the particular case of the eigenstates $\psi_\hbar$. We choose L large enough such that $\hbar^{L-K_0-2K_1}$ is negligible in comparison with $\hbar^{-(d-1+c\delta)}$.

Proposition 3.7. *Let $(\psi_\hbar)_{\hbar\to 0}$ be any sequence of eigenstates* (2.2)*. Then, in the semiclassical limit, the pressures of $\psi_\hbar$ satisfy*

$$p_{n,\alpha}(\psi_\hbar) + p_{n,\beta}(\psi_\hbar) \geq 2(d-1+c\delta)\log\hbar + \mathcal{O}(1) \geq -2\frac{(d-1+c\delta)\lambda_{\max}}{(1-\delta')}\, n + \mathcal{O}(1)\,. \tag{3.20}$$

3.3.7. *Subadditivity until the Ehrenfest time.* Before taking the limit $\hbar \to 0$, we prove that a similar lower bound holds if we replace $n \asymp |\log\hbar|$ by some fixed n_o, and $\mathcal{P}^{(n)}$ by the corresponding partition $\mathcal{P}^{(n_o)}$. This is due to the following *subadditivity property*, which is the semiclassical analogue of the classical subadditivity of pressures for invariant measures.

Proposition 3.8 (Subadditivity)**.** *Let $\delta' > 0$. There is a function $R(n_o,\hbar)$, and a real number $R > 0$ independent of δ', such that, for all integer n_o,*

$$\limsup_{\hbar\longrightarrow 0} |R(n_o,\hbar)| \leq R$$

and such that, for all n_o, $n \in \mathbb{N}$ with $n_o + n \leq n_E(\hbar) = \frac{(1-\delta')|\log\hbar|}{\lambda_{\max}}$, for any $(\psi_\hbar)$ normalized eigenstates satisfying (2.2)*, the following inequality holds:*

$$p_{n_o+n,\alpha}(\psi_\hbar) \leq p_{n_o,\alpha}(\psi_\hbar) + p_{n-1,\alpha}(\psi_\hbar) + R(n_o,\hbar)\,.$$

A similar inequality holds for $p_{n_o+n,\beta}(\psi_\hbar)$.

The non–commutative dynamical system formed by (U^t) acting on pseudodifferential operators is (approximately) commutative on time intervals of length $n_E(\hbar)$:

$$\|[\mathrm{Op}_\hbar(a)(t), \mathrm{Op}_\hbar(b)]\|_{L^2(M)} = \mathcal{O}(\hbar^{c\delta'}),$$

for any time $|t| \leq n_E(\hbar)$. On such a time interval, it can be treated as a commutative dynamical system, up to small errors tending to 0 with $\hbar$. This explains why the quantum entropy (or pressure) $p_{n_o+n,\alpha}(\psi_\hbar)$ has the same subadditivity property as the classical entropy (3.1), up to small errors, as long as n_o+n remains bounded by the Ehrenfest time.

Thanks to this subadditivity, we may finish the proof of Theorem 2.4. Let $n_o \in \mathbb{N}$ be fixed and $n = n_E(\hbar)$. Using the Euclidean division $n = q(n_o+1) + r$ (with $r \leq n_o$), Proposition 3.8 implies that for $\hbar$ small enough,

$$\frac{p_{n,\alpha}(\psi_\hbar)}{n} \leq \frac{p_{n_o,\alpha}(\psi_\hbar)}{n_o} + \frac{p_{r,\alpha}(\psi_\hbar)}{n} + \frac{R(n_o,\hbar)}{n_o}\,.$$

Using (3.20) and the fact that $p_{r,\alpha}(\psi_\hbar) + p_{r,\beta}(\psi_\hbar)$ stays uniformly bounded (by a quantity depending on n_o) when $\hbar \to 0$, we find

$$\frac{p_{n_o,\alpha}(\psi_\hbar)}{n_o} + \frac{p_{n_o,\beta}(\psi_\hbar)}{n_o} \geq -2\frac{(d-1+c\delta)\lambda_{\max}}{(1-\delta')} - 2\frac{R(n_o,\hbar)}{n_o} + \mathcal{O}_{n_o}(1/n)\,. \tag{3.21}$$

We are now dealing with the partition $\mathcal{P}^{(n_o)}$, n_0 being fixed.

3.3.8. *End of the proof.* Let us take a subsequence of $(\psi_{\hbar_k})$ such that the Wigner measures $W_k = W_{\psi_{\hbar_k}}$ converge to a semiclassical measure μ on $\mathcal{E}$, invariant under the geodesic flow (see Prop. 1.2). We may take the limit $\hbar_k \to 0$ (so that $n \to \infty$) in the above expression. The norms appearing in the definition of $h_{n_o}(\psi_{\hbar_k})$ can be written as

$$\|P^*_{\boldsymbol{\epsilon}}\, \psi_{\hbar_k}\| = \|P_{\epsilon_0} P_{\epsilon_1}(1) \cdots P_{\epsilon_{n_o}}(n_o)\, \psi_{\hbar_k}\|\,. \tag{3.22}$$

For any sequence $\boldsymbol{\epsilon}$ of length n_o, the laws of pseudodifferential calculus imply the convergence of $\|P^*_{\boldsymbol{\epsilon}}\, \psi_{\hbar_k}\|^2$ to $\mu(\{\boldsymbol{\epsilon}\})$, where $\{\boldsymbol{\epsilon}\}$ is the function $P^2_{\epsilon_0}\,(P^2_{\epsilon_1} \circ g^1) \ldots (P^2_{\epsilon_{n_o}} \circ g^{n_o})$ on T^*M. Thus $h_{n_o}(\psi_{\hbar_k})$ semiclassically converges to the classical entropy

$$h_{n_o}(\mu) = h_{n_o}(\mu, (P^2_k)) = -\sum_{|\boldsymbol{\epsilon}|=n_o} \mu(\{\boldsymbol{\epsilon}\}) \log \mu(\{\boldsymbol{\epsilon}\})\,.$$

As a result, the left hand side of (3.21) converges to

$$\frac{2}{n_o}\, h_{n_o}(\mu) + \frac{3}{n_o} \sum_{|\boldsymbol{\epsilon}|=n_o} \mu(\{\boldsymbol{\epsilon}\})\ \log J^u_{n_o}(\boldsymbol{\epsilon})\,. \tag{3.23}$$

Since μ is g^t-invariant and $J^u_{n_o}$ has the multiplicative structure (3.14), the second term in (3.23) can be simplified:

$$\sum_{|\boldsymbol{\epsilon}|=n_o} \mu(\{\boldsymbol{\epsilon}\})\ \log J^u_{n_o}(\boldsymbol{\epsilon}) = n_o \sum_{\epsilon_0,\epsilon_1} \mu(\{\epsilon_0\epsilon_1\})\ \log J^u_1(\epsilon_0,\epsilon_1)\,.$$

We have thus obtained the lower bound

$$\frac{h_{n_o}(\mu)}{n_o} \geq -\frac{3}{2} \sum_{\epsilon_0,\epsilon_1} \mu(\{\epsilon_0\epsilon_1\})\ \log J^u_1(\epsilon_0,\epsilon_1) - \frac{(d-1+c\delta)\lambda_{\max}}{(1-\delta')} - 2\frac{R}{n_o}\,. \tag{3.24}$$

δ and δ' could be taken arbitrarily small, and at this stage they can be let vanish.

The Kolmogorov–Sinai entropy of μ is by definition the limit of the first term $\frac{h_{n_o}(\mu)}{n_o}$ when n_o goes to infinity, with the notable difference that the smooth functions P_k should be replaced by characteristic functions associated with some partition of M, $M = \bigsqcup_k Q_k$. Thus, let us consider such a partition of diameter $\leq \varepsilon/2$, such that μ does not charge the boundaries of the Q_k. By convolution we can smooth the characteristic functions $(\mathbb{1}_{Q_k})$ into a smooth partition of unity $(P_k = \mathbb{1}^{sm}_{Q_k})$ satisfying the conditions of section 3.3.1. The lower bound (3.24) holds with respect to the smooth partition (P^2_k), and does not depend on the derivatives of the P_k: as a result, the same bound carries over to the characteristic functions $(\mathbb{1}_{Q_k})$.

We can finally let n_o tend to $+\infty$, then let the diameter $\varepsilon/2$ of the partition tend to 0. From the definition (3.13), the first term in the right hand side of (3.24) converges to the integral $-\frac{3}{2}\int_{\mathcal{E}} \log J^u(\rho) d\mu(\rho)$ as $\varepsilon \to 0$, which proves (2.3).

□

4. Concluding remark.

In the course of the proof, we evaluated the entropy of $\psi_\hbar$ at a finite time $n_E(\hbar) \underset{\hbar\longrightarrow 0}{\longrightarrow} +\infty$ called the Ehrenfest time : it is the largest time interval on which the Schrödinger flow acting on $\hbar$–pseudodifferential operators can be treated as a commutative dynamical system.

In constant curvature, we were lucky to find that the Ehrenfest time is large enough to capture positive entropy. In other words, for $n = n_E(\hbar)$, the upper bound (3.17) is a positive power of $\hbar$.

In variable curvature, due to the fact that the Lyapunov exponents are not the same everywhere, the Ehrenfest time should be defined as a quantity depending of the point in phase space, and we took for $n_E(\hbar)$ the infimum of all these "local" Ehrenfest times. This is certainly not a sharp way of dealing with the Ehrenfest time issue. It can now happen that the upper bound (3.17) is a negative power of $\hbar$, in which case it contains no information.

The method of [1] was much less sensitive to the lack of optimality of certain estimates, since it required to evaluate the entropy much beyond the Ehrenfest time

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