Centre de Mathématiques Laurent Schwartz
Ecole Polytechnique

SEMINAIRE

Equations aux Dérivées Partielles

2005-2006

Karine Beauchard

Controllability of Schrödinger equations

Séminaire É. D. P. (2005-2006), Exposé n° IX, 18 p.

<http://sedp.cedram.org/item?id=SEDP_2005-2006____A9_0>

U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

Controllability of Schrödinger equations

Karine Beauchard
CMLA ENS Cachan
94230 Cachan *

Abstract

One considers a quantum particle in a 1D moving infinite square potential well. It is a nonlinear control system in which the state is the wave function of the particle and the control is the acceleration of the potential well. One proves the local controllability around any eigenstate, and the steady state controllability (controllability between eigenstates) of this control system. In particular, the wave function can be moved from one eigenstate to another one, exactly and in finite time, by moving the potential well in a suitable way.

The proof uses moment theory, a Nash-Moser theorem, Coron's return method and expansions to the second order.

This article summarizes two works : [4] and a joint work with Jean-Michel Coron [5].

1 Introduction

1.1 The system

A quantum particle, in a 1D space, is represented by its wave function

$$\begin{array}{rccccc} \phi: & \mathbb{R} & \times & \mathbb{R} & \to & \mathbb{C} \\ & (t & , & z) & \mapsto & \phi(t,z). \end{array}$$

The physical meaning of $|\phi(t,z)|^2 dz$ is the probability of the particle to be in the volume dz surrounding the point z at time t. Thus, at any time t, one has

$$\int_{\mathbb{R}} |\phi(t,z)|^2 dz = 1.$$

When the particle is in a potential $V(z)$, its wave function solves the Schrödinger equation

$$i\hbar \frac{\partial \phi}{\partial t}(t,z) = -\frac{\hbar^2}{2m}\frac{\partial^2 \phi}{\partial z^2}(t,z) + V(z)\phi(t,z).$$

Let us consider a particle in an infinite square potential well

$$V(z) = 0 \text{ when } z \in I := (-1/2, 1/2), V(z) = +\infty \text{ when } z \notin I$$

*Karine.Beauchard@cmla.ens-cachan.fr

which moves in $\mathbb{R}$ along time. Up to renormalisation, the wave function solves

$$i\frac{\partial \phi}{\partial t}(t,z) = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2}(t,z) + V(z - D(t))\phi(t,z),$$

where $D(t)$ is the position of the potential well. The change of independent variables $(t,z) \to (t,q)$ and wave function $\phi \to \psi$, defined by

$$q := z - D(t),$$

$$\psi(t,q) := \phi(t,z)e^{i((D-z)\dot{D} - \frac{1}{2}\int_0^t \dot{D}^2)},$$

transforms the previous equation into

$$(\Sigma) \begin{cases} i\frac{\partial \psi}{\partial t}(t,q) = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2}(t,q) - u(t)q\psi(t,q), q \in I, t \in \mathbb{R}_+, \\ \psi(t, \pm 1/2) = 0, \end{cases}$$

where $u := -\ddot{D}$ is the acceleration of the well (see [20]). This equation gives the dynamic of the wave function in the moving system of reference. The system (Σ) is a control system in which

- the state is the wave function ψ of the particle, that, for every $t \in \mathbb{R}_+$, belongs to the $L^2(I,\mathbb{C})$-unitary sphere $\mathcal{S}$,
- the control is the acceleration $t \mapsto u(t) \in \mathbb{R}$ of the well.

This control system is nonlinear : it is bilinear with respect to the couple (ψ, u).

1.2 Main results of this article

In order to state the main results of this article, let us introduce the definition of a solution of (Σ), the definition of a trajectory of (Σ) and few notations.

Definition 1 *Let $T_1, T_2 \in \mathbb{R}$ with $T_1 < T_2$, $u : [T_1, T_2] \to \mathbb{R}$ be a continuous function, $\psi_0 \in H^2 \cap H_0^1(I,\mathbb{C}) \cap \mathcal{S}$. A function $\psi : [T_1,T_2] \times I \to \mathbb{C}$ is a solution of*

$$\begin{cases} i\frac{\partial \psi}{\partial t}(t,q) = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2}(t,q) - u(t)q\psi(t,q), q \in I, t \in \mathbb{R}_+, \\ \psi(t, \pm 1/2) = 0, \\ \psi(T_1, q) = \psi_0(q), q \in I, \end{cases} \tag{1}$$

if

- *$\psi \in C^0([T_1,T_2], H^2 \cap H_0^1(I,\mathbb{C})) \cap C^1([T_1,T_2], L^2(I,\mathbb{C}))$,*
- *the first equality of (1) holds in $L^2(I,\mathbb{C})$ for every $t \in [T_1,T_2]$,*
- *the third equality of (1) holds in $H^2 \cap H_0^1(I,\mathbb{C})$.*

Then, one says that (ψ, u) is a trajectory of the control system (Σ).

One introduces the operator A defined on

$$D(A) := (H^2 \cap H^1_0)(I,\mathbb{C}), \text{ by } A\varphi := -\frac{1}{2}\frac{d^2\varphi}{dq^2}.$$

The eigenvectors and eigenvalues of this operator are, for $k \in \mathbb{N}^*$,

$$\varphi_k(q) := \begin{cases} \sqrt{2}\sin(k\pi q), & \text{when } k \text{ is even}, \\ \sqrt{2}\cos(k\pi q), & \text{when } k \text{ is odd}, \end{cases} \qquad \lambda_k := \tfrac{1}{2}(k\pi)^2.$$

One calls “eigenstates”the particular solution of the free system ($u \equiv 0$ in (Σ))

$$\psi_k(t,q) := \varphi_k(q) e^{-i\lambda_k t}$$

and “ground state”the eigenstate corresponding to $k = 1$. One introduces the spaces

$$H^s_{(0)}(I,\mathbb{C}) := D(A^{s/2})$$

for $s \in \mathbb{R}^*_+$ and the unitary $L^2(I,\mathbb{C})$-sphere

$$\mathcal{S} := \{\varphi \in L^2(I,\mathbb{C}); \|\varphi\|_{L^2(I,\mathbb{C})} = 1\}.$$

In [4], one proves the local controllability of the system (Σ) around the ground state for $u \equiv 0$. This behavior was conjectured by Rouchon in [20]. The same result holds with the same proof around any eigenstate.

Theorem 1 *Let $\phi_0, \phi_1 \in \mathbb{R}$. There exist $T > 0$ and $\eta > 0$ such that, for every ψ_0, ψ_f in $\mathcal{S} \cap H^7_{(0)}(I,\mathbb{C})$ satisfying*

$$\|\psi_0 - \varphi_1 e^{i\phi_0}\|_{H^7(I,\mathbb{C})} < \eta, \quad \|\psi_f - \varphi_1 e^{i\phi_1}\|_{H^7(I,\mathbb{C})} < \eta,$$

there exists a trajectory (ψ, u) of the control system (Σ) on $[0,T]$ such that $\psi(0) = \psi_0$, $\psi(T) = \psi_f$ and $u \in H^1_0((0,T),\mathbb{R})$.

In [5], we study the same physical system, but we control not only the wave function ψ of the particle but also the position D and the speed S of the potential well. Thus, the studied control system is

$$(\Sigma_0) \begin{cases} i\frac{\partial\psi}{\partial t}(t,q) = -\frac{1}{2}\frac{\partial^2\psi}{\partial q^2}(t,q) - < u(t), q > \psi(t,q), q \in (-1/2,1/2), t \in \mathbb{R}_+, \\ \psi(t,\pm 1/2) = 0, \\ \dot{S} = u, \\ \dot{D} = S, \end{cases}$$

in which

- the state is $(\psi, S, D) \in \mathcal{S} \times \mathbb{R} \times \mathbb{R}$
- the control is the real valued function $t \mapsto u(t)$.

The main result of [5] is the following one.

Theorem 2 *For every $n_0, n_f \in \mathbb{N}^*$, there exists a time $\mathcal{T} > 0$ and a trajectory (ψ, S, D, u) of (Σ_0) on $[0, \mathcal{T}]$ such that $(\psi(0), S(0), D(0)) = (\varphi_{n_0}, 0, 0)$, $(\psi(\mathcal{T}), S(\mathcal{T}), D(\mathcal{T})) = (\varphi_{n_f}, 0, 0)$, and $u \in H_0^1((0, \mathcal{T}), \mathbb{R})$.*

Therefore, one can change the energy level of the particle by moving the potential well in a suitable way.

In section 2, one states a previous non controllability result for the system (Σ), due to Turinici [21], with arguments from Ball Marsden and Slemrod [1]. One explains why a non controllability result and a positive controllability result can hold simultaneously for the same equation.

Section 3 presents the proof of Theorem 1 in [4]. In subsection 3.1, one explains why the classical linear test does not work. This is the first difficulty of this work. Subsection 3.2 presents the strategy developed to solve this difficulty, it relies on Coron's return method and quasi-static deformations. In subsections 3.3, 3.4, 3.5 and 3.6, one details the steps of the this strategy. Coron's return method needs the local controllability of (Σ) around some trajectory $(\widetilde{\psi}, \widetilde{u})$. This local controllability is proved thanks to the linear test. Unfortunately, the classical inverse mapping theorem does not give the conclusion because of a loss of regularity. This is the second difficulty of this work. One concludes thanks to the Nash-Moser theorem. Finally, subsection 3.7 contains remarks, conjectures and open problems dealing with Theorem 1.

Section 4 presents the steps of the proof of Theorem 2 in [5]. This proof relies on many local controllability results, got thanks to similar arguments as in section 3. However, an additional difficulty appears, and we use expansions to the second order.

Finally, others PDEs have the same pathology as (Σ) : they are known to be not controllable (in some functional spaces) thanks to the argument of [1], but affirmative controllability results have been proved (in other functional spaces) with the technic introduced for the study of (Σ). Such results are mentioned in section 5

2 A previous non controllability result for (Σ)

In [1], Ball, Marsden and Slemrod discuss the controllability of infinite dimensional bilinear control systems of the form

$$\dot{w}(t) = \mathcal{A}w(t) + p(t)\mathcal{B}(w(t)), \tag{2}$$

where the state is w and the control is p. Thanks to Baire lemma, they prove the following non controllability result.

Theorem 3 *Let X be a Banach space with $dim(X) = +\infty$. Let $\mathcal{A}$ generate a C^0-semi group of bounded linear operators on X and $\mathcal{B} : X \to X$ be a bounded linear operator. Let $w_0 \in X$ be fixed and let $w(t; p, w_0)$ denote the unique solution of (2) for $p \in L^1_{loc}([0, +\infty), \mathbb{R})$ with $w(0) = w_0$. The set of states accessible from w_0 defined by*

$$S(w_0) := \{w(t; p, w_0); t \geqslant 0, p \in L^r_{loc}([0, \infty), \mathbb{R}), r > 1\}$$

is contained in a countable union of compact subsets of X and, in particular, has dense complement.

As noticed by Turinici in [21], Theorem 3 shows that, for (Σ), given $\psi_0 \in X := \mathcal{S} \cap H^2_{(0)}((0,1),\mathbb{C})$, the set of $\psi(t)$ in X accessible from the initial condition ψ_0, by using controls $u \in L^r_{loc}([0,\infty),\mathbb{R})$, $r > 1$, has dense complement in X. Thus, the system (Σ) is not controllable in $\mathcal{S} \cap H^2_{(0)}(I,\mathbb{C})$, with control functions u in $H^1_0((0,T),\mathbb{R})$, $T > 0$.

However, there is no obstruction for having controllability in other spaces. For example, Theorem 3 does not apply with

$$\widetilde{X} := H^3_{(0)}(I,\mathbb{C})$$

instead of X because the operator $\mathcal{B}$, defined by $\mathcal{B}\varphi := q\varphi$, does not map $\widetilde{X}$ into $\widetilde{X}$. Thus, there is no obstruction for controllability to hold in $H^3_{(0)}(I,\mathbb{C})$ with controls $u \in L^2_{\text{loc}}([0,+\infty),\mathbb{C})$.

In this article, one proves local controllability results in $H^7_{(0)}(I,\mathbb{C})$, with control functions u in $H^1_0((0,T),\mathbb{R})$, with $T > 0$. Thus, the negative result proved by G. Turinici relies on a choice of functional spaces which does not allow controllability. In order to state affirmative controllability results, one must

- either control ψ in $H^2_{(0)}(I,\mathbb{C})$ but with a control functions set larger than $L^2((0,T),\mathbb{R})$, for example $H^{-1}((0,T),\mathbb{R})$,
- or control ψ using the control functions set $L^2((0,T),\mathbb{R})$, but in a smaller space than $H^2_{(0)}(I,\mathbb{C})$, for example $H^3_{(0)}(I,\mathbb{C})$.

3 Local controllability of (Σ) around the ground state

3.1 Failure of the linear test

A classical approach to get local controllability around a trajectory consists in proving the controllability of the linearized control system around this trajectory and concluding thanks to an inverse mapping theorem. But this classical approach does not work : Rouchon proved in [20] that, around any state of definite energy, the linear tangent approximate system is not controllable.

Indeed, the linearized control system around the eigen state ψ_k is

$$(\Sigma_{L,k}) \begin{cases} i\frac{\partial \Psi}{\partial t}(t,q) = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial q^2}(t,q) - v(t)q\psi_k(t,q), q \in I, t \in \mathbb{R}_+, \\ \Psi(t,\pm 1/2) = 0, \end{cases}$$

It is a control system in which

- the state is the function Ψ, that belongs, at any time t to the tangent space $T_{\mathcal{S}}\psi_k(t)$ to the sphere $\mathcal{S}$, at the point $\psi_k(t)$,

- the control is the real valued function $t \mapsto v(t)$.

The non controllability of $(\Sigma_{L,k})$ is clear : if k is odd (resp. even), and if the initial condition Ψ_0 satisfies $\Psi_0(-x) = -\Psi_0(x)$ (resp. $\Psi_0(-x) = \Psi_0(x)$) for every $x \in I$, then, $\Psi(t)$ has the same parity property for every t, whatever u is. In particular, one cannot reach any final state Ψ_f that does not satisfy $\Psi_f(-x) = -\Psi_f(x)$ (resp. $\Psi_f(-x) = \Psi_f(x)$) for every $x \in I$, so $(\Sigma_{L,k})$ is not controllable. Therefore, **the main reason why $(\Sigma_{L,k})$ is not controllable is the parity of the eigenvectors of A.**

3.2 Strategy : return method and quasi-static deformations

The proof of Theorem 1 relies on Coron's return method and quasi-static deformations. This method was introduced by Coron in [6] in order to solve a stabilization problem. It has been used in order to get controllability results for partial differential equations by Coron in [9], [7], [8], by Coron and Fursikov in [12], by Fursikov and Imanuvilov in [13], by Glass in [14], [15], and by Horsin in [17].

This method is in two steps. First, one finds a trajectory $(\widetilde{\psi}, \widetilde{u})$ of the control system (Σ) such that the linearized control system around $(\widetilde{\psi}, \widetilde{u})$ is controllable in time T. Then, using an implicit function theorem, one gets the local controllability in time T of the nonlinear system (Σ) around $(\widetilde{\psi}(0), \widetilde{\psi}(T))$: there exist neighborhoods V_0 of $\widetilde{\psi}(0)$ and V_T of $\widetilde{\psi}(T)$ such that the system (Σ) can be moved in time T from any point $\xi \in V_0$ to any point $\zeta \in V_T$.

In a second step, given two points ψ_0, ψ_f closed enough to $\varphi_1 e^{i\phi_0}$, $\varphi_1 e^{i\phi_1}$, one proves that the system (Σ) can be moved

- from ψ_0 to a point $\xi \in V_0$, using quasi-static deformations,
- from one point $\zeta \in V_T$ to ψ_f, using again quasi-static deformations,
- from ξ to ζ using the local controllability around $(\widetilde{\psi}(0), \widetilde{\psi}(T))$.

The trajectory $(\widetilde{\psi}, \widetilde{u})$ used in [4] is the ground state for a constant acceleration $u \equiv \gamma$, $\gamma \in \mathbb{R}^*$,

$$\psi_{1,\gamma}(t, q) := \varphi_{1,\gamma} e^{-i\lambda_{1,\gamma} t}.$$

Here, $\lambda_{1,\gamma}$ is the first eigenvalue of the operator A_γ defined on

$$D(A_\gamma) := (H^2 \cap H_0^1)(I, \mathbb{C}), \text{ by } A_\gamma \varphi := -\frac{1}{2}\frac{d^2\varphi}{dq^2} - \gamma q \varphi.$$

and $\varphi_{1,\gamma}$ is an associated eigenvector.

In section 3.3, one justifies the controllability of the linearized system around $(\psi_{1,\gamma}, u \equiv \gamma)$, for $\gamma \in \mathbb{R}^*$ small enough. Unfortunately, the classical inverse mapping theorem is not sufficient to conclude the local controllability of (Σ) around $\psi_{1,\gamma}$, because of a loss of regularity in the controllability of the linearized system. This is explained in section 3.4. One deals with this difficulty by applying a Nash-Moser theorem. The local controllability result proved, by doing this, is stated in section 3.5. In section 3.6, one gives explicitly the quasi-static deformations used in the second step of the return method. Finally, in section 3.7, one gives some remarks, open problems and conjectures about Theorem 1

3.3 Controllability of the linearized system around the trajectory $(\psi_{1,\gamma}, u \equiv \gamma)$

Let $\gamma \neq 0$. The linearized system around $(\psi_{1,\gamma}, u \equiv \gamma)$ is

$$(\Sigma_{L,\gamma}) \begin{cases} i\frac{\partial \Psi}{\partial t}(t,q) = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial q^2}(t,q) - (\gamma + v)(t) q \psi_{1,\gamma}(t,q), q \in I, t \in \mathbb{R}_+, \\ \Psi(t, \pm 1/2) = 0, \end{cases}$$

It is a control system in which

- the state is the function Ψ, with, for every t, $\Psi(t) \in T_{\mathcal{S}}\psi_{1,\gamma}(t)$,
- the control is the real valued function $t \mapsto v(t)$.

In [4], one proves the following controllability result for $(\Sigma_{L,\gamma})$.

Proposition 1 *There exists $\gamma^* > 0$ such that, for every $\gamma \in (-\gamma^*, \gamma^*) - \{0\}$, for every $T > 0$, the linear system $(\Sigma_{L,\gamma})$ is controllable in $H^3_{(0)}(I, \mathbb{C})$ with control functions $v \in L^2((0,T), \mathbb{R})$.*

Proof : Let us introduce the nondecreasing sequence of eigenvalues of A_γ, $(\lambda_{n,\gamma})_{n \in \mathbb{N}^*}$ and associated orthonormal eigenvectors $(\varphi_{n,\gamma})_{n \in \mathbb{N}^*}$. Since $(\Sigma_{L,\gamma})$ is a linear control system, one can take $\Psi(0) = 0$. Then, for every $n \in \mathbb{N}^*$, one has

$$< \Psi(t), \varphi_n >= i < q\varphi_{1,\gamma}, \varphi_{n,\gamma} > e^{-i\lambda_{n,\gamma} t} \int_0^t v(t) e^{i(\lambda_{n,\gamma} - \lambda_{1,\gamma})\tau} d\tau,$$

where $< .,. >$ denotes the $L^2(I, \mathbb{C})$-scalar product.

Let $T > 0$ and $\Psi_f \in T_{\mathcal{S}}\psi_{1,\gamma}(T)$. We look for $v : [0,T] \to \mathbb{R}$ such that $\Psi(T) = \Psi_f$. This equality is equivalent to the following **trigonometric moment problem** on v,

$$\frac{< \Psi_f, \varphi_n > e^{i\lambda_{n,\gamma} T}}{i b_{n,\gamma}} = \int_0^T v(\tau) e^{i(\lambda_{n,\gamma} - \lambda_{1,\gamma})\tau} d\tau, \forall n \in \mathbb{N}^*, \tag{3}$$

where $b_{n,\gamma} :=< q\varphi_{1,\gamma}, \varphi_{n,\gamma} >$. Thanks to the analyticity of the functions $\gamma \mapsto \lambda_{n,\gamma}$ and $\gamma \mapsto \varphi_{n,\gamma}$, one proves the following proposition.

Proposition 2 *There exists $\gamma_0 > 0$ such that, for every $\gamma \in (0, \gamma_0]$ and for every $k \in \mathbb{N}^*$, $b_{k,\gamma} \neq 0$.*
There exist $\gamma_1 > 0$ and $C > 0$ such that, for every $\gamma \in (0, \gamma_1]$ and for every even integer $k \geqslant 2$,

$$|b_{k,\gamma} - \frac{(-1)^{\frac{k}{2}+1} 8k}{\pi^2 (k^2-1)^2}| < \frac{C\gamma}{k^3},$$

and for every odd integer $k \geq 3$,

$$|b_{k,\gamma} - \gamma \frac{2(-1)^{\frac{k-1}{2}}(k^2+1)}{\pi^4 k (k^2-1)^2}| < \frac{C\gamma^2}{k^3}.$$

Reasoning as in [18, chap 1.2], with the Ingham inequality of [16], one gets the existence of a solution $v \in L^2((0,T),\mathbb{R})$ to the moment problem (3) as soon as $\Psi_f \in D(A_\gamma^{3/2})$ and $T > 0$. $\Box$

The controllability of the linearized system around $(\psi_{1,\gamma}, u \equiv \gamma)$ is possible when $\gamma \neq 0$ because **the introduction of a parameter $\gamma \neq 0$ breaks the parity properties of the eigenvectors of the operator A_γ**.

3.4 The inverse mapping theorem does not work

Unfortunately, the controllability result for the linearized system around $(\psi_{1,\gamma}, u \equiv \gamma)$ of Proposition 1 is not sufficient to get the local controllability of the nonlinear system (Σ) around $\psi_{1,\gamma}$ by applying the classical inverse mapping theorem. Indeed, the map Φ which associates to any couple of initial condition and control (ψ_0, v) the couple of initial and final conditions (ψ_0, ψ_T) for the system (Σ) with $u = \gamma + v$, is well defined and of class C^1 between the following spaces,

$$\begin{array}{lccccccc} \Phi : & [S \cap D(A_\gamma^{3/2})] & \times & H^1((0,T),\mathbb{R}) & \to & [S \cap D(A_\gamma^{3/2})] & \times & [S \cap D(A_\gamma^{3/2})] \\ & (\psi_0 & , & v) & \mapsto & (\psi_0 & , & \psi_T). \end{array}$$

Its differential map $d\Phi(\varphi_{1,\gamma}, 0)$ at the point $(\varphi_{1,\gamma}, 0)$ maps the space

$$E := [T_S(\varphi_{1,\gamma}) \cap D(A_\gamma^{3/2})] \times H^1((0,T),\mathbb{R})$$

into the space

$$F := [T_S(\psi_{1,\gamma}(0)) \cap D(A_\gamma^{3/2})] \times [T_S(\psi_{1,\gamma}(T)) \cap D(A_\gamma^{3/2})],$$

where $T_S(\xi)$ is the tangent space to the L^2-sphere $\mathcal{S}$ at the point ξ. It admits a right inverse, $d\Phi(\varphi_{1,\gamma}, 0)^{-1}$, but it does not map F into E : it only maps F into

$$[T_S(\psi_{1,\gamma}(0)) \cap D(A_\gamma^{3/2})] \times L^2((0,T),\mathbb{R}).$$

One looses regularity in the controllability of the linearized system around $(\psi_{1,\gamma}, u \equiv \gamma)$

3.5 Application of the Nash-Moser theorem

One deals with this loss of regularity by using a **Nash-Moser implicit function theorem** adapted from [19] and one gets the following theorem.

Theorem 4 *Let $T := 4/\pi$. There exists $\gamma^* > 0$ such that, for every $\gamma \in (0, \gamma^*)$, there exists $\eta > 0$ such that, for every $(\psi_0, \psi_T) \in \mathcal{S} \cap D(A_\gamma^{7/2})$ satisfying*

$$\|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7(I,\mathbb{C})} \leqslant \eta, \quad \|\psi_T - \psi_{1,\gamma}(T)\|_{H^7(I,\mathbb{C})} \leqslant \eta,$$

there exists a trajectory (ψ, u) of the control system (Σ) such that $\psi(0) = \psi_0$, $\psi(T) = \psi_T$ and $(u - \gamma) \in H_0^1((0,T),\mathbb{R})$.

The application of the Nash-Moser is rather technical so we refer to [4] for the proof of this theorem.

It is at this step of the proof that the regularity assumption H^7 appears : in order to prove the convergence of the Nash iterations, one needs regularity on the initial and final conditions.

The Nash-Moser theorem is useful because one works on Sobolev spaces. However, it is perhaps possible to prove a local controllability result for (Σ) by applying the inverse mapping theorem, but one would need to use other functional spaces.

3.6 Quasi-static deformations

In the second step of the return method, one constructs explicitly, for $\gamma > 0$ small enough, trajectories

$$(\psi, u) : [0, T^1] \to H^7(I, \mathbb{C}) \times \mathbb{R}$$

such that

$$u(0) = 0, \quad u(T^1) = \gamma, \quad \psi(0) = \varphi_1 e^{i\phi_0}, \quad \psi(T^1) \in D(A_\gamma^{7/2}),$$
$$\|\psi(T^1) - \varphi_{1,\gamma}\|_{H^7} < \eta/2.$$

Then, for $\psi_0 \in D(A^{7/2})$ closed enough to $\varphi_1 e^{i\phi_0}$, the same control moves the system from ψ_0 to ξ which satisfies

$$\xi \in D(A_\gamma^{7/2}) \text{ and } \|\xi - \varphi_{1,\gamma}\|_{H^7} < \eta,$$

thanks to the continuity with respect to initial conditions. The control used in this step moves slowly from 0 to γ : one makes **quasi-static deformations**, as in [9]. More precisely, one proves the following theorem in [4].

Theorem 5 *Let $\gamma_0 \in \mathbb{R}$. One considers the solution ψ_ϵ of the following system*

$$\begin{cases} i\dot{\psi}_\epsilon = -\frac{1}{2}\psi_\epsilon'' - \gamma_0 f(\epsilon t) q \psi_\epsilon, \\ \psi_\epsilon(0) = \varphi_1 e^{i\phi_0}, \\ \psi_\epsilon(t, -1/2) = \psi_\epsilon(t, 1/2) = 0, \end{cases}$$

where $f \in C^\infty([0,1], \mathbb{R})$ satisfies $f^{(k)}(0) = 0$ for every $k \in \mathbb{N}$, $f(1) = 1$, $0 \leqslant f \leqslant 1$ and $\phi_0 \in [0, 2\pi)$. Let $(\epsilon_n)_{n \in \mathbb{N}^*}$ be defined by

$$\frac{1}{\epsilon_n} \int_0^1 \lambda_{1,\gamma_0 f(t)} dt = \phi_0 + 2n\pi,$$

for every $n \in \mathbb{N}^*$. There exists $\gamma^* > 0$ such that, for every $\gamma_0 \in (-\gamma^*, \gamma^*)$, for every $s \in \mathbb{N}$, $(\psi_{\epsilon_n}(1/\epsilon_n))_{n \in \mathbb{N}}$ converges to φ_{1,γ_0} in $H^s(I, \mathbb{C})$.

3.7 Remarks, open problems, conjectures

The proof given in [4] gives the controllability of (Σ) in H^7 with control functions in H_0^1. The exponent 7 is only technical and related to the application of the Nash-Moser theorem. With the same strategy and another version of the Nash-Moser theorem (see [3] or [2]), it is now possible to prove the same theorem with everywhere $H^7_{(0)}(I,\mathbb{C})$ replaced by $H^{5+\epsilon}_{(0)}(I,\mathbb{C})$, $\epsilon > 0$.

One conjectures that the optimal results for (Σ) are the local controllability

- in H^3 with control functions $u \in L^2$,
- in H^5 with control functions $u \in H^1_0$,
- in H^7 with control functions $u \in H^2_0$, etc.

Indeed, these are the optimal results for the controllability of the linearized system around $(\psi_{1,\gamma}, u \equiv \gamma)$.

In theorem 4, one proves the local controllability of (Σ) around $\psi_{1,\gamma}$ in time $T = 4/\pi$. This assumption is probably only technical too. Since the linearized system around $(\psi_{1,\gamma}, u \equiv \gamma)$ is controllable in any positive time (see Proposition 1), one expects the non linear system to be locally controllable around $\psi_{1,\gamma}$ also in any positive time. However, this point is an open problem.

Another interesting question about (Σ) is the existence of a minimal time for the local controllability around ψ_1. An affirmative answer is given in [10], the value of this minimal time is still open.

4 Steady-state controllability of (Σ)

The main result of [5] is a little bit stronger than Theorem 2. It is the following one.

Theorem 6 *For every $n \in \mathbb{N}^*$, there exists $\eta_n > 0$ such that, for every $n_0, n_f \in \mathbb{N}^*$, for every $(\psi_0, S_0, D_0), (\psi_f, S_f, D_f) \in [\mathcal{S} \cap H^7_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R}$ with*

$$\|\psi_0 - \varphi_{n_0}\|_{H^7} + |S_0| + |D_0| < \eta_{n_0}, \quad \|\psi_f - \varphi_{n_f}\|_{H^7} + |S_f| + |D_f| < \eta_{n_f},$$

there exists a time $\mathcal{T} > 0$ and a trajectory (ψ, S, D, u) of (Σ_0) on $[0, \mathcal{T}]$, which satisfies $(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0)$, $(\psi(\mathcal{T}), S(\mathcal{T}), D(\mathcal{T})) = (\psi_f, S_f, D_f)$ and $u \in H^1_0((0,\mathcal{T}),\mathbb{R})$.

Theorem 2 is a corollary of this theorem. In the following subsections, one details the steps of the proof of Theorem 6, in [5].

4.1 Global strategy

Thanks to the reversibility of the control system (Σ_0), in order to get Theorem 6, it is sufficient to prove it with $n_f = n_0 + 1$. We prove it with $n_0 = 1$ and $n_f = 2$ to simplify the notations.

First, we prove the local controllability of (Σ_0) around the trajectory $(Y^{\theta,0,0}, u \equiv 0)$ for every $\theta \in [0,1]$, where

$$Y^{\theta,0,0}(t) := (\psi_\theta(t), S(t) \equiv 0, D(t) \equiv 0),$$

$$\psi_\theta(t) := \sqrt{1-\theta}\psi_1(t) + \sqrt{\theta}\psi_2(t) \text{ for } \theta \in (0,1),$$

$$Y^{k,0,0}(t) = (\psi_{k+1}(t), S(t) \equiv 0, D(t) \equiv 0) \text{ for } k = 0, 1.$$

Thus we know that

- there exists an open ball V_0 (resp. V_1) centered at $Y^{0,0,0}(0)$ (resp. $Y^{1,0,0}$) such that (Σ_0) can be moved in finite time between any two points in V_0 (resp. V_1),
- for every $\theta \in (0,1)$, there exists an open ball V_θ centered at $Y^{\theta,0,0}(0)$ such that (Σ_0) can be moved in finite time between any two points in V_θ.

Then, we conclude thanks to a compactness argument : the segment

$$[Y^{0,0,0}(0), Y^{1,0,0}(0)] := \{\sqrt{\lambda}Y^{0,0,0}(0) + \sqrt{1-\lambda}Y^{1,0,0}(0); \lambda \in [0,1]\}$$

is compact in $\mathcal{S}\times\mathbb{R}\times\mathbb{R}$ and covered by $\cup_{0\leqslant\theta\leqslant 1} V_\theta$ thus there exists a increasing finite family $(\theta_n)_{1\leqslant n\leqslant N}$ such that $[Y^{0,0,0}(0), Y^{1,0,0}(0)]$ is covered by $\cup_{1\leqslant n\leqslant N} V_{\theta_n}$. We can assume $V_{\theta_n} \cap V_{\theta_{n+1}} \neq \emptyset$ for $n = 1, ..., N-1$. Given $Y_0 \in V_{\theta_1}$ and $Y_f \in V_{\theta_N}$, we move (Σ_0) from Y_0 to a point $Y_1 \in V_{\theta_1} \cap V_{\theta_2}$ in finite time, from Y_1 to a point $Y_2 \in V_{\theta_2} \cap V_{\theta_3}$ in finite time...etc and we reach Y_f in finite time.

Now, let us explain the proof of the local controllability of (Σ_0) around $Y^{\theta,0,0}$ for every $\theta \in [0,1]$. The strategy for $\theta \in (0,1)$ is different from the one for $\theta \in \{0,1\}$ but involves the same ideas. In the next section, one details the case $\theta \in (0,1)$ which is the simplest one. One refers to [5] for the case $\theta \in \{0,1\}$.

4.2 Local controllability of (Σ_0) around $Y^{\theta,0,0}$ for $\theta \in (0,1)$

In this subsection, one presents the steps of the proof of the following theorem.

Theorem 7 *Let $\theta \in (0,1)$. Let $T := 4/\pi$. There exists $\eta > 0$ such that, for every $(\psi_0, S_0, D_0), (\psi_f, S_f, D_f) \in [\mathcal{S} \cap H^7_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R}$ with*

$$\|\psi_0 - \psi_\theta(0)\|_{H^7} + |S_0| + |D_0| < \eta,$$

$$\|\psi_f - \psi_\theta(T)\|_{H^7} + |S_f| + |D_f| < \eta,$$

there exists a trajectory (ψ, S, D) of (Σ_0) on $[0, 2T]$ such that

$$(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0),$$

$$(\psi(2T), S(2T), D(2T)) = (\psi_f, S_f, D_f),$$

and $u \in H^1_0((0,2T),\mathbb{R})$.

4.2.1 Controllability up to codimension one of the linearized system around $Y^{\theta,0,0}$ for $\theta \in (0,1)$

First, let us notice that the linear test is not sufficient to conclude because the linearized system around $(Y^{\theta,0,0}(t), u \equiv 0)$ is not controllable. Indeed, one has the following proposition.

Proposition 3 *Let* $\theta \in (0,1)$.*Let* $T>0$ *and* (Ψ, s, d) *be a trajectory of*

$$(\Sigma_\theta^l) \begin{cases} i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial q^2} - wq\psi_\theta, \\ \Psi(t, \pm 1/2) = 0, \\ \dot{s} = w, \\ \dot{d} = s. \end{cases}$$

on $[0,T]$. *Then, the function*

$$t \mapsto \Im(< \Psi(t), \sqrt{1-\theta}\psi_1(t) - \sqrt{\theta}\psi_2(t) >)$$

is constant on $[0,T]$. *Thus, the control system* (Σ_θ^l) *is not controllable.*

Proof : Let us consider the function $\xi_\theta(t) := \sqrt{1-\theta}\psi_1(t) - \sqrt{\theta}\psi_2(t)$. We have

$$i\frac{\partial \xi_\theta}{\partial t} = -\frac{1}{2}\frac{\partial^2 \xi_\theta}{\partial q^2},$$

$$\frac{d}{dt}\left(\Im < \Psi(t), \xi_\theta(t) >\right) = \Im(iw < q\psi_\theta(t), \xi_\theta(t) >).$$

The explicit expressions of ψ_θ and ξ_θ provide, for every t,

$$< q\psi_\theta(t), \xi_\theta(t) > \in i\mathbb{R},$$

which gives the conclusion.

Let $T>0$, and $\Psi_0 \in T_{\mathcal{S}}(\psi_\theta(0))$, $\Psi_f \in T_{\mathcal{S}}(\psi_\theta(T))$. A necessary condition for the existence of a trajectory of (Σ_θ^l) satisfying $\Psi(0) = \Psi_0$ and $\Psi(T) = \Psi_f$ is

$$\Im(< \Psi_f, \sqrt{1-\theta}\psi_1(T) - \sqrt{\theta}\psi_2(T) >) = \Im(< \Psi_0, \sqrt{1-\theta}\varphi_1 - \sqrt{\theta}\varphi_2 >).$$

This equality does not happen for an arbitrary choice of Ψ_0 and Ψ_f. Thus (Σ_θ^l) is not controllable.□

However, the previous proposition provides the only invariant quantity for the linear system (Σ_θ^l). Indeed, one has the following proposition.

Proposition 4 *Let* $\theta \in (0,1)$. *Let* $T>0$, $(\Psi_0, s_0, d_0), (\Psi_f, s_f, d_f) \in H^3_{(0)}(I, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ *be such that*

$$\Re < \Psi_0, \psi_\theta(0) >= \Re < \Psi_f, \psi_\theta(T) >= 0, \quad (4)$$

$$\Im < \Psi_f, \sqrt{1-\theta}\varphi_1 e^{-i\lambda_1 T} - \sqrt{\theta}\varphi_2 e^{-i\lambda_2 T} >= \Im < \Psi_0, \sqrt{1-\theta}\varphi_1 - \sqrt{\theta}\varphi_2 > . \quad (5)$$

There exists $w \in L^2((0,T), \mathbb{R})$ *such that the solution of* (Σ_θ^l) *with control* w *and such that* $(\Psi(0), s(0), d(0)) = (\Psi_0, s_0, d_0)$ *satisfies* $(\Psi(T), s(T), d(T)) = (\Psi_f, s_f, d_f)$.

Remark 1 *The condition (5) means that we miss exactly two directions, which are* $(\Psi, s, d) = (\pm i\xi_\theta, 0, 0)$. *Thus, if we want to control the components* $< \Psi, \varphi_k >$ *for* $k \geqslant 2$ *and* $\Re < \Psi, \varphi_1 >$ *then, we cannot control* $\Im < \Psi, \varphi_1 >$. *This is why we say that we miss the two directions* $(\Psi, s, d) = (\pm i\varphi_1, 0, 0)$. *We call this situation "controllability up to codimension one".*

4.2.2 Local controllability up to codimension one of (Σ_0) around $Y^{\theta,0,0}$ for $\theta \in (0,1)$

Let us introduce the following closed subspace of $L^2(I,\mathbb{C})$

$$V := \overline{\text{Span}\{\varphi_k; k \geqslant 2\}}$$

and the orthogonal projection $\mathcal{P} : L^2(I,\mathbb{C}) \to V$.

We want to get from the previous proposition the local controllability up to codimension one of the nonlinear system (Σ_0) around $Y^{\theta,0,0}$. Again, because of a loss of regularity, the inverse mapping theorem is not sufficient to conclude and we use a Nash-Moser theorem. Since we need a continuity property in the end of the proof of Theorem 7, which is not given in [19], we adapt the theorem and the proof of [19]. By doing this, we get the following theorem.

Theorem 8 *Let $\theta \in (0,1)$. Let $T := 4/\pi$. There exists $C > 0$, $\delta > 0$ and a continuous map*

$$\begin{array}{llcll} \Gamma : & \mathcal{V}(0) & \times & \mathcal{V}(T) & \to & H^1_0((0,T),\mathbb{R}) \\ & ((\psi_0, S_0, D_0) & , & (\widetilde{\psi_f}, S_f, D_f)) & \mapsto & u \end{array}$$

where

$$\mathcal{V}(0) := \{(\psi_0, S_0, D_0) \in [\mathcal{S} \cap H^7_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R}; \|\psi_0 - \psi_\theta(0)\|_{H^7} + |S_0| + |D_0| < \delta\},$$

$$\mathcal{V}(T) := \{(\widetilde{\psi_f}, S_f, D_f) \in [H^7_{(0)}(I,\mathbb{C}) \cap V \cap B_{L^2}(0,1)] \times \mathbb{R} \times \mathbb{R}; \|\widetilde{\psi_f} - \mathcal{P}\psi_\theta(T)\|_{H^7} + |S_f| + |D_f| < \delta\},$$

such that, for every $((\psi_0, S_0, D_0), (\widetilde{\psi_f}, S_f, D_f)) \in \mathcal{V}(0) \times \mathcal{V}(T)$, the trajectory of (Σ_0) with control $\Gamma(\psi_0, S_0, D_0, \widetilde{\psi_f}, S_f, D_f)$ such that $(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0)$ satisfies

$$(\mathcal{P}\psi(T), S(T), D(T)) = (\widetilde{\psi_f}, S_f, D_f)$$

and

$$\begin{aligned} \|\Gamma(\psi_0, S_0, D_0, \widetilde{\psi_f}, S_f, D_f)\|_{H^1_0((0,T),\mathbb{R})} \leqslant\ & C[\|\mathcal{P}(\psi_0 - \psi_\theta(0))\|_{H^7(I,\mathbb{C})} + |S_0| + |D_0| + \\ & \|\widetilde{\psi_f} - \mathcal{P}\psi_\theta(T)\|_{H^7(I,\mathbb{C})} + |S_f| + |D_f|]. \end{aligned}$$

4.2.3 Motion in the directions $(\psi, S, D) = (\pm i\varphi_1, 0, 0)$ for the second order term

In [5], we prove the following proposition, which is the key point to prove that the nonlinear system (Σ_0) can also be moved in the to directions which are not given by Theorem 8.

Theorem 9 *Let $\theta \in (0,1)$. Let $T := 4/\pi$. There exists $w_\pm \in H^4 \cap H^3_0((0,T),\mathbb{R})$, $\nu_\pm \in H^3_0((0,T),\mathbb{R})$ such that the solutions of*

$$\begin{cases} i\dot{\Psi}_\pm = -\frac{1}{2}\Psi''_\pm - w_\pm q \psi_\theta, \\ \Psi_\pm(0) = 0, \\ \Psi_\pm(t,-1/2) = \Psi_\pm(t,1/2) = 0, \\ \dot{s}_\pm = w_\pm, s_\pm(0) = 0, \\ \dot{d}_\pm = s_\pm, d_\pm(0) = 0, \end{cases} \tag{6}$$

$$
\begin{cases}
i\dot{\xi}_\pm = -\frac{1}{2}\xi_\pm'' - w_\pm q\Psi_\pm - \nu_\pm q\psi_\theta, \\
\xi_\pm(0) = 0, \\
\xi_\pm(t,-1/2) = \xi_\pm(t,1/2) = 0, \\
\dot{\sigma}_\pm = \nu_\pm, \sigma_\pm(0) = 0, \\
\dot{\delta}_\pm = \sigma_\pm, \delta_\pm(0) = 0,
\end{cases} \tag{7}
$$

satisfy $\Psi_\pm(T) = 0$, $s_\pm(T) = 0$, $d_\pm(T) = 0$, $\xi_\pm(T) = \pm i\varphi_1$, $\sigma_\pm = 0$, $\delta_\pm = 0$.

We detail in the next subsection how this result on the second order term allows to conclude for the nonlinear system.

This strategy has already been used by Coron and Crepeau in [11], for KdV equation. In their case, the second order term was not sufficient, they used the third one.

4.2.4 Proof of Theorem 7

In all this section $T := 4/\pi$. Let $\rho \in \mathbb{R}$, $\psi_0, \psi_f \in H^7_{(0)}(I,\mathbb{C})$, $S_0, D_0, S_f, D_f \in \mathbb{R}$. Let us consider, for $t \in [0,T]$

$$u(t) := \sqrt{|\rho|}w + |\rho|\nu,$$

where $w := w_+, \nu := \nu_+$ if $\rho \geqslant 0$ and $w := w_-, \nu := \nu_-$ if $\rho \leqslant 0$ and $w_\pm, \nu_\pm$ are defined in Theorem 9. Let (ψ, S, D) be the solution of (Σ_0) on $[0,T]$ with control u and such that

$$(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0).$$

Then, we have

$$S(T) = S_0, \quad D(T) = D_0.$$

We have $u \in W^{3,1}((0,T),\mathbb{R})$ and $u(0) = u(T) = \dot{u}(0) = \dot{u}(T) = 0$ thus (see [4, Appendix B Proposition 51]) the function $\psi(T)$ belongs to $H^7_{(0)}(I,\mathbb{C})$.

In [5], we prove the following proposition.

Proposition 5 *There exists $\mathcal{C} > 0$ such that, for every $\rho \in (-1,1)$, we have*

$$\|\psi(T) - (\psi_\theta(T) + i\rho\varphi_1)\|_{H^7(I,\mathbb{C})} \leqslant \mathcal{C}[\|\psi_0 - \psi_\theta(0)\|_{H^7(I,\mathbb{C})} + |\rho|^{3/2}].$$

Now, we use the local controllability up to codimension one around Y_θ. Let $\delta > 0$ be as in Theorem 8. We assume

$$\|\psi_0 - \psi_\theta(0)\|_{H^7(I,\mathbb{C})} < \frac{\delta}{4\mathcal{C}},$$

$$|S_0| + |D_0| < \frac{\delta}{2},$$

$$\|\mathcal{P}[\psi_f - \psi_\theta(2T)]\|_{H^7} + |S_f| + |D_f| < \delta.$$

When ρ satisfies

$$|\rho| < \eta := \min\{1; \frac{\delta}{4(\|\varphi_1\|_{H^7} + \mathcal{C})}\},$$

the previous proposition proves that

$$\|\psi(T)-\psi_\theta(0)\|_{H^7} \leqslant (\|\varphi_1\|_{H^7}+\mathcal{C})|\rho|^{3/2}+\frac{\delta}{4}<\frac{\delta}{2}.$$

Thus $(\psi(T),S_0,D_0)\in\mathcal{V}(0)$ and $(\mathcal{P}\psi_f,S_f,D_f)\in\mathcal{V}(T)$. Thanks to Theorem 8, there exists

$$\tilde{u} := \Gamma(\psi(T),S_0,D_0,\mathcal{P}\psi_f,S_f,D_f)\in H_0^1((T,2T),\mathbb{R})$$

such that

$$(\mathcal{P}\psi(2T),S(2T),D(2T))=(\mathcal{P}\psi_f,S_f,D_f),$$

where (ψ,S,D) is the solution of (Σ_0) with control u on $[0,2T]$, with u extended to $[0,2T]$ by $u:=\tilde{u}$ on $[T,2T]$. The Theorem 8 and the previous proposition give the existence of a constant C such that

$$\|u\|_{H^1((T,2T),\mathbb{R})} \leqslant C[|\rho|^{3/2}+\|\psi_0-\psi_\theta(0)\|_{H^7}+|S_0|+|D_0|+\|\mathcal{P}(\psi_f-\psi_\theta(2T))\|_{H^7}+|S_f|+|D_f|]. \tag{8}$$

We define the map

$$\begin{array}{rccc} F: & (-\eta,\eta) & \to & \mathbb{R} \\ & \rho & \mapsto & \Im(<\psi(2T),\varphi_1>). \end{array}$$

Thanks to Theorem 8, F is continuous on $(-\eta,\eta)$. We can assume δ is small enough so that

$$\Re(<\psi(2T),\varphi_1>)>0,$$

because ψ is closed enough to ψ_θ. Since $\psi\in\mathcal{S}$ and $\Re(<\psi(2T),\varphi_1>)$ is positive, we have

$$\psi(2T)=\psi_f \text{ if and only if } F(\rho)=\Im(<\psi_f,\varphi_1>).$$

Therefore, in order to get Theorem 7, it is sufficient to prove that F is surjective on a neighborhood of 0.

Let $x(t):=<\psi(t),\varphi_1>$ on $[T,2T]$. We have

$$x(2T)=x(T)+i\int_T^{2T}u(t)<q\psi(t),\varphi_1>e^{i\lambda_1 t}dt.$$

Thus

$$F(\rho)=\rho+[\Im(x(T))-\rho]+\Im\left(i\int_T^{2T}u(t)<q\psi(t),\varphi_1>e^{i\lambda_1 t}\right),$$

where

$$|\Im(x(T))-\rho|\leqslant\|\psi(T)-(\psi_\theta(T)+i\rho)\|_{L^2}\leqslant C[|\rho|^{3/2}+\|\psi_0-\psi_\theta(0)\|_{H^7}],$$

$$|\int_T^{2T}u(t)<q\psi(t),\varphi_1>e^{i\lambda_1 t}dt|\leqslant T\|u\|_{L^\infty((T,2T),\mathbb{R})}.$$

Using (8), we get the existence of a constant K such that

$$|F(\rho)-\rho|\leqslant K[|\rho|^{3/2}+\|\psi_0-\psi_\theta(0)]\|_{H^7}+\|\mathcal{P}[\psi_f-\psi_\theta(2T)]\|_{H^7}+|S_f|+|D_f|+|S_0|+|D_0|].$$

There exists $\tau \in (0,\eta)$ such that

$$K|\tau|^{3/2} < \frac{\tau}{3}.$$

Let us assume that

$$K[\|\psi_0 - \psi_\theta(0)]\|_{H^7} + \|\mathcal{P}[\psi_f - \psi_\theta(2T)]\|_{H^7} + |S_f| + |D_f| + |S_0| + |D_0|] < \frac{\tau}{3}.$$

Then

$$F(\tau) > \frac{\tau}{3} \text{ and } F(-\tau) < -\frac{\tau}{3},$$

thus the intermediate values theorem guarantees that F is surjective on a neighborhood of zero, this ends the proof of Theorem 7.

4.3 Remark, open problem

Theorem 2 states a steady-state controllability result in long time. No bound for the time of control is known. The existence of a minimal time for such motions is an open problem.

5 The same technic on other PDEs

The technic introduced for the study of (Σ) gave positive controllability results for others PDEs. Note that these PDEs were known to be not controllable in particular functional spaces, thanks to the argument of [1]. One proves their controllability in other spaces.

The first PDE is the following 1D beam equation

$$(\mathcal{P}) \begin{cases} u_{tt} + u_{xxxx} + p(t)u_{xx} = 0, x \in (0,1), t \in \mathbb{R}_+ \\ u(t,.) = u_x(t,.) = 0 \text{ at } x = 0,1, \end{cases}$$

in which the state variable is $u(t,x)$ and the control is the function $p(t)$. Thanks to the Nash-Moser theorem, one proves, in [3], the local controllability of $(\mathcal{P})$ in $H^{5+\epsilon} \times H^{3+\epsilon}((0,1),\mathbb{R})$, $\epsilon > 0$ around reference trajectories of the form

$$(u_{\text{ref}}(t,x) := v_k(x)\sin(\sqrt{\lambda_k}t) + v_{k+1}(x)\sin(\sqrt{\lambda_{k+1}}t), u \equiv 0),$$

where, for every $n \in \mathbb{N}^*$,

$$\tfrac{d^4}{dx^4}v_n = \lambda_n v_n, \quad v_n(0) = v_n(1) = v_n'(0) = v_n'(1) = 0.$$

The second PDE represents a quantum particle in a 1D infinite square potential well with variable length. The state variable is the wave function ψ of the particle and the control is the length $l(t)$ of the potential. After changes of variable and wave functions, one works on the equivalent control system

$$(\mathcal{V}) \begin{cases} i\dot{\psi} = -\psi'' + (\dot{u} - u^2)x^2\psi, x \in (0,1), t \in \mathbb{R}_+, \\ \psi(t,0) = \psi(t,1) = 0, \end{cases}$$

in which the state is ψ and the control u is subjected to $u(0) = u(T) = \int_0^T u(t)dt = 0$. Thanks to the Nash-Moser theorem and expansion to the second order, one proves, in [2], local controllability results in $H^{5+\epsilon}((0,1),\mathbb{C})$ for $(\mathcal{V})$. Then, a compactness argument provides steady state controllability.

6 Conclusion

We have proved the local controllability around any eigenstate and the steady-state controllability of (Σ).

The technic introduced for the study of (Σ) is general enough to be applied on other equations. Thus, it should also give affirmative controllability results for the bilinear control systems which are known to be not controllable (in particular spaces) in the general framework of [1]. This constitutes an open problem.

References

[1] J.M. Ball, J.E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. *SIAM J. Control and Optim.*, 20, July 1982.

[2] K. Beauchard. Controllability of a quantum particle in a 1D infinite square potential well with variable length. *ESAIM:COCV (accepted).*

[3] K. Beauchard. Local Controllability of a 1-D beam equation (submitted).

[4] K. Beauchard. Local Controllability of a 1-D Schrödinger equation. *J. Math. Pures et Appl.*, 84:851–956, July 2005.

[5] K. Beauchard and J.-M. Coron. Controllability of a quantum particle in a moving potential well. *accepted in J. Functional Analysis*, 2005.

[6] J.-M. Coron. Global asymptotic stabilization for controllable systems without drift. *Math. Control Signals Systems*, 5:295–312, 1992.

[7] J.-M. Coron. Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles bidimensionnels. *C. R. Acad. Sci. Paris*, 317:271–276, 1993.

[8] J.-M. Coron. On the controllability of 2-D incompressible perfect fluids. *J. Math. Pures Appl.*, 75:155–188, 1996.

[9] J.-M. Coron. Local Controllability of a 1-D Tank Containing a Fluid Modeled by the shallow water equations. *ESAIM: COCV*, 8:513–554, June 2002.

[10] J.-M. Coron. On the small-time local controllability of a quantum particule in a moving one-dimensional infinite square potential well. *C.R.A.S. (to appear)*, 2005.

[11] J.-M. Coron and E. Crépeau. Exact boundary controllability of a nonlinear KdV equation with critical lengths. *J. Eur. Math. Soc.*, 6:367–398, 2004.

[12] J.-M. Coron and A. Fursikov. Global exact controllability of the 2D Navier-Stokes equation on a manifold without boundary. *Russian Journal of Mathematical Physics*, 4:429–448, 1996.

[13] A. V. Fursikov and O. Yu. Imanuvilov. Exact controllability of the Navier-Stokes and Boussinesq equations. *Russian Math. Surveys*, 54:565–618, 1999.

[14] O. Glass. Exact boundary controllability of 3-D Euler equation. *ESAIM: COCV*, 5:1–44, 2000.

[15] O. Glass. On the controllability of the Vlasov-Poisson system. *Journal of Differential Equations*, 195:332–379, 2003.

[16] A. Haraux. Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire. *J. Math. Pures et Appl.*, 68:457–465, 1989.

[17] Th. Horsin. On the controllability of the Burgers equation. *ESAIM: COCV*, 3:83–95, 1998.

[18] W. Krabs. *On moment theory and controllability of one-dimensional vibrating systems and heating processes*. Springer Verlag, 1992.

[19] L. Hörmander. On the Nash-Moser Implicit Function Theorem. *Annales Academiae Scientiarum Fennicae*, pages 255–259, 1985.

[20] P. Rouchon. Control of a quantum particule in a moving potential well. *2nd IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control, Seville*, 2003.

[21] G. Turinici. On the controllability of bilinear quantum systems. *In C. Le Bris and M. Defranceschi, editors, Mathematical Models and Methods for Ab Initio Quantum Chemistry*, volume 74 of Lecture Notes in Chemistry.