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Magnetic Schrödinger Operator: Geometry, Classical and Quantum Dynamics and Spectral Asymptotics

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Magnetic Schrödinger Operator: Geometry, Classical and Quantum Dynamics and Spectral Asymptotics

Victor Ivrii

Abstract

I study the Schrödinger operator with the strong magnetic field, considering links between geometry of magnetic field, classical and quantum dynamics associated with operator and spectral asymptotics. In particular, I will discuss the role of short periodic trajectories.

0 Preface

I will consider *Magnetic Schrödinger operator*

$$(1) \quad H = \frac{1}{2} \left(\sum_{j,k} P_j g^{jk}(x) P_k - V \right), \quad P_j = hD_j - \mu A_j$$

where g^{jk} , A_j , V are smooth real-valued functions of $x \in \mathbb{R}^d$ and (g^{jk}) is a positive-definite matrix, $0 < h \ll 1$ is a Planck parameter and $\mu \gg 1$ is a coupling parameter. I assume that H is a self-adjoint operator.

2-dimensional magnetic Schrödinger is very different from 3-dimensional, all others could be close to one of these cases but are more complicated.

I am interested in the geometry of magnetic field, classical and quantum dynamics associated with operator (1) and spectral asymptotics

$$(2) \quad \int e(x, x, 0) \psi(x) dx$$

as $h \rightarrow +0$, $\mu \rightarrow +\infty$ where $e(x, y, \tau)$ is the Schwartz kernel of the spectral projector of H and $\psi(x)$ is cut-off function. Everything is assumed to be C^∞ .

1 Geometry of Magnetic Field

Magnetic Intensity Magnetic field is described by a form

$$(3) \quad \sigma = d\left(\sum_k A_k dx_k\right) = \frac{1}{2} \sum_{j,k} F_{jk} dx_j \wedge dx_k$$

with

$$(4) \quad F_{jk} = \partial_j V_k - \partial_k V_j.$$

So σ does not change after *gauge transformation* $\vec{A} \mapsto \vec{A} + \vec{\nabla}\phi$ and this transformation does not affect other objects I am interesting in as well.

I am discussing local things and *Aharonov-Bohm effect* which demonstrates that knowledge of σ , g_{jk} , V is not sufficient to characterize spectral properties of H is beyond my analysis.

Canonical forms If σ is of maximum rank $2r = 2[d/2]$ one can reduce it locally to the *Darboux canonical form*

$$(5) \quad \sigma = \sum_{1 \leq j \leq r} dx_{2j-1} \wedge dx_{2j}.$$

So, (5) is a canonical form of σ near *generic* point for *generic* \vec{A} . However situation becomes much more complicated near *general* point for *generic* \vec{A} . Complete results are not known. Assuming $d = 2r$ and σ is generic J. Martinet [Ma] had shown that $\Sigma_k = \{x, \text{rank } F(x) \leq d - k\}$ are submanifolds and calculated their codimensions. In particular, $\text{codim } \Sigma_1 = 1$. Moreover, $\Sigma_2 = \emptyset$ as $d = 2, 4$ (not true for $d \geq 6$).

As $d = 2$ generic 2-form σ has a local canonical form

$$(6) \quad \sigma = x_1 dx_1 \wedge dx_2, \quad \Sigma = \{x_1 = 0\}.$$

However, as $d = 4$ not all points of $\Sigma = \Sigma_1$ are equal: $\Lambda = \{x \in \Sigma, \text{Ker } F(x) \subset T_x \Sigma\}$ is submanifold of dimension 1. As $\bar{x} \in \Sigma \setminus \Lambda$ $\dim(\text{Ker } F(x) \cap T_{\bar{x}} \Sigma) = 1$ and in its vicinity one can reduce σ to

$$(7) \quad \sigma = x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

while in the vicinity of $\bar{x} \in \Lambda$ canonical form is

$$(8) \quad \sigma = dx_1 \wedge dx_2 - x_4 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_4 + x_3 dx_2 \wedge dx_3 + x_4 dx_2 \wedge dx_4 + 2\left(x_1 - \frac{1}{2}(x_3^2 + x_4^2)\right) dx_3 \wedge dx_4$$

(R. Roussarie [Rou], modified by $x_2 \mapsto x_2 - \frac{1}{2}x_3x_4$).

Magnetic lines *Magnetic lines* are described by

$$(9) \quad \frac{dx}{dt} \in \text{Ker } F(x) \cap T_x \Sigma$$

where one can skip $\cap T_x \Sigma$ without changing the substance of the definition.

As $\text{rank } F = d$ (and thus d is even) there are no magnetic lines. As $\text{rank } F = d - 1$ (and thus d is odd) through each point passes exactly 1 magnetic line.

As $d = 2$ and σ is defined by (6) the only magnetic line is $\{x_1 = 0\}$. As $d = 4$ and σ is defined by (7) magnetic lines are straight lines $\{x_1 = 0, x_3 = \text{const}, x_4 = \text{const}\}$. As $d = 4$ and σ is defined by (8) $\Lambda = \{x_1 = x_3 = x_4 = 0\}$ and magnetic lines are helices $\{x_1 = 0, x_3 = r \cos \theta, x_4 = r \sin \theta, x_2 = \text{const} - r^2 \theta / 2\}$ (with $r = \text{const}$), winging around Λ .

True geometry From the point of view of operator H simultaneous analysis of form σ and metrics (g^{lj}) should be crucial, but I am not aware about any results. It appears, however, that only eigenvalues $\pm if_j$ and eigenspaces of matrix $(F'_k) = (\sum_j g^{lj} F_{jk})$ are really important, and in the case of the generic magnetic field they are not very difficult to examine.

As $d = 2$

$$(10) \quad f_1 = F_{12} / \sqrt{g}, \quad g = \det(g^{jk})^{-1}$$

while for $d = 3$

$$(11) \quad f_1 = \frac{1}{2} \left(\sum_{j,k,l,m} g^{jk} g^{lm} F_{jl} F_{km} \right)^{1/2} = \left(\sum_{j,k,l,m} g_{jk} F^j F^k \right)^{1/2}$$

where $F^j = \frac{1}{2} \sum_{k,l} \varepsilon^{jkl} F_{kl}$ is a vector intensity of magnetic field, ε^{jkl} is an absolutely skew-symmetric tensor with $\varepsilon^{123} = 1/\sqrt{g}$.

2 Classical Dynamics

2.1 Constant case

This case has been well-known long ago.

2D case Assume first that g^{jk} , F_{jk} and V are constant. Then with no loss of the generality one can assume that $g^{jk} = \delta_{jk}$, skew-symmetric matrix (F_{jk}) is reduced to the canonical form:

$$(12) \quad F_{jk} = \begin{cases} f_j & j = 1, \dots, r, k = j + r \\ -f_k & j = r + 1, \dots, 2r, k = j - r, \\ 0 & \text{otherwise} \end{cases}$$

$f_j > 0$ and $V = 0$; moreover, one can select $A_j(x)$ as linear functions.

Then as $d = 2$, $f_1 > 0$ classical particle described by Hamiltonian

$$(13) \quad H(x, \xi) = \frac{1}{2} \left(\sum_{j,k} g^{jk}(x) (\xi_j - \mu A_j(x)) (\xi_k - \mu A_k(x)) - V(x) \right)$$

moves along *cyclotrons* which in this case are circles of radius $\rho_1 = (\mu f_1)^{-1} \sqrt{2E}$ with the angular velocity $\omega_1 = \mu f_1$ on energy level $\{H(x, \xi) = E\}$.

3D case As $d = 3$, $f_1 > 0$ there are a *cyclotron movement* along circles of radii $\rho_1 = (\mu f_1)^{-1} \sqrt{2E_1}$ with the angular velocity $\omega_1 = \mu f_1$ and a *free movement* along magnetic lines (which are straight lines along $\text{Ker } F$) with a speed $\sqrt{2E_f}$ and energy E is split into two constant arbitrary parts $E = E_1 + E_f$.

Multidimensional case Multidimensional case with $d = 2r = \text{rank } F$ is a combination of 2D cases: there are r cyclotron movements with angular velocities $\omega_k = \mu f_k$ and radii $\rho_k = (\mu f_k)^{-1} \sqrt{2E_k}$ where energy E is split into r constant arbitrary parts $E = E_1 + E_2 + \dots + E_r$. The exact nature of the trajectories depends on the comeasurability of f_1, \dots, f_r .

As $d > 2r = \text{rank } F$ in addition to the cyclotronic movements described above appears a free movement along any constant direction $\vec{v} \in \text{Ker } F$ with a speed $\sqrt{2E_f}$ where energy E is split into $r + 1$ constant arbitrary parts $E = E_1 + E_2 + \dots + E_r + E_f$.

This difference between cases $d = 2r = \text{rank } F$ and $d > 2r = \text{rank } F$ will be traced through the whole paper.

2.2 Full rank case

Assume now only that $d = \text{rank } F$ (see f.e [Ivr2, Ivr3]). In addition, assume temporarily that F_{jk} and g^{jk} are constant but potential $V(x)$ is linear. Then cyclotronic movement(s) is combined with the *magnetic drift* described by equation

$$(14) \quad \frac{dx_j}{dt} = (2\mu)^{-1} \sum_k \Phi^{jk} \partial_k V$$

where $(\Phi^{jk}) = (F_{jk})^{-1}$.

As $d = 2$ it will be movement along cycloid and multidimensional movement will be combination of those.

Not assuming anymore that V is linear we get a bit more complicated picture:

- Equation (14) holds modulo $O(\mu^{-2})$; modulo error $O(\mu^{-2}t)$;
- As $d = 2$ cycloid is replaced by a more complicated curve drifting along $V = \text{const}$ and thus cyclotron radius $\rho = (\mu f_1)^{-1} \sqrt{2E + V}$ would be preserved;
- In higher dimensions all cyclotron radii are preserved as well.

Without assumption that g^{jk} and F_{jk} are constant picture becomes even more complicated:

- As $d = 2$ cycloid is replaced by a more complicated curve drifting along $f^{-1}(V + 2E) = \text{const}$ (thus preserving *angular momentum* $\omega_1 \rho_1^2$ according to equation

$$(15) \quad \frac{dx}{dt} = (2\mu)^{-1} (\nabla f^{-1}(V + 2E))^\perp$$

where $^\perp$ means clockwise rotation by $\pi/2$ assuming that at point in question $g^{jk} = \delta_{jk}$;

- In higher dimensions (at least as *non-resonance conditions* $f_j \neq f_k$ for all $j \neq k$ and $f_j \neq f_k + f_l$ for all j, k, l are fulfilled) one can split potential $V = V_1 + \dots + V_k$ so that similar equations hold in each eigenspace of (F_k^j) and both separate energies and angular momenta are (almost) preserved.

2.3 3D case

As $d > 2r = \text{rank } F$ the free movement is the main source of the spatial displacement and the most interesting case is $2r = d - 1$ and especially $d = 3, r = 1$.

In this case the *magnetic angular momentum* M is (almost) preserved; thus *kinetic energy* of magnetic rotation is $\frac{1}{2}f^{-1}M^2$; therefore in the coordinate system such that $g^{1j} = \delta_{1j}$ the free movement is described by 1D Hamiltonian

$$(16) \quad H_1(x_1, \xi_1; x', M) = \frac{1}{2}\xi_1^2 - \frac{1}{2}V_{\text{eff}}$$

with effective potential $V_{\text{eff}}(x_1, x') = V - f^{-1}M^2, x = (x_1, x')$.

Thus the particle does not necessarily run the whole magnetic line and the helix winging around it does not necessarily have constant the step or radius. Effect of the magnetic drift is rather minor.

2.4 2D case: variable rank

Situation becomes really complicated for variable $\text{rank } F$. I am going to consider only $d = 2, 4$ and a generic magnetic form σ . Let me start from the model Hamiltonian as $d = 2$:

$$(17) \quad H^0 = \frac{1}{2} \left(\xi_1^2 + (\xi_2 - \mu x_1^\nu / \nu)^2 - 1 \right);$$

the drift equation is

$$(18) \quad \frac{dx_1}{dt} = 0, \quad \frac{dx_2}{dt} = \frac{1}{2}(\nu - 1)\mu^{-1}x_1^{-\nu}$$

and for $|x_1| \gg \bar{\gamma} = \mu^{-1/(\nu+1)}$ gives a proper description of the picture.

For the model Hamiltonian (17) with $\mu = 1$ (otherwise one can scale $x_1 \mapsto \mu^{1/2}x_1, \xi_2 \mapsto \mu k$) we can consider also 1-dimensional movement along x_1 with potential

$$(19) \quad \mathcal{V}(x_1; k) = 1 - (k - x_1^\nu / \nu)^2, \quad k = \xi_2;$$

Then for odd ν

- \mathcal{V} is one-well potential;

- As $k = \pm 1$ one of its extremes is 0 where $\frac{dV}{dx_1}(0) = 0$;
- Well is more to the right/left from 0 as $\pm k > 0$; as $k = 0$ well becomes symmetric.

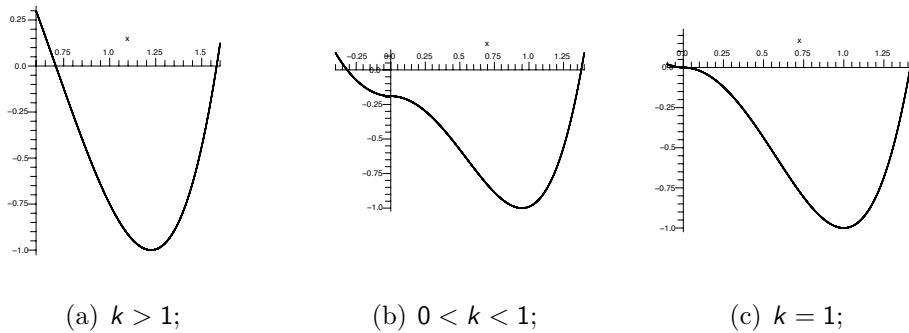


Figure 1: Effective potential for odd ν .

On the other hand, for even ν potential is always symmetric and

- We have two-well potential with the central bump above surface if $k > 1$
- and below it as $0 < k < 1$:
- As $k = \pm 1$ one of its extremes is 0 where $\frac{dV}{dx_1}(0) = 0$;
- Well is more to the right/left from 0 as $\pm k > 0$; as $k = 0$ well becomes symmetric.

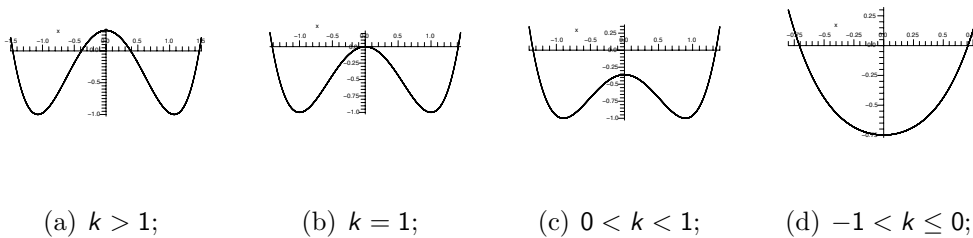


Figure 2: Effective potential for even ν .

Let us consider trajectories *on the energy level 0*. From the analysis of the potential one can conclude that for $k \neq \pm 1$ the movement along x_1 is periodic with the period

$$(20) \quad T(k) = 2 \int_{x_1^-(k)}^{x_1^+(k)} \frac{dx_1}{\sqrt{2\mathcal{V}(x_1; k)}};$$

however one needs to analyze the increment of x_2 during this period

$$(21) \quad I(k) = 2 \int_{x_1^-(k)}^{x_1^+(k)} \frac{(k - x_1^\nu/\nu) dx_1}{\sqrt{2\mathcal{V}(x_1; k)}}.$$

One can prove that $I(k) \geq 0$ as $k \geq k^*$ with $0 < k^* < 1$ for even ν and $k^* = 0$ for odd ν . In particular, $k^* \approx 0.65$ for $\nu = 2$. Further, $I(k) \asymp (k - k^*)$ as $k \approx k^*$.

On figures 3–5 are shown trajectories on (x_1, x_2) -plane plotted by Maple in the outer zone (these trajectories have mirror-symmetric or central-symmetric for even or odd ν in zone $x_1 < 0$), in the inner zone for even ν and in the inner zone for odd ν respectively. For the spectral asymptotics periodic trajectories are very important, especially the short ones. Periodic trajectories shown above are *very unstable* and taking $V = 1 - \alpha x_1$ instead of x_1 breaks them down (figure 6). One can find details in section 1, [Ivr5].

2.5 4D case: variable rank

The most natural model operator corresponding to the canonical form (7) is $H^0 + H''$ with H^0 as above and $2H'' = \xi_3^2 + (\xi_4 - x_3)^2$. Then H'' is a movement integral. Therefore the dynamics is split into dynamics in $(x', \xi') = (x_1, x_2, \xi_1, \xi_2)$ described above with potential $W = V - 2E$ and the standard cyclotron movement with energy E in $(x'', \xi'') = (x_3, x_4, \xi_3, \xi_4)$.

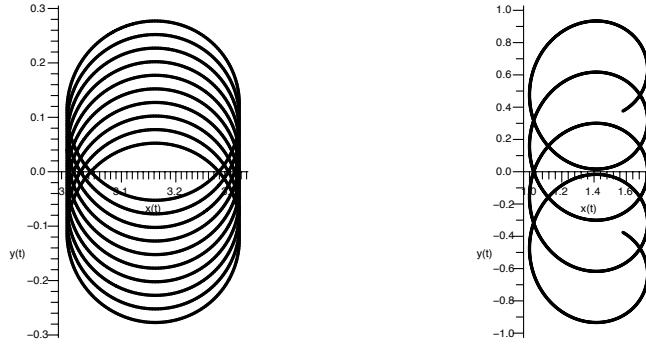
Situation actually is way more complicated: considering $H^0 + (1 + \alpha x_1)E$ we arrive to the 1-D potential $\mathcal{V} - 1(1 + \alpha x_1)$ and playing with E and α one can kill the drift even for $k \gg 1$ leading to many periodic trajectories.

Consider canonical form (8) which in polar coordinates in (x_3, x_4) becomes

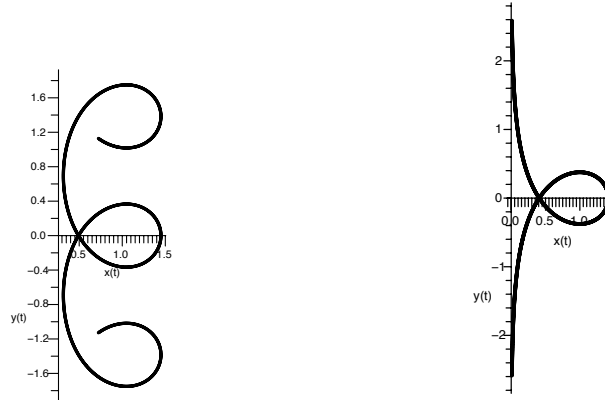
$$(22) \quad \sigma = d\left(\left(x_1 - \frac{1}{2}\rho^2\right)dx_2 + \left(x_1 - \frac{1}{4}\rho^2\right)\rho^2 d\theta\right).$$

The most natural classical Hamiltonian corresponding to this form is

$$(23) \quad 2H = \xi_1^2 + \left(\xi_2 - \mu\left(x_1 - \frac{1}{2}\rho^2\right)\right)^2 + \varrho^2 + r^{-2}\left(\vartheta - \mu\left(x_1 - \frac{1}{4}\rho^2\right)\rho^2\right)^2 - 1$$



(a) $k \gg 1$; as $x_1 > 0$ trajectory moves up and rotates clockwise (b) k decreases, still $k > 1$. Trajectory becomes less tight; actual size of cyclotrons increases;



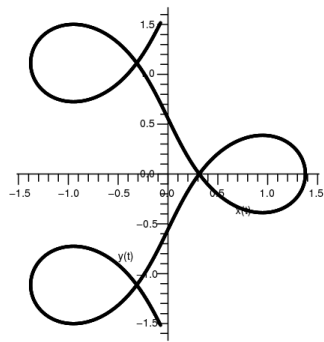
(c) k further decreases, still $k > 1$; (d) $k = 1$. Trajectory contains just one cyclotron.

Figure 3: Movements in the outer zone

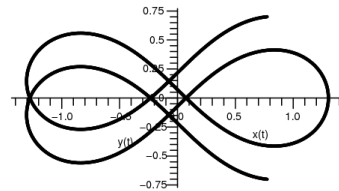
with ϱ, ϑ dual to ρ, θ .

Note that ξ_2 and ϑ are movement integrals and therefore $x_1 - \frac{1}{2}\rho^2$ is preserved modulo $O(\mu^{-1})$. Based on this one can prove that

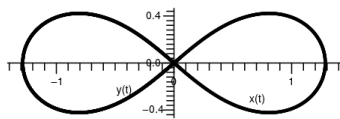
- There is a cyclotronic movement with the angular velocity $\asymp \mu^{-1}$ in the normal direction to paraboloid $\{-x_1 + \frac{1}{2}\rho^2 = \frac{1}{2}\bar{\rho}^2\}$



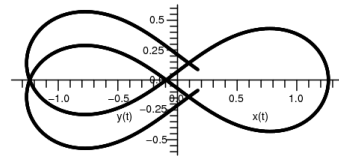
(a) $k < 1$ slightly



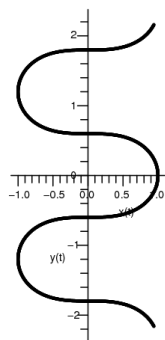
(b) k further decays but still larger than k^* . Drift slows down



(c) $k = k^*$. No drift; trajectory becomes periodic



(d) $k < k^*$. Drift now is down!

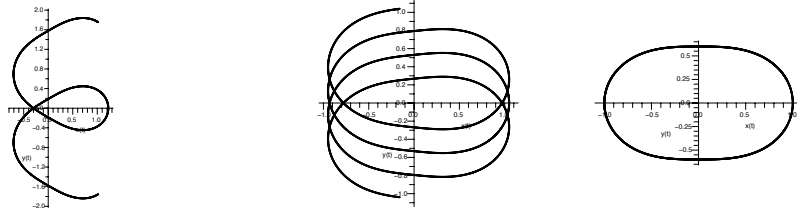


(e) k decays further. Drift down accelerates.



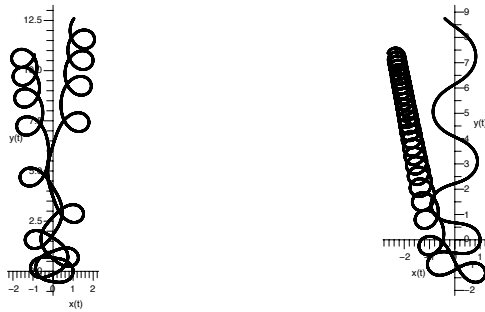
(f) and further; as $k = -1$ we have a straight line down

Figure 4: Even ν ; movements in the inner zone



(a) $k < 1$ slightly. Drift is up and the fastest
 (b) k decays further. Drift up slows down Trajectory is periodic further
 (c) $k = 0$. No drift.

Figure 5: Odd ν ; movements in the inner zone



(a) Even ν

(b) Odd ν

Figure 6: Breaking periodic trajectories by a linear potential

- combined in the zone $\{|x_1| \leq c\rho^2\}$ with the movement similar to one described in 2D case in (ρ, θ) -coordinates (with $\{x_1 = 0\}$ now equivalent to $\{\rho = \bar{\rho}\}$) on the surface of this ellipsoid
- and also combined some movement along x_2 ;
- I did not consider zone $\{|x_1| \geq c\rho^2\}$ since it was not needed for the spectral asymptotics.

One can find details in section 1, [Ivr8].

3 Quantum Dynamics

Microlocal canonical form (*Birkhoff normal form*) play a crucial role in the analysis of the quantum dynamics and spectral asymptotics.

3.1 Canonical forms. I

In the case $d = 2$ and a full-rank magnetic field canonical form of magnetic Schrödinger operator is (one half of)

$$(24) \quad \omega_1(x_1, \mu^{-1}hD_1)(h^2D_2^2 + \mu^2x_2^2) - W(x_1, \mu^{-1}hD_1) + \sum_{m+k+l \geq 2} a_{mkl}(x_1, \mu^{-1}hD_1)(h^2D_2^2 + \mu^2x_2^2)^m \mu^{2-2m-2k-l} h^l$$

with $\omega_j = f_j \circ \Psi$, $W = V \circ \Psi$ with some map Ψ . The first line is *main part* of the canonical form.

In the case $d = 3$ and a maximal-rank magnetic field microlocal canonical form of magnetic Schrödinger operator is one half of

$$(25) \quad \omega_1(x_1, x_2, \mu^{-1}hD_2)(h^2D_3^2 + \mu^2x_3^2) + h^2D_1^2 - W(x_1, x_2, \mu^{-1}hD_2) + \sum_{m+n+k+l \geq 2} a_{mnkl}(x_2, \mu^{-1}hD_2)(h^2D_3^2 + \mu^2x_3^2)^m D_1^n \times \mu^{2-2m-2k-l-n} h^{l+n}$$

Again, the first line is *main part* of the canonical form.

In the case $d \geq 4$ and a constant rank magnetic field microlocal canonical form of Magnetic Schrödinger operator is of the similar type provided we can avoid some obstacles:

If f_j have constant multiplicities (say, f_j are simple, for the sake of simplicity) then the main part is

$$(26) \quad \sum_{1 \leq j \leq r} \omega_j(x', x'', \mu^{-1}hD'')(h^2D_{r+q+j}^2 + \mu^2x_{r+q+j}^2) + h^2D'^2 - W(x', x'', \mu^{-1}hD'');$$

where $x' = (x_1, \dots, x_q)$, $x'' = (x_{q+1}, \dots, x_{q+r})$, $2r = \text{rank } F$, $q = d - 2r$.

Next terms appear if one can avoid higher order *resonances*: $\sum_j p_j f_j(x) = 0$ with $p_j \in \mathbb{Z}$; $3 \leq \sum_j |p_j|$ is called *the order of the resonance*.

After operator is reduced to the canonical form one can decompose functions as

$$(27) \quad u(x) = \sum_{\alpha \in \mathbb{Z}^{+r}} u_{\alpha}(x', x'') \Upsilon_{p_1}(x_{r+q+1}) \cdots \Upsilon_{p_r}(x_d)$$

where Υ are eigenfunctions of harmonic oscillator $\hbar^2 D^2 + \mu^2 x^2$ (i.e. scaled Hermite functions).

Then as $2r = d$ one gets a family of r -dimensional $\mu^{-1}\hbar$ -PDOs and for $2r < d$ one gets a family of q -dimensional Schrödinger operators with potentials which are r -dimensional $\mu^{-1}\hbar$ -PDOs.

The similar approach also works for 2 and 4-dimensional Schrödinger operators with the degenerate magnetic field of the types I considered before but only in the *far outer zone* $\{\gamma(x) \stackrel{\text{def}}{=} |x_1| \gg \mu^{-1/\nu}\}$ and to this form operator is reduced in balls $B(\bar{x}, \frac{1}{2}\gamma(\bar{x}))$.

3.2 Canonical forms. II

As $d = 2, 4$ and magnetic field degenerates there is also a more global canonical form.

As $d = 2$ in zone $\{|x_1| \ll 1\}$ this form is (after multiplication by some non-vanishing function)

$$(28) \quad \hbar^2 D_1^2 + (\hbar D_2 - \mu x_1^{\nu}/\nu)^2 - W(x) + \text{perturbation}$$

with $W = V\phi^{-2/(\nu+1)}$ (if originally $f_1 \sim \phi \text{dist}(x, \Sigma)^{\nu-1}$, $\Sigma = \{x_1 = 0\}$).

For $d = 4$ one can separate a cyclotron part corresponding to the non-vanishing eigenvalue f_2 ; after this one gets a 3-dimensional second-order DO (+ perturbation) with the principal part which is the quadratic form of rank 2 and a free term $V - (2\alpha + 1)\mu\hbar f_2$ where $\alpha \in \mathbb{Z}^+$ is a corresponding *magnetic quantum number*.

This operator could be reduced to the form similar to (28) (at least away from $\Lambda = \{x_3 = x_4 = 0\}$); here $W = (V - (2\alpha + 1)\mu\hbar f_2)\phi^{-2/3}$.

Close to Λ but as $|x_1| \leq C\rho^2$ one can get a similar form but with θ instead of x_2 and $\mu\rho$ instead of μ .

3.3 Periodic orbits

One can prove that semiclassical quantum dynamics follows the classical one long enough to recover sharp remainder estimates but the notion of *periodic*

orbit should be adjusted to reflect *logarithmic uncertainty principle*

$$(29) \quad |\text{osc}(x)| \cdot |\text{osc}(D)| \geq C\hbar |\log \hbar|$$

where \hbar is effective Plank constant (it could be h or $\mu^{-1}h$ or one of them scaled depending on the particular situation).

I need a logarithm because I am interesting in the *size of the box* outside of which function is negligible rather than in the mean quadratic deviation. Function $\exp(-|x|^2/2\hbar)$ scaled shows that *boxing* requires a logarithmic factor.

So instead of individual trajectories I consider their beams satisfying logarithmic uncertainty principle. One can see that the classical trajectory is not periodic but cannot say this about the semiclassical beam until much larger time, after few “periods”.

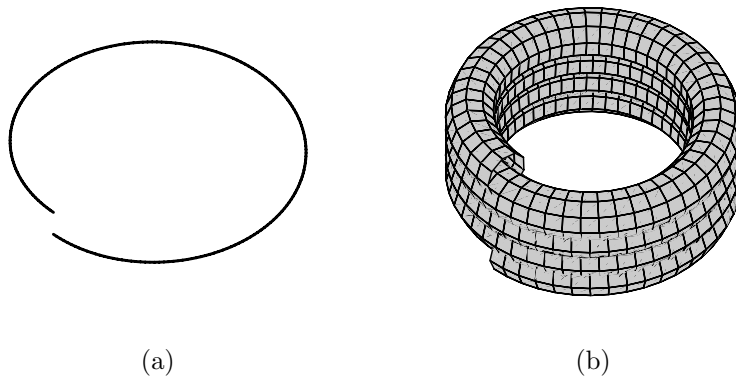


Figure 7: Classical (a) and semiclassical (b) periodicity.

4 Spectral asymptotics

4.1 Tauberian method

Problem I am looking at asymptotics as $h \rightarrow +0$, $\mu \rightarrow +\infty$ of

$$(30) \quad \Gamma(Qe)(0) = \int (e(\cdot, \cdot, 0)Q_y^t)_{x=y} dy = \text{Tr}(QE(0))$$

where $e(x, y, \tau)$ is the Schwartz kernel of the spectral projector $E(\tau)$ of operator H and Q is a pseudo-differential operator, Q^t means a dual operator.

As $Q = I$ we get $\text{Tr } E(0)$ which is the number of negative eigenvalues of H (and $+\infty$ if there is an essential spectrum of H below 0).

I hope to construct this expression (30) with $Q = I$ from itself for elements of the partition of unity with self-adjoint $Q \geq 0$.

Tauberian method *Tauberian method Fourier* says that the *main part* of $\Gamma(eQ_y^t)$ is given by expression

$$(31) \quad h^{-1} \int_{-\infty}^0 \left(F_{t \rightarrow h^{-1}\tau} \bar{\chi}_T(t) \Gamma(uQ_y^t) \right) d\tau$$

while the *remainder* does not exceed $C \frac{M}{T} + C'h^s$ where

$$(32) \quad M = M_T = \sup_{|\tau| \leq \epsilon} \left| \left(F_{t \rightarrow h^{-1}\tau} \bar{\chi}_T(t) \Gamma(uQ_y^t) \right) \right|$$

and s is large, C does not depend on ϵ, T, h, μ and s while C' depends on $\epsilon > 0, T, s$.

Here and below $u(x, y, t)$ is the Schwartz kernel of the propagator $e^{ih^{-1}tH}$, $\bar{\chi} \in C_0^\infty([-1, 1])$ equal 1 at $[-\frac{1}{2}, \frac{1}{2}]$, $\chi \in C_0^\infty([-1, 1])$ equal 0 at $[-\frac{1}{2}, \frac{1}{2}]$, $\chi_T(t) = \chi(t/T)$, $T > 0$ and $F_{t \rightarrow h^{-1}\tau}$ is h -Fourier transform.

Actually this remainder estimate persists if one replaces T by any larger number T' *only* in expression (31).

So, I want to increase T without (significantly) increasing M_T in (32).

Evil of periodic trajectories *Microlocal analysis* says that if there are no periodic trajectories with periods in $[\frac{T}{2}, T]$ on energy levels in $[-2\epsilon, 2\epsilon]$ then

$$(33) \quad \sup_{|\tau| \leq \epsilon} \left| \left(F_{t \rightarrow h^{-1}\tau} \chi_T(t) \Gamma(uQ_y^t) \right) \right| \leq C'h^s.$$

Therefore if there are no periodic trajectories with periods in $[T, T']$ on energy levels in $[-2\epsilon, 2\epsilon]$, then one can retain T in (31), M_T in (32) but the remainder estimate improves to $C \frac{M}{T'} + C'h^s$.

So, periodic trajectories are one of the main obstacles in getting a good remainder estimate. For example, if all trajectories are periodic with the

period $T = T_{\square}$ then it can happen that $M_T \asymp TT_{\square}^{-1}M_{T_{\square}}$ as $T \geq T_{\square}$ and increasing T does not bring any improvement.

For example, let $\mu \leq 1$, $V \asymp 1$. Then there are no periodic trajectories with periods in $[T_0, T_1]$, $T_0 = Ch|\log h|$ and $T_1 = \epsilon$ because $\text{dist}(x(t), x(0)) \asymp T$ as $T \leq T_1$ and this distance is observable as $T \geq T_0$.

Then $M \leq Ch^{-d}T_0 = Ch^{1-d}|\log h|$ and the remainder estimate is $O(h^{1-d}|\log h|)$. Actually one can get rid off this log factor in $M = Ch^{1-d}$ and the remainder estimate becomes $O(h^{1-d})$. This remainder estimate cannot be improved without geometric assumptions of the global nature.

Also, taking T really small in (31) allows us to calculate u and (31) by a crude *successive approximation method* with an unperturbed operator H having coefficients frozen as $x = y$.

As $\mu \geq 1$ the same arguments are true but $T_1 = \epsilon\mu^{-1}$ and the remainder estimate is $O(\mu h^{1-d})$. This remainder estimate cannot be improved as $d = 2$, g^{jk} , f_1 and V are constant.

In our arguments: because all trajectories are periodic (pure cyclotronic movement).

From direct calculations: as domain is \mathbb{R}^2 all eigenvalues are *Landau levels* $(\alpha + \frac{1}{2})\mu h - \frac{1}{2}V$ of infinite multiplicity ($\alpha \in \mathbb{Z}^+$) and

$$(34) \quad e(x, x, \tau) = \frac{1}{2\pi} \sum_{n \geq 0} \theta(2\tau + V - (2n + 1)\mu hf) \mu h^{-1} f \sqrt{g}$$

with jumps $\asymp \mu h^{-1}$ at Landau levels.

However in many cases one can improve remainder estimate $O(\mu h^{1-d})$. The idea is to show that actually periodicity is broken.

From the point of view of applications one should take Q with support (with respect to x) in ball $B(0, \frac{1}{2})$ (then rescaling arguments could be applied) and impose condition on operator *only in the circle of light* $B(0, 1)$ with the self-adjointness being the *only* condition outside of it.

So anything out of $B(0, 1)$ is a *dark territory* and we *must* take $T \leq T^*$ which is the time for which trajectory which started from $\text{supp } Q$ remains in $B(0, 1)$. But we can chose the time direction and we can chose it for every beam individually.

Now, as $d = 3$ the typical trajectory is non-periodic because of the free movement and

- one must take $T \leq T^* \asymp 1$;

- but for most of the trajectories one can take $T_1 \asymp T^*$ retaining $T_0 = Ch|\log h|$

and the remainder estimate is $O(h^{1-d})$ (under very mild assumptions).

Let $d = 2$ and f_1 do not vanish. Then

- since the drift speed is $O(\mu^{-1})$ one can take $T \asymp \mu$; *under certain non-degeneracy conditions breaking periodicity of the cyclotronic movement* one can take $T_1 = T^*$ retaining $T_0 = Ch|\log h|$

and the remainder estimate is $O(\mu^{-1}h^{1-d})$.

When tamed, our worst enemy (periodic trajectories) could become our best friend!

4.2 Results: Constant-rank case

Results: “Constant” case

Theorem 1. *Let g^{jk}, F_{jk} and V be constant and domain be \mathbb{R}^d . Then*

$$(35) \quad \mathcal{E}_d^{\text{MW}}(x, E) \stackrel{\text{def}}{=} \Omega_{d-2r}(2\pi)^{-d+r} \mu^r h^{-d+r} \times \\ \sum_{\alpha \in \mathbb{Z}^{+r}} \left(2E + V - \sum_j (2\alpha_j + 1) f_j \mu h - V \right)_+^{\frac{d}{2}-r} f_1 \cdots f_r \sqrt{g}$$

where Ω_k is a volume of unit ball in \mathbb{R}^k .

In particular, the spectrum is pure point if and only if $r = 0$.

One can prove this theorem easily by direct calculations. *Magnetic Weyl Expression* $cE_d^{\text{MW}}(x, E)$ becomes our candidate for the main part of asymptotics in the general case.

Results: $d = 2$ As $d = 2$ formula (35) provides a good approximation and the non-degeneracy condition below breaks periodicity and provides a good remainder estimate [Ivr7]:

Theorem 2. *Let $d = 2$ and g^{jk}, F_{jk}, V be smooth in $B(0, 1)$, f_1 non-vanishing there and $\psi \in C_0^\infty(B(0, 1))$. Let assume that all critical values of V/f_1 are non-degenerate. Then*

$$(36) \quad \left| \int (e(x, x, 0) - \mathcal{E}_d^{\text{MW}}(x, 0)) \psi(x) dx \right| \leq C \mu^{-1} h^{1-d}$$

as $\mu \leq ch^{-1}$.

Remark 3. (i) If in the general case magnetic field spoils remainder estimate $O(\mu h^{1-d})$, but in the covered case magnetic field improves it;

(ii) Estimate (36) holds in multidimensional full-rank case as well but non-degeneracy condition is pretty complicated and is not generic [Ivr3];

(iii) As $\epsilon_1 \leq \mu h \leq c$ nondegeneracy condition changes; as $d = 2$ it reads: $(2\alpha + 1)\mu h$ is not a degenerate critical value of V/f for any $\alpha \in \mathbb{Z}^+$;

(iv) As $f_1 + \dots + f_r \geq \epsilon > 0$, $e(x, x, 0)$ is negligible and $\mathcal{E}_d^{\text{MW}} = 0$ for $\mu \geq ch^{-1}$.

Results: Non full-rank case Let us assume that $\text{rank } F$ is constant but less than d . In this case remainder estimate cannot be better than $O(h^{1-d})$ but it also cannot be much worse [Ivr4]:

Theorem 4. Let g^{jk}, F_{jk}, V be smooth in $B(0, 1)$, $\text{rank}(F) = 2r$, $0 < 2r < d$ so f_1, \dots, f_r do not vanish there and $\psi \in C_0^\infty(B(0, 1))$. Then

(i) As either $2r = d - 1$ and some very mild non-degeneracy condition is fulfilled or $2r = d - 2$ or $\mu \leq h^{\delta-1}$ with $\delta > 0$

$$(37) \quad \left| \int (e(x, x, 0) - \mathcal{E}_d^{\text{MW}}(x, 0)) \psi(x) dx \right| \leq Ch^{1-d};$$

(ii) As $2r = d - 1$ the left-hand expression does not exceed $C\mu h^{2-\delta-d} + Ch^{1-d}$ with arbitrarily small $\delta > 0$.

4.3 Results: Degenerating 2D case

Consider case $d = 2$, $f_1 \asymp |x_1|^{\nu-1}$ with $\nu \geq 2$ assuming that

$$(38) \quad V \geq \epsilon_0 > 0.$$

We consider ϵ -vicinity of $\{x_1 = 0\}$ with small enough constant $\epsilon > 0$.

Then in the *outer zone* $\{\bar{\gamma} = C\mu^{-1/\nu} \leq |x_1| \leq \epsilon\}$ there is a drift with the speed $\mu^{-1}\gamma^{-\nu}$, this drift breaks periodicity and therefore contribution of the strip $\{|x_1| \asymp \gamma\}$ with $\gamma \in (\bar{\gamma}, \epsilon)$ to the remainder estimate does not exceed $Ch^{1-d} \times \gamma \times \mu^{-1}\gamma^{-\nu}$ where the second factor is the width of the strip and the third one is the inverse ‘‘control time’’. Then the total contribution of the outer zone to the remainder estimate does not exceed the same expression as $\gamma = \bar{\gamma}$ which is $C\mu^{-1/\nu}h^{1-d}$; this is our best shot.

In the *inner zone* $\{|x_1| \leq \bar{\gamma}\}$ or equivalently $\{|\xi_2| \leq C_0\}$ the similar arguments work as long as $\rho \asymp |\xi_2 - k^* V^{1/2}| \geq \epsilon$.

Otherwise there seems to be no drift to save the day. However it is not that bad. Really, period in x_1 is $\asymp \bar{\gamma}$ and if

$$(39) \quad |\xi_2 - k^* V^{1/2}| \asymp \rho$$

the speed of the drift is $\asymp \rho$, the shift with respect to x_2 is $\asymp \rho \bar{\gamma}$ and in order to be observable it must satisfy logarithmic uncertainty principle $\rho \bar{\gamma} \times \rho \geq Ch |\log h|$ because characteristic scale in ξ_2 is ρ now. So, periodicity is broken provided

$$(40) \quad \rho \geq \bar{\rho}_1 = C(\bar{\gamma}^{-1} h |\log h|)^{\frac{1}{2}},$$

which leaves us with much smaller *periodic zone*

$$(41) \quad \mathcal{Z}_{\text{per}} = \{|\xi_2 - k^* V^{1/2}| \leq \bar{\rho}_1\}.$$

And in this periodic zone picking up $T_1 \asymp \bar{\gamma}$ we can derive remainder estimate $O(h^{-1} \bar{\rho}_1)$ which does not exceed our dream estimate $C\mu^{-1/\nu} h^{-1}$ provided $\bar{\rho}_1 \leq \bar{\gamma}$ or

$$(42) \quad \mu \leq C(h |\log h|)^{-\frac{\nu}{3}}.$$

Actually for the general operator rather than the model one we need to assume that $\rho \geq C\bar{\gamma}$ but this does not spoil our dream estimate.

So we need to consider periodic zone defined by (41) assuming that (42) does not hold.

Inside of Periodic Zone Even in the periodic zone \mathcal{Z}_{per} periodicity of trajectories can be broken as $W = V\phi^{-2/(\nu+1)}|_{x_1=0}$ is “variable enough” which leads us [Ivr5, Ivr6] to

Theorem 5. *Let $d = 2$, $f_1 = \phi(x)|x_1|^{\nu-1}$ with $\nu \geq 2$ and condition (38) be fulfilled. Then as $\psi \in C_0^\infty(B(0, 1) \cap \{|x_1| \leq \epsilon\})$*

$$(43) \quad \left| \int (e(x, x, 0) - \mathcal{E}_d^{\text{MW}}(x, 0)) \psi(x) dx \right| \leq C(\mu^{-1/\nu} + \hbar^{(q+1)/2}) h^{1-d}$$

where here and below $\hbar = \mu^{1/\nu} h$, $q = 0$ in the general case and $q = 1$ under assumption “ W does not have degenerate critical points”.

To improve this remainder estimate one should take in account the short periodic trajectories. Actually, periodicity of the trajectories close to them is broken but only after time $T_0 = C\rho^{-2}h|\log h|$ (see our discussion in subsection 3.3). Skipping details [Ivr5, Ivr6]

Theorem 6. *Let $d = 2$, $f_1 = \phi(x)|x_1|^{\nu-1}$ with $\nu \geq 2$ and condition (38) be fulfilled. Then as $\psi \in C_0^\infty(B(0, 1) \cap \{|x_1| \leq \epsilon\})$*

$$(44) \quad \left| \int (\mathbf{e}(x, x, 0) - \mathcal{E}_d^{\text{MW}}(x, 0))\psi(x) dx - \int \mathcal{E}_{\text{corr}}^{\text{MW}}(x_2, 0)\psi(0, x_2) dx_2 \right| \leq C\mu^{-1/\nu}h^{1-d}$$

provided either some very mild nondegeneracy condition is fulfilled or $\mu \leq ch^{\delta-\nu}$ where $\mathcal{E}_{\text{corr}}^{\text{MW}}(x_2, 0)$ is defined below.

Remark 7. (i) In the case we are considering right now (and no other case considered in this article) condition $f_1 + \dots + f_r \geq \epsilon_0$ fails and therefore $\mathbf{e}(x, x, 0)$ is not negligible as $\mu \geq ch^{-1}$;

(ii) On the other hand, $\mathbf{e}(x, x, 0)$ is negligible as $\mu \geq ch^{-\nu}$;

(iii) As $ch^{-1} \leq \mu \leq ch^{-\nu}$, $\mathcal{E}^{\text{MW}}(x, 0)$ is supported in the strip $\{|x_1| \leq \bar{\gamma}_1 \stackrel{\text{def}}{=} C_0(\mu h)^{-1/(\nu-1)}\}$ where $\bar{\gamma}_1 \geq \bar{\gamma} = c\mu^{-1/\nu}$; therefore the main part of the spectral asymptotics (after integration) is of magnitude $(\mu h)^{-1/(\nu-1)}h^{-d}$.

In theorem 6 the correction term is defined by

$$(45) \quad \mathcal{E}_{\text{corr}}^{\text{MW}}(x_2, \tau) = (2\pi h)^{-1} \int \mathbf{n}_0(\tau; x_2, \xi_2, \hbar) d\xi_2 - \int \mathcal{E}_0^{\text{MW}}(\tau; x_1, x_2, \hbar) dx_1$$

where \mathbf{n}_0 is an eigenvalue counting function for an auxillary 1D-operator

$$(46) \quad \mathbf{a}_0(x_2, \xi_2, \hbar) = \frac{1}{2} \left(\hbar^2 D_1^2 + (\xi_2 - x_1'/\nu)^2 - W(x_2) \right)$$

and $\mathcal{E}_0^{\text{MW}}$ is magnetic Weyl approximation for the related 2-dimensional operator.

Using Bohr-Sommerfeld approximation one can find eigenvalues of $\mathbf{a}_0(x, \xi_2, \hbar)$ with $O(\hbar^s)$ precision and $\mathcal{E}_{\text{corr}}^{\text{MW}}(x_2, \tau)$ with $O(\hbar^{-1}\hbar^s)$ precision. In particular, modulo $O(\hbar^{-1}\hbar) = O(\bar{\gamma}^{-1})$

$$(47) \quad \mathcal{E}_{\text{corr}}^{\text{MW}}(x_2, 0) \equiv \varkappa h^{-1} \hbar^{\frac{1}{2}} W^{\frac{1}{4} - \frac{1}{4\nu}} G \left(\frac{S_0 W^{\frac{1}{2} + \frac{1}{2\nu}}}{2\pi \hbar} \right)$$

with some constants \varkappa and S_0 and function G defined by

$$(48) \quad G(t) = \int_{\mathbb{R}} \left(t + \frac{1}{2}\eta^2 - \lfloor t + \frac{1}{2}\eta^2 + \frac{1}{2} \rfloor \right) d\eta$$

with the converging integral in the right-hand expression. One can prove easily that

$$(49) \quad G \neq 0, \quad G(t+1) = G(t), \quad \int_0^1 G(t) dt = 0, \quad G \in C^{\frac{1}{2}}.$$

This is one of examples of the short periodic trajectories really contributing to the asymptotics.

4.4 Results: Degenerating 4D case

4D case is way more complicated than 2D one. But there are some good news: since $f_1 + f_2 \geq \epsilon$ in the generic case, we need to consider only $\mu \leq ch^{-1}$.

The main difficulty in 4D case comes from the *outer zone* $\{\bar{\gamma} = C\mu^{-1/\nu} \leq |x_1| \leq \epsilon\}$ because there could be short periodic trajectories. In other words: Landau level $(2\alpha_1 + 1)\mu hf_1 + (2\alpha_2 + 1)\mu hf_2$ could be flat 0. Still in contrast to the very general case when this can happen for up to $(\mu h)^{-1}$ pairs $\alpha \in \mathbb{Z}^{+2}$, in the assumptions of theorem below it can happen only for no more than C pairs and I was able to prove [Ivr8]:

Theorem 8. *Let F is of Martinet-Roussarie type and condition (38) be fulfilled. Then as ψ is supported in $B(0, 1) \cap \{|x_1| \leq \epsilon\}$*

$$(50) \quad \left| \int (e(x, x, 0) - \mathcal{E}_4^{\text{MW}}(x, 0))\psi(x) dx \right| \leq C\mu^{-1/2}h^{-3} + C\mu^2h^{-2}.$$

One can improve this result under extra condition breaking flat Landau levels [Ivr8]:

Theorem 9. *In frames of above theorem assume that $(V/f_2)_{x_1=0}$ does not have degenerate critical points. Then as ψ is supported in $B(0, 1) \cap \{|x_1| \leq \epsilon\}$*

$$(51) \quad \left| \int (e(x, x, 0) - \mathcal{E}_4^{\text{MW}}(x, 0) - \mathcal{E}_{\text{corr}}^{\text{MW}}(x, 0))\psi(x) dx \right| \leq C\mu^{-1/2}h^{-3}.$$

Here $\mathcal{E}_{\text{corr}}^{\text{MW}} = O(\mu^{5/4}h^{-3/2})$ is associated with periodic zone $\{|x_1| \leq c\mu^{-1/2}\}$, and is the sum of similar expressions in 2D case for $V_\beta = V - (2\beta + 1)\mu hf_2$ with $\beta \in \mathbb{Z}^+$; locally all of them but one could be dropped.

4.5 Results: Nondegenerating 4D case revisited

As I mentioned, even if magnetic field does not degenerate, non-degeneracy condition of [Ivr3] is not generic (it defines open but not everywhere dense set). Recently I was able to prove [Ivr9]

Theorem 10. *Let us consider a 4D Schrödinger operator with non-degenerate generic magnetic field. Then*

- (i) *For generic potential V asymptotics holds with the principal part as in (51) and the remainder estimate $O(\mu^{-1}h^{-3})$;*
- (ii) *For general potential V asymptotics holds with the principal part as in (50) and the remainder estimate $O(\mu^{-1}h^{-3} + \mu^2h^{-2})$; correction term could be skipped with no penalty unless $h^{-1/3+\delta} \leq \mu \leq h^{-1/3-\delta}$ in which case it can be skipped with the penalty $O(h^{-8/3-\delta})$ where $\delta > 0$ is arbitrarily small.*

(added December 7, 2007)

All results here are proven in the series of articles at

<http://www.math.toronto.edu/ivrii/Research/Preprints.php>

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