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Colin Guillarmou

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SCATTERING AND RESOLVENT ON GEOMETRICALLY FINITE HYPERBOLIC MANIFOLDS WITH RATIONAL CUSPS

COLIN GUILLARMOU

Abstract. These notes summarize the papers [8, 9] on the analysis of resolvent, Eisenstein series and scattering operator for geometrically finite hyperbolic quotients with rational non-maximal rank cusps. They complete somehow the talk given at the PDE seminar of Ecole Polytechnique in october 2005.

1. Introduction

In this talk we describe recent results of [8, 9] about geometric scattering theory on infinite volume hyperbolic manifolds with rational non-maximal rank cusps.

It is well-known that the resolvent of the Laplacian extends meromorphically through the continuous spectrum in some geometric situations, the first obvious case is of course the Euclidean Laplacian and (fairly general) compactly supported perturbations. Poles of the this extension are called resonances and contain informations about classical dynamic and geometry of the considered system. Many deep studies of resonances and scattering objects shed light on these relations between classical dynamic-trapped set and quantum data. In hyperbolic geometry the most striking example is the Selberg zeta function for convex co-compact hyperbolic manifolds which is defined as an infinite products over the length of closed geodesics and whose singularities are essentially the resonances (see [24, 10]).

It is however important to observe that very few cases of Riemannian manifolds enjoy this property of meromorphic extension of the resolvent through the essential spectrum. It turns out that this property is extremely sensible to the asymptotic behaviour of the geometry near infinity, and it is believed that slight perturbations of locally symmetric spaces have this property, see recent works of Mazzeo-Vasy in this direction [21]. For geometrically finite hyperbolic manifolds $\Gamma\backslash\mathbb{H}^{n+1}$, i.e. those groups Γ which have a finite-sided fundamental domain in the natural compactification of hyperbolic space $\mathbb{H}^{n+1}$, Bunke and Olbrich [1] show in the most general case that one can define a scattering operator acting on the infinity of the manifold and it extends to $\mathbb{C}$ meromorphically. This extension is in general equivalent to the extension of the resolvent. When the manifold is convex-compact, that is without cusps, the first proof of the extension of the resolvent and its careful pseudo-differential analysis goes back to Mazzeo-Melrose [18] and the bound on the counting function of resonances was given by Guillopé-Zworski [12], improved by Patterson-Perry [24] using thermodynamic formalism of Fried-Ruelle and Selberg zeta function. The study of scattering operator in this precise case is done by Guillopé-Zworski [11, 13] in dimension 2, by Perry [26] in higher dimension and by Joshi-Sa Barreto [14] for a more general class of manifolds called asymptotically hyperbolic. We finally recall that Froese-Hislop-Perry [2] did analyze the general geometrically finite case in dimension 3, but without giving bounds on resonances.

We propose another proof of the result of Bunke-Olbrich (in any dimension) in a simpler geometric case. We deal directly with the resolvent, so that we are able to estimates the growth

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of the counting function of resonances; this allows us to describe what type of operator is the scattering operator through the analysis of its Schwartz kernel and results in the spirit of Graham-Zworski [6] are investigated too. Our geometric condition lies in the cusps, indeed we consider only cusps that have no irrational rotational part, this fact will be explained later and allows to have relatively explicit model formulae for the resolvent at infinity.

We consider an infinite volume hyperbolic quotient $X := \Gamma\backslash\mathbb{H}^{n+1}$ where Γ is a discrete group of isometries of $\mathbb{H}^{n+1}$ which admits a fundamental domain with finitely many sides, and such that each parabolic subgroup of Γ does not contain irrational rotation. This last condition is always satisfied in dimension $n+1=3$ and, in general, can be reduced to the case where each parabolic subgroup is conjugate to a lattice of translations in $\mathbb{R}^n$ (in the model $\mathbb{H}^{n+1} = (0,\infty)\times\mathbb{R}^n$), possibly by passing to a finite cover, thus resolvent, scattering operator and Eisenstein functions are obtained as a finite sum on the cover. Similarly, elliptic elements of Γ can also be excluded by passing to a finite cover, X is then a smooth manifold, and since the presence of maximal-rank cusps do not add difficulties (they are finite volume ends), we will avoid them for simplicity of exposition. The manifold X equipped with the hyperbolic metric is complete and the spectrum of the Laplacian Δ_X splits into continuous spectrum $[\frac{n^2}{4},\infty)$ and a finite number of L^2 eigenvalues included in $(0,\frac{n^2}{4})$ which form the point spectrum $\sigma_{pp}(\Delta_X)$ (see Lax-Phillips [15]). In [8] we proved that

Theorem 1.1. *With previous assumptions on the manifold $X = \Gamma\backslash\mathbb{H}^{n+1}$, the modified resolvent*

$$R(\lambda) := (\Delta_X - \lambda(n-\lambda))^{-1}$$

extends from $\{\Re(\lambda) > \frac{n}{2}\}$ to $\mathbb{C}$ meromorphically with poles of finite multiplicity (i.e. the rank of the polar part in the Laurent expansion at each pole is finite) from $L^2_{comp}(X)$ to $L^2_{loc}(X)$, the counting funtion for resonances satisfies

$$N(R) := \sharp\{\lambda \text{ resonances }; |\lambda - \frac{n}{2}| < R\} = O(R^{\dim X+1})$$

The non-optimal bound is a consequence of the method and was already a problem in the simpler case with no cusps in [12], the optimal bound being obtained in that case by dynamic arguments [24]. To prove this theorem, we proceed in way similar to [12], and we can deal with rational cusps since an explicit formula for the resolvent is available for "models cusps" using decomposition of variables.

Once the resolvent has been studied, we can define a Poisson operator, Eisenstein functions, and scattering operator, then we show that they have meromorphic extension to $\mathbb{C}$.

To explain the results, we have to recall the geometry at infinity of the manifold X. This is actually described for instance in Section 2 of Mazzeo-Phillips [20]. In our case with rational cusps, X (or actually a finite cover) can be seen as the interior of a smooth compact manifold with boundary $\bar{X}$. If ρ is a boundary defining function of the boundary $\partial\bar{X}$ and if g is the hyperbolic metric on X, then $\rho^2 g$ extends as a smooth non-negative tensor on $\bar{X}$, which is only positively definite outside some submanifolds of the boundary $\partial\bar{X}$ where it degenerates. Each one of these submanifolds arises from a cusp point of X (i.e. a fixed point at infinity of $\mathbb{H}^{n+1}$ for a parabolic subgroup of Γ) and is diffeomorphic to a k-dimensional torus T^k if the parabolic subgroup has rank k. If we note c the union of these submanifolds, $B = \partial\bar{X}\setminus c$ is a non-compact manifold which can be thought as the infinity of X, and $B = \Gamma\backslash\Omega$ where $\Omega\subset S^n$ is the domain of discontinuity of Γ. It turns out that B has ends diffeomorphic to $(\mathbb{R}^{n-k}_y\setminus\{|y|<1\})\times T^k$, each end arising from a rank-k parabolic subgroup of Γ fixing a point at infinity of $\mathbb{H}^{n+1}$. The natural compactification $\bar{B}$ of B corresponds to the radial compactification in the y variable in each end thus $\bar{B}$ is a fibred boundary manifold in the sense of Mazzeo-Melrose [19], the trivial fibrations being the projections

$$\phi_k : S^{n-k-1} \times T^k \to S^{n-k-1}.$$

When equipped with the metric $h_0 := \rho^2 g|_B$, (B, h_0) is conformal to the 'exact Φ-type metric' (as defined in [19]) equal to

$$\frac{dr^2}{r^4} + \frac{d\theta^2}{r^2} + dz^2, \text{ on } \quad (0,\epsilon)_r \times S_\theta^{n-k-1} \times T_z^k$$

near its infinity, the conformal factor decreasing enough to make the volume of B finite.

Poisson and scattering operators $\mathcal{P}(\lambda), S(\lambda)$ are constructed in [9] through a Poisson problem (i.e. a generalized eigenvalue problem) in a way similar to that introduced on Euclidean manifolds by Melrose (see [23]). The structure of the metric near the cusps c is quite degenerate and $\mathcal{P}(\lambda), S(\lambda)$ do not act naturally on $C^\infty(\partial\bar{X})$ as it would be the case when there is no cusps. Indeed, they acts much on subspaces related to this cusp structure. We define the subalgebra $C^\infty_{\rm acc}(\bar{X})$ of $C^\infty(\bar{X})$ of functions which are asymptotically constant in the cusps, these are the $f \in C^\infty(\bar{X})$ such that

$$Z(f|_c) = 0, \quad Z((X_1 \dots X_N f)|_c) = 0$$

for all smooth vector fields $X_1, \dots, X_N$ on $\bar{X}$ ($\forall N \in \mathbb{N}$) and all smooth vector fields Z on c. In other words, these are the functions whose restrictions at the cusp submanifolds are locally constant and similarly for all derivatives. It is actually possible to find a boundary defining function ρ for $\bar{X}$ in this subalgebra. The volume form ${\rm dvol}_g$ of g can be expressed by $\rho^{-n-1}R_c^2\mu_{\bar{X}}$ for a function R_c which is smooth positive in $\bar{X} \setminus c$ with $R_c^2 \in C^\infty_{\rm acc}(\bar{X})$ vanishing at order $2k$ at each k-dimensional component of c and where $\mu_{\bar{X}}$ is a smooth volume density on $\bar{X}$. We can define $C^\infty_{\rm acc}(\partial\bar{X})$ and $R_c^{-1}C^\infty_{\rm acc}(\partial\bar{X})$ by restriction of $C^\infty_{\rm acc}(\bar{X})$ and $R_c^{-1}C^\infty_{\rm acc}(\bar{X})$ at $\partial\bar{X}$ and $B = \partial\bar{X} \setminus c$ (here we use the same notation for R_c and its restriction $R_c|_{\partial\bar{X}}$). For any boundary defining function $\rho \in C^\infty_{\rm acc}(\bar{X})$, one can define the Poisson operator $\mathcal{P}(\lambda)$ by solving the generalized eigenvalue problem: if $\Re(\lambda) \geq \frac{n}{2}$ and $\lambda \notin \frac{n}{2} + \mathbb{N}$, then for $f \in R_c^{-1}C^\infty_{\rm acc}(\partial\bar{X})$ there exists a unique solution $\mathcal{P}(\lambda)f$ of the following

$$\begin{cases} (\Delta_X - \lambda(n-\lambda))\mathcal{P}(\lambda)f = 0 \\ \mathcal{P}(\lambda)f = \rho^{n-\lambda}F(\lambda, f) + \rho^\lambda G(\lambda, f) \\ F(\lambda, f), G(\lambda, f) \in R_c^{-1}C^\infty_{\rm acc}(\bar{X}) \\ F(\lambda, f)|_{\rho=0} = f \end{cases}.$$

The solution is constructed by approximation using an indicial equation for Δ_X and then by correcting the error through the resolvent; for that we need the precise mapping property of the extended resolvent of Theorem 1.1:

$$R(\lambda) : \dot{C}^\infty(\bar{X}) \to \rho^\lambda R_c^{-1} C^\infty_{\rm acc}(\bar{X}).$$

where $\dot{C}^\infty(\bar{X})$ is the set of functions in $C^\infty(\bar{X})$ vanishing at all order at $\partial\bar{X}$ (a kind of Schwartz space).

Next we analyze Eisenstein functions in [9]. The metric h_0 induces an $L^2(B)$ Hilbert space on B and we prove

Theorem 1.2. *If $R(\lambda; w; w')$ denotes the Schwartz kernel of the modified resolvent, the Eisenstein function*

$$E(\lambda; b; w') := \lim_{w \to b}[\rho(w)^{-\lambda}R(\lambda; w; w')], \quad b \in B, w' \in X$$

is a smooth function on $B \times X$ if λ is not a resonance. There exists $C > 0$ such that for all $N > 0$ it is the Schwartz kernel of a meromorphic operator

$$E(\lambda) : \rho^N L^2(X) \to L^2(B)$$

in $\Re(\lambda) > \frac{n}{2} - CN$ with poles of finite multiplicity, satisfying $\mathcal{P}(\lambda) = (2\lambda - n)^t E(\lambda)$ on $R_c^{-1}C^\infty_{acc}(\partial\bar{X})$. Except possibly at $\{\lambda; \Re(\lambda) < \frac{n}{2}, \lambda(n-\lambda) \in \sigma_{pp}(\Delta_X)\}$, the set of poles of $E(\lambda)$ coincides with the set of resonances.

Using the asymptotic expression of $\mathcal{P}(\lambda)f$, the scattering operator is defined (with same notations) by

$$S(\lambda):\left\{\begin{array}{ccc} R_c^{-1}C^\infty_{\rm acc}(\partial\bar{X}) & \to & R_c^{-1}C^\infty_{\rm acc}(\partial\bar{X}) \\ f & \to & F(\lambda,f)|_{\rho=0} \end{array}\right. .$$

For $\Re(\lambda)=\frac{n}{2}$, $S(\lambda)$ can be extended to $L^2(B)$ as a unitary operator and it gives, in a sense, a parametrization of the absolutely continuous spectrum of Δ_X. When the manifold has no cusp, it is proved in [26, 14] that $S(\lambda)$ is an elliptic pseudo-differential operator of order $2\lambda-n$, a fact that contrasts with the Euclidean geometry where it is a Fourier integral operator.

We prove in [9] the following result

Theorem 1.3. *The scattering operator $S(\lambda)$ extends meromorphically to $\mathbb{C}$ as a family of pseudo-differential operators in the full Φ-calculus on the manifold with fibred boundary $\bar{B}$ in the sense of Mazzeo-Melrose* [19]. *In $\{\Re(\lambda)\leq\frac{n}{2},\lambda(n-\lambda)\notin\sigma_{pp}(\Delta_X)\}$, λ_0 is a pole of $S(\lambda)$ if and only if λ_0 is a resonance and it has finite multiplicity. In $\{\Re(\lambda)>\frac{n}{2}\}$, $S(\lambda)$ has only first order poles whose residue is*

$$Res_{\lambda_0}S(\lambda)=\left\{\begin{array}{ll} -\frac{(-1)^{j+1}2^{-2j}}{j!(j-1)!}P_j+\Pi_{\lambda_0} & \text{if } \lambda_0=\frac{n}{2}+j, j\in\mathbb{N} \\ \Pi_{\lambda_0} & \text{if } \lambda_0\notin\frac{n}{2}+\mathbb{N} \end{array}\right.$$

where P_j is the j-th GJMS conformal Laplacian of [5] *on (B,h_0) and Π_{λ_0} is an operator with rank* $\dim\ker_{L^2}(\Delta_X-\lambda_0(n-\lambda_0))$.

Note that the GJMS conformal Laplacians P_j in [5] are well-defined for all j if $n\geq 3$ (resp. for $j\leq 1$ if $n=2$) if the manifold is locally conformally flat, which is the case for B.

In [9], we also prove some results similar to Graham-Zworski theorems in [6] for this class of manifolds. We actualy study conformally covariance of $S(\lambda)$ under change of boundary defining function in $C^\infty_{\rm acc}(\bar{X})$. We also show that Branson's Q curvature is a scattering object in some sense

$$Q=\frac{(-1)^{\frac{n}{2}}2^{-n}}{\frac{n}{2}!(\frac{n}{2}-1)!}S(n)1.$$

Moreover Q is in $L^1(B,\mathrm{dvol}_{h_0})$ and

$$\frac{(-1)^{\frac{n}{2}}2^{1-n}}{\frac{n}{2}!(\frac{n}{2}-1)!}\int_B Q\ \mathrm{dvol}_{h_0}=L$$

where L is the log term, independent of ρ, appearing in the expansion of the volume

$$\mathrm{vol}_X(\{\rho>\epsilon\})\sim c_0\epsilon^{-n}+\dots+c_{n-2}\epsilon^{-2}+L\log(\epsilon^{-1})+V+o(1).$$

The case of irrational cusps is more technically involved and will probably be carried out in a following paper. It is worthy to add that this analysis could be used to study the divisors of Selberg's zeta function as Patterson-Perry [24] did for convex co-compact hyperbolic manifolds.

2. Geometry, covering and asymptotically constant functions in the cusps

We describe here with more details the assumptions about the cusps discussed roughly in the introduction; we strongly use Section 2 of Mazzeo-Phillips [20]. Let Γ a discrete subgroup of orientation preserving isometries of the hyperbolic space $\mathbb{H}^{n+1}$. Recall that Γ acts also on the natural compactification $\bar{\mathbb{H}}^{n+1}=\{m\in\mathbb{R}^{n+1};||m||\leq 1\}$ of $\mathbb{H}^{n+1}$ and on its boundary S^n; an element $\gamma\in\Gamma$ is called hyperbolic if it fixes two points on S^n and no point in $\mathbb{H}^{n+1}$, parabolic if it fixes one point on S^n and no point in $\mathbb{H}^{n+1}$, then γ is elliptic if it fixes a point of $\mathbb{H}^{n+1}$. The limit set of the group Λ is the set of accumulation points of $\Gamma.m$ on S^n where m is any fixed point in $\mathbb{H}^{n+1}$, the domain of dicontinuity is $\Omega:=S^n\setminus\Lambda$.

If Γ contains elliptic elements, there exists a subgroup Γ_0 of finite index of Γ without elliptic elements thus X is finitely covered by $\Gamma_0\backslash\mathbb{H}^{n+1}$, the latter being a smooth manifold. Since we study resolvent of the Laplacian and other related objects, we can always pass to a finite cover without difficulties: objects on X can indeed be obtained by summing on a finite set objects on the finite cover. Thus we exclude elliptic elements in Γ. We suppose that Γ is geometrically finite, which means here that it admits a fundamental domain F with finitely many sides in $\bar{\mathbb{H}}^{n+1}$.

If the group has no parabolic elements and is non-compact, the manifold $X := \Gamma\backslash\mathbb{H}^{n+1}$ is said convex co-compact. In particular it is the interior of a smooth compact manifold with boundary $\bar{X} := X \cup \partial\bar{X}$ where $\partial\bar{X}$ is the compact manifold $\Gamma\backslash\Omega$, Ω being the domain of discontinuity of Γ. The metric is asymptotically hyperbolic in this case, which means that there exists a diffeomorphism ψ from $[0,\epsilon)_\rho \times \partial\bar{X}$ to its image in $\bar{X}$ such that $\psi_*(\rho)$ is a boundary defining function of $\partial\bar{X}$ in $\bar{X}$

$$\psi^* g = \frac{d\rho^2 + h(\rho)}{\rho^2}$$

where $h(\rho)$ is a family of metrics on $\partial\bar{X}$ depending smoothly on $\rho \in [0,\epsilon)$.

If Γ contains parabolic elements, each fixed point $p \in S^n$ of a parabolic element of Γ is called a cusp point, and for each cusp point p, let Γ_p be the subgoup of Γ fixing p. Actually Γ_p contains only parabolic elements and it can be shown that there is a Γ_p invariant neighbourhood U_p of p such that $\Gamma\backslash(F \cap U_p)$ is isometric to a neighbourhood of p in $\Gamma_p\backslash(F \cap U_p)$. The subgroup Γ_p has a maximal free abelian subgoup Γ_a with rank k, the rank of the cusp p is defined to be the integer k. We suppose that $k \leq n-1$ for each p since this case is well known in term of scattering theory. Using now conjugation, it suffices to look at the case where $p = \infty$ in the upper half model $\mathbb{H}^{n+1} = \mathbb{R}^+ \times \mathbb{R}^n$. Section 2 of [20] (the arguments come from Thurston's lecture notes) shows that there is an affine subspace $\mathbb{R}^k \subset \mathbb{R}^n$ fixed by Γ_∞ on which Γ_a acts as a group of k translations. Note that k is called the rank of the cusp. Every $\gamma \in \Gamma_a$ acts as

$$\gamma(y,z) = (R_\gamma y, z + b_\gamma) \text{ on } \mathbb{R}_y^{n-k-1} \oplus \mathbb{R}_z^k$$

for some $R_\gamma \in O(n-k-1)$ and $b_\gamma \in \mathbb{R}^k$. There is a flat torus $\Gamma_a\backslash\mathbb{R}^k \simeq T^k$ such that $\Gamma_a\backslash\mathbb{R}^n$ is a flat vector bundle with basis this torus. We assume that the holonomy representation $\gamma \to R_\gamma \in O(n-k-1)$ of this bundle has a finite image, so that all rotations R_γ have rational angle $p\pi/q$ for some $p,q \in \mathbb{N}$. Then there is a finite cover of this bundle which is isometric to $T^k \times \mathbb{R}^{n-k}$, and we are reduced to study the case where each R_γ is the identity.

Example: to fix the ideas, we give the most basic case of geometrically finite hyperbolic manifold with rational non-maximal rank cusp (actually with no rotational part), this is the quotient $X = \Gamma\backslash\mathbb{H}^3$ where $\Gamma = \langle\gamma\rangle$ is the group generated by the element

$$\gamma : (x,y,z) \in \mathbb{H}^3 = \mathbb{R}^+ \times \mathbb{R}^2 \to (x,y,z+1) \in \mathbb{H}^3.$$

It is topologically $\mathbb{R}_x^+ \times \mathbb{R}_y \times S_z^1$, the domain of discontinuity is $\Omega = \mathbb{R}^2 = S^2 \setminus \{\infty\}$ and the natural boundary $B := \Gamma\backslash\Omega \simeq \mathbb{R}_y \times S_z^1$. Note that a natural compactification $\bar{X}$ of X is to consider $X \simeq S_z^1 \times \mathbb{H}^2_{(x,y)}$ and compactify $\mathbb{H}^2$ into the ball $\bar{\mathbb{H}}^2 := \{m \in \mathbb{R}^2; |m| \leq 1\}$. The boundary of $\bar{X}$ is a torus $T^2 := S^1 \times S^1$ of genus 1 and the cusp (the fixed point at infinity of γ) becomes a circle S^1, $\partial\bar{X}$ is also the compactification of B obtained by compactifying the line $\mathbb{R}_y$ into a circle.

We get back to the general case. The construction of the extension of the resolvent is done by parametrix construction and Fredholm theorem, as usual. In our case we do not have the explicit expression of a model resolvent that gives directly the parametrix, contrary to Euclidean case or Riemann surfaces [11, 13]. However we are able to have fairly good local parametrix on the neighbourhoods of infinity, in particular near the cusps. We need to describe precisely a covering by neighbourhood of infinity.

By assumptions on the cusps and using [2, 26] we obtain a covering of the manifold X by model charts. There exists a compact K of X such that $X \setminus K$ is covered by a finite number of charts isometric to either a regular neighbourhood (M_r, g_r) or a rank-k cusp neighbourhood (M_k, g_k) where

$$M_r := \{(x,y) \in (0,\infty) \times \mathbb{R}^n; x^2 + |y|^2 < 1, \}, \quad g_r = x^{-2}(dx^2 + dy^2),$$

$$M_k := \{(x,y,z) \in (0,\infty) \times \mathbb{R}^{n-k} \times T^k; x^2 + |y|^2 > 1\}, \quad g_k = x^{-2}(dx^2 + dy^2 + dz^2)$$

for $k = 1, \dots, n-1$ with (T^k, dz^2) a k-dimensional flat torus.

One can choose the covering such that the cusps neighbourhood do not intersect, possibly by adding regular neighbourhoods. We will make as if there was only one neighbourhood of each type to simplify notations, moreover we consider by abuse of notation M_r, M_k as sets of X.

The model M_k can be considered as a subset of the quotient $X_k = \Gamma_\infty \backslash \mathbb{H}^{n+1}$ of $\mathbb{H}^{n+1}$ by a rank-k parabolic subgroup Γ_∞ of Γ which fixes the point at infinity of $\mathbb{H}^{n+1}$. Moreover from previous discussion and since we work on the finite cover, we can consider the subgroup Γ_∞ to be a group of translations ating on rr^k; X_k is isometric to $\mathbb{R}^+_x \times \mathbb{R}^{n-k}_y \times T^k_z$ equipped with the metric

$$g_k = \frac{dx^2 + dy^2 + dz^2}{x^2}$$

dz^2 being the flat metric on a k-dimensional torus T^k. Hence X_k can be compactified into the compact manifold with boundary $\bar{X}_k = \bar{\mathbb{H}}^{n-k+1} \times T^k$ where $\bar{\mathbb{H}}^{n-k+1}$ is the ball $\{|w| \leq 1\}$ in $\mathbb{R}^{n-k+1}$. With the new coordinates

$$t := \frac{x}{x^2 + |y|^2}, \quad u := \frac{-y}{x^2 + |y|^2} \tag{2.1}$$

(M_k, g_k) is isometric to

$$\{(t,u,z) \in (0,\infty) \times \mathbb{R}^{n-k} \times T^k; t^2 + |u|^2 < 1\}$$

equipped with the metric

$$\frac{dt^2 + du^2 + (t^2 + |u|^2)^2 dz^2}{t^2}. \tag{2.2}$$

These coordinates compactify M_k by adding $\{t = 0\}$, since t and u extend smoothly to $\bar{X}_k \setminus \{x = y = 0\}$, the infinity of X in the chart M_k is $\{t = 0\}$. Also we will call cusp submanifold the submanifold $\{t = u = 0\}$ of M_k, it will be denoted c_k and we remark that $c_k \simeq \infty \times T^k \simeq T^k$ in M_k where ∞ is the point at infinity in the half-space model of $\mathbb{H}^{n-k+1}$. Let us finally denote $\bar{M}_k = M_k \cup \{t = 0\}$.

The model M_r is simpler and can be considered as a subset of $\mathbb{H}^{n+1}$, we define as for M_k $\bar{M}_r := M_r \cup \{x = 0\}$.

There exist some smooth functions $\chi, \chi^r, \chi^1, \dots, \chi^{n-1}$ on respectively $X, M_r, M_1, \dots, M_{n-1}$ which, through the isometric charts $I_r, I_1, \dots, I_n$, satisfy

$$I_r^* \chi^r + \sum_{k=1}^{n-1} I_k^* \chi^k + \chi = 1 \tag{2.3}$$

with χ having compact support in X.

Using the previous discussion, one obtains a compactification of X as a smooth compact manifold with boundary $\bar{X}$. Moreover, with no loss of generality one can choose a boundary defining function ρ which is equal to the function t in each neighbourhood $\bar{M}_k$. The boundary $\partial\bar{X}$ is covered by some charts $B_1, \dots, B_{n-1}, B_r$ induced by $M_1, \dots, M_{n-1}, M_r$ by taking

$$B_k := \bar{M}_k \cap \partial\bar{X} \simeq \{(u,z) \in \mathbb{R}^{n-k} \times T^k; |u|^2 < 1\}$$

$$B_r := \bar{M}_r \cap \partial\bar{X} \simeq \{y \in \mathbb{R}^n; |y|^2 < 1\}.$$

From the discussion above, we see that the metric on X can be expressed by

$$g = \frac{H}{\rho^2}$$

with H a smooth non-negative symmetric 2-tensor on $\bar{X}$ which degenerates at the cusps submanifolds $(c_k)_{k=1,\dots,n-1}$. Let us define $c := (\cup_k c_k) \subset \partial\bar{X} \subset \bar{X}$, and $B := \partial\bar{X} \setminus c$, then the restriction

$$h_0 := H|_B = (\rho^2 g)|_B \tag{2.4}$$

is a smooth metric on the non-compact manifold B.

We will also need to use functions representing the distance to the cusps submanifolds as follows: for $k = 1, \dots, n-1$, let r_{c_k} be a continuous non-negative function in $\bar{X}$, smooth and positive in $\bar{X} \setminus c_k$ which satisfies

$$I_{k*}(r_{c_k}) = \sqrt{t^2 + |u|^2}$$

in $\bar{M}_k$ and is equal to 1 in M_j when $j \neq k$. Then we define the functions

$$r_c := \prod_{k=1}^{n-1} r_{c_k}, \quad R_c := \prod_{k=1}^{n-1} (r_{c_k})^k \tag{2.5}$$

on $\bar{X}$ and we will also denote by r_{c_k}, r_c and R_c their restriction to $\partial\bar{X}$. It can easily be checked that B equipped with the metric h_0 of (2.4) has a volume density dvol_{h_0} which is of the form

$$\mathrm{dvol}_{h_0} = R_c^2 \mu_{\partial\bar{X}} \tag{2.6}$$

with $\mu_{\partial\bar{X}}$ a smooth non-vanishing density (volume density) on $\partial\bar{X}$. Similarly the volume density dvol_g on X can be expressed by

$$\mathrm{dvol}_g = \rho^{-n-1} R_c^2 \mu_{\bar{X}} \tag{2.7}$$

for a smooth volume density $\mu_{\bar{X}}$ on $\bar{X}$. In what follows, we will write $L^2(X)$ and $L^2(B)$ for the Hilbert spaces of square integrable functions on X and B with respect to the volume densities dvol_g and dvol_{h_0}.

For a compact manifold $\bar{M}$ with boundary $\partial\bar{M}$, we denote by $\dot{C}^\infty(\bar{M})$ the set of smooth functions on $\bar{M}$ which vanish at all orders at $\partial\bar{M}$. There will be a special set of smooth functions on $\bar{X}, \partial\bar{X}$ which will play an important role for what follows, these are the functions which are "asymptotically constant in the cusp variables". To give a precise definition we begin by introducing the sets $\mathcal{C}(T\bar{X})$, $\mathcal{C}(T\partial\bar{X})$ and $\mathcal{C}(Tc)$ of smooth vector fields on $\bar{X}, \partial\bar{X}, c$. Then we set

$$C^\infty_{\mathrm{acc}}(\bar{X}) := \{f \in C^\infty(\bar{X}); \forall X_1, \dots, X_N \in \mathcal{C}(T\bar{X}), \forall Z \in \mathcal{C}(Tc), Z(f|_c) = 0, Z(X_1 \dots X_N f|_c) = 0\}$$

and $C^\infty_{\mathrm{acc}}(\partial\bar{X})$ is defined similarly by replacing $\bar{X}$ by $\partial\bar{X}$. These functions are constant on each cusp submanifold c_k and their derivatives as well. In local coordinates (t, u, z) near the cusp $c_k = \{t = u = 0\}$, one can check by a Taylor expansion at $(0, 0, z) \in c_k$ and Borel Lemma that a function $f \in C^\infty_{\mathrm{acc}}(\bar{X})$ can be decomposed locally as a sum

$$f(t, u, z) = f_0(t, u) + O((t^2 + |u|^2)^\infty) = f_0(t, u) + O(r_c^\infty) \tag{2.8}$$

for some f_0 smooth. The class $C^\infty_{\mathrm{acc}}(\bar{X})$ forms a subalgebra of $C^\infty(\bar{X})$ which is stable under the action of $\mathcal{C}(T\bar{X})$. It can be topologized by the semi-norms of $C^\infty(\bar{X})$ and it is closed in $C^\infty(\bar{X})$.

Observe also that r_c^2 and R_c^2 defined by (2.5) are in $C^\infty_{\mathrm{acc}}(\bar{X})$. Actually one can check (see [9]) that the space $R_c^{-1} C^\infty_{\mathrm{acc}}(\bar{X})$ does not depend on the choices of ρ, R_c^2 in $C^\infty_{\mathrm{acc}}(\bar{X})$, this space will be useful later.

To construct the solution of the Poisson problem, we need a kind of model form for the metric. It is actually possible to prove that there exists a collar neighbourhood $(0,\epsilon)_\rho \times \partial\bar{X}$ of $\partial\bar{X}$ induced by a boundary defining function $\rho \in C^\infty_{\rm acc}(\bar{X})$ such that

$$g = \frac{d\rho^2 + h(\rho)}{\rho^2} \tag{2.9}$$

for some smooth family of symmetric tensors $h(\rho)$ on $\partial\bar{X}$, depending smoothly on ρ, positive for $\rho > 0$, with $h(0) = h_0$ positive on B and satisfying

$$h(\rho) = du^2 + (\rho^2 + |u|^2)^2 dz^2$$

in each $\bar{M}_k$. The proof is detailed in [9, Sec. 2.5] and we actually describe the set of boundary defining functions in $C^\infty_{\rm acc}(\bar{X})$ that puts the metric under an "almost model" form, a fact that is necessary to investigate the conformal invariance properties of scattering operator and related objects as in [6].

3. The resolvent

We begin by recalling results known in the convex co-compact case. First Mazzeo-Melrose [18] and Guillopé-Zworski [12] results can be summarize as

Theorem 3.1. *If $X := \Gamma\backslash\mathbb{H}^{n+1}$ is convex co-compact, then the resolvent $R(\lambda) = (\Delta_X - \lambda(n-\lambda))^{-1}$ extends meromorphically from $\Re(\lambda) > \frac{n}{2}$ to $\mathbb{C}$ with poles of finite multiplicity. The counting function for resonances $N(R)$ satisfies*

$$N(R) = O(R^{\dim X + 1})$$

Note that the bound is improved in $O(R^{\dim X})$ by Patterson-Perry [24] using thermodynamic formalism of Fried-Ruelle, a formalism which seems not well adapted to cases with cusps.

Example: we return to our example given in last section. The Laplacian is

$$\Delta_X = -(x\partial_x)^2 + 2x\partial_x + x^2\Delta_{\mathbb{R}\times S^1}$$

and by decomposing in spherical harmonics on S^1 and Fourier transform in $\mathbb{R}$ the Laplacian on the cylinder $\Delta_{\mathbb{R}\times S^1}$ becomes $|\xi|^2 + \omega_m^2$ where ω_m^2 are the eigenvalues of Δ_{S^1}. Now we have a one dimensional spectral problem on the half line $(0,\infty)_x$, the resolvent can be computed by Sturm-Liouville theory, the Green kernel for $\Re(\lambda) > 1$ is

$$G_{\lambda,\xi,m}(x,x') = \left\{ \begin{array}{ll} -K_{\lambda-1}(x|\xi_m|)I_{\lambda-1}(x'|\xi_m|) & \text{if } x > x' \\ -K_{\lambda-1}(x'|\xi_m|)I_{\lambda-1}(x|\xi_m|) & \text{if } x \leq x' \end{array} \right.$$

where K_s, I_s are the modified Bessel functions and $\xi_m = (\xi, \omega_m) \in \mathbb{R}^2$. The resolvent Schwartz kernel is then obtained by

$$R(\lambda; x, y, z; x', y', z') = -xx' \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} e^{i\xi.(y-y') + i\omega_m(z-z')} G_{\lambda,\xi,m}(x,x') d\xi$$

and can be proved to extend meromorphically in $\mathbb{C}$ as operator from $C_0^\infty(X) \to C^\infty(X)$. The generalized eigenfunctions can be obtained from this formula by considering the asymptotics Bessel functions: for $f \in C_0^\infty(B)$ (recall $B = \mathbb{R}\times S^1$) and $\Re(\lambda) = 1$, the function

$$P(\lambda)f = \frac{-2^{1-\lambda}}{\Gamma(\lambda)} x \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}^{n-k}} e^{i\xi_m.(y-y',z-z')} |\xi_m|^{\lambda-\frac{n}{2}} K_{\lambda-\frac{n}{2}}(|\xi_m|x) f(y',z') dy'dz'd\xi$$

is a generalized eigenfunction of Δ for spectral parameter $\lambda(2-\lambda)$ and with an asymptotic

$$P(\lambda)f = x^{2-\lambda} f(y,z) + x^\lambda (S(\lambda)f)(y,z) + O(x^2), \quad x \to 0, |y| < C \tag{3.1}$$

where $S(\lambda)f \in C^\infty(B)$. This defines $S(\lambda)$, the scattering operator, and again the asymptotics of $K_s(z)$ as $z \to 0$ gives that $S(\lambda) = c(\lambda)\Delta_{S^1\times\mathbb{R}}^{\lambda-1}$ for some explicit meromorphic constant $c(\lambda)$. Actually the function x blows-up at the cusp in $\bar{X}$, thus it is not really a "good" boundary defining function to use to obtain uniform asymptotics at infinity of X, the function $\rho = x/(1+x^2+|y|^2)$ is more adpated for that in the sense that it is a smooth boundary defining function for the compactification $\bar{X}$. Of course replacing x by ρ in (3.1) would change $S(\lambda)$, though only by conformal multiplications on the right and left. Remark that $S(\lambda)$ is a pseudo-differential operator on the cylinder $B = S^1 \times \mathbb{R}$, but the terms coming from $\omega_m = 0$ and $\omega_m \neq 0$ are of very different nature when we approach the infinity of B (two circles), corresponding to the cusp. The second ones are rapidly decreasing in space ("almost properly supported") as $|y - y'| \to \infty$ wheras the first one has conormal singularities as $|y - y'| \to \infty$. In other words, the action of $S(\lambda)$ on constant functions in variable z has a very different behaviour than the action on functions whose zeroth Fourier coefficient in the circle vanishes, which is also a typical fact of maximal rank cusps (i.e. finite volume cusps) for instance. This is roughly the reason of introducing this class $C^\infty_{\rm acc}$ of functions asymptotically constant in the cusps.

In our general case, a good parametrix can be found in the regular neighbourhood M_r of infinity by the method of Guillopé-Zworski [12], which is a simplification of that of Mazzeo-Melrose [18] for constant curvature manifolds.

A parametrix in the cusps neighbourhood can actually be obtained by an explicit calculation of the resolvent on $X_k = \Gamma_\infty \backslash \mathbb{H}^{n+1}$ with Γ_∞ a rank-k parabolic subgroup (of translations) fixing ∞, pretty much as in the previous example. Note that Γ_∞ is the image of the lattice $\mathbb{Z}^k$ by a map $A_k \in GL_k(\mathbb{R})$. By Fourier decomposition on the torus $T^k = \Gamma_\infty \backslash \mathbb{R}^k$ and conjugating by $x^{\frac{k}{2}}$, the operator $\Delta_{X_k} - \lambda(n-\lambda)$ acts on

$$L^2(X_k) = \bigoplus_{m \in \mathbb{Z}^k} \mathcal{H}_m, \quad \mathcal{H}_m \simeq L^2(\mathbb{R}^{n-k}_y \times \mathbb{R}^+_x, x^{-(n-k+1)} dy dx) = L^2(\mathbb{H}^{n-k+1})$$

as a family of operators

$$P_m(\lambda) := -x^2 \partial_x^2 + (n-k-1)x\partial_x + x^2(\Delta_y + |\omega_m|^2) - s(n-k-s)$$

where $\omega_m = 2\pi {}^t(A_k^{-1})m$ for $m \in \mathbb{Z}^k$ are the (square root of) eigenvalues of the Laplacian on T_k with eigenfunctions $e^{i\omega_m \cdot z}$ and $s := \lambda - \frac{k}{2}$ is a shifted spectral parameter. The problem is reduced to study a family of Schrödinger operators on hyperbolic spaces of lower dimension $\mathbb{H}^{n-k+1}$. The resolvent $R_{X_k}(\lambda) = (\Delta_{X_k} - \lambda(n-\lambda))^{-1}$ for the Laplacian on X_k is for $\Re(\lambda) > \frac{n}{2}$

$$R_{X_k}(\lambda) = \bigoplus_{m \in \mathbb{Z}^k} R_m(\lambda) \text{ on } L^2(X_k) = \bigoplus_{m \in \mathbb{Z}^k} \mathcal{H}_m \tag{3.2}$$

with

$$R_m(\lambda; x, y, x', y') = |A_k|^{-\frac{1}{2}} (xx')^{-\frac{k}{2}} \int_{\mathbb{R}^k} R_{\mathbb{H}^{n+1}}(\lambda; x, y, z; x', y', 0) e^{i\omega_m \cdot z} dz \tag{3.3}$$

where $R_{\mathbb{H}^{n+1}}(\lambda) = (\Delta_{\mathbb{H}^{n+1}} - \lambda(n-\lambda))^{-1}$ is the resolvent for the Laplacian on $\mathbb{H}^{n+1}$ and $|A_k| := |\det(A_k)|$. We set

$$r = (|y-y'|^2 + x^2 + x'^2)^{\frac{1}{2}}, \quad d = \frac{xx'}{r^2}, \quad \tau = \frac{xx'}{r^2 + |z|^2} = d(1 + \frac{|z|^2}{r^2})^{-1} \tag{3.4}$$

and recall (see e.g. [12], [26]) that the resolvent on $\mathbb{H}^{n+1}$ can be written for all $J \in \mathbb{N} \cup \infty$

$$R_{\mathbb{H}^{n+1}}(\lambda; x, y, z; x', y', 0) = \tau^\lambda \sum_{j=0}^{J-1} \alpha_{j,n}(\lambda) \tau^{2j} + \tau^{\lambda+2J} G_{J,n}(\lambda, \tau) \tag{3.5}$$

$$\alpha_{j,n}(\lambda) := \frac{2^{-1}\pi^{-\frac{n}{2}} \Gamma(\lambda + 2j)}{\Gamma(\lambda - \frac{n}{2} + 1 + j)\Gamma(j+1)}$$

with $G_{J,n}(\lambda, \tau)$ a smooth function in $\tau \in [0, \frac{1}{2})$ with a conormal singularity at $\tau = \frac{1}{2}$ and $G_{\infty,n}(\lambda, \tau) = 0$. Note that the sum (3.5) converges locally uniformly in $\tau \in [0, \frac{1}{2})$ if $J = \infty$.

From (3.3) and (3.5) it is easy to see, by the change of variable $w = z/r$, that for $m \neq 0$ and setting $s := \lambda - \frac{k}{2}$

$$R_m(\lambda) = d^s \sum_{j=0}^{J-1} \alpha_{j,n}(\lambda) d^{2j} F_{j,\lambda}(r\omega_m) + d^{s+2J} \int_{\mathbb{R}^k} e^{-ir\omega_m . z} \frac{G_{J,n}(\lambda, d(1+|z|^2)^{-1})}{(1+|z|^2)^{\lambda+2J}} dz \tag{3.6}$$

(3.7)

$$F_{j,\lambda}(u) = |A_k|^{-\frac{1}{2}} \int_{\mathbb{R}^k} e^{-iu.w}(1+|w|^2)^{-\lambda-2j} dw = |A_k|^{-\frac{1}{2}} \frac{2^{-\lambda-2j+1}(2\pi)^{\frac{k}{2}}}{\Gamma(\lambda+2j)} |u|^{s+2j} K_{-s-2j}(|u|)$$

when $\Re(\lambda) > \frac{n}{2}$ (see e.g. [3] for the last formula), $K_s(z)$ being the Bessel function defined by

$$K_s(z) := \int_0^\infty \cosh(st) e^{-z\cosh(t)} dt, \quad z > 0. \tag{3.8}$$

It is easy to see that the sum (3.6) with $J = \infty$ converges uniformly for $r > 0$ and $d \in [0, \frac{1}{2})$. When $m = 0$, $R_0(\lambda)$ is the shifted Green kernel of the Laplacian on $\mathbb{H}^{n-k+1}$, that is

$$R_0(\lambda) = d^s \sum_{j=0}^{J-1} \alpha_{j,n-k}(s) d^{2j} + d^{s+2J} G_{J,n-k}(\lambda, d), \quad s = \lambda - \frac{k}{2}. \tag{3.9}$$

The representations (3.6) and (3.9) give a meromorphic extension of $R_m(\lambda)$ to $\mathbb{C}$, with poles on $\frac{k}{2} - \mathbb{N}_0$ of finite rank which only come from the case $m = 0$ when $n - k + 1$ is even. The continuity property of the extented operators on weighted L^2 spaces are a bit technical and checked in [8].

Once we have this resolvent, we can construct, using partition of unity (2.3), a parametrix $E_1(\lambda)$ such that

$$(\Delta_X - \lambda(n-\lambda)) E_1(\lambda) = 1 + K_1(\lambda)$$

with $K_1(\lambda)$ compact on $\rho^a L^2(X)$ for some $a > 0$ in $\Re(\lambda) > \frac{n-1}{2}$.

The final parametrix uses similar arguments as in [18, 12], in particular it just involves the indicial operator of the Laplacian far from the cusps submanifolds. Indeed our parametrix in the cusps is so good that it captures all the singularities in a neighbourhood of the cusps and it just remains to deal with errors which are supported (in the left factor on $\bar{X} \times \bar{X}$) in a compact set of $\bar{X}$ that does not touch the cusp submanifolds, where the Laplacian is an elliptic 0-operator in the sense of [18]. Indeed in M_k we have

$$\Delta_X = -(\rho\partial_\rho)^2 + n\rho\partial_\rho - 2k(\rho^2 + |u|^2)^{-1}\rho^3\partial_\rho + \rho^2 \Delta_{h(\rho)}$$

with $h(\rho) = du^2 + (\rho^2 + |u|^2)^2 dz^2$ a metric on $\{0 < |u| < 1\} \times T^k_z$. The indicial equation is

$$(\Delta_X - \lambda(n-\lambda))\rho^{n-\lambda+j} f - j(2\lambda - n - j)\rho^{n-\lambda+j} f \in \rho^{n-\lambda+j+1} C^\infty(\bar{X}).$$

in $\{\rho < \epsilon\}$ if $f \in C_0^\infty(M)$ (recall $M = \partial\bar{X} \setminus c$).

This allows to construct by induction $E_N(\lambda)$ for any $N > 0$ such that

$$K_N(\lambda) := (\Delta_X - \lambda(n-\lambda)) E_N(\lambda) - 1$$

is compact (trace class) on $\rho^N L^2(X)$ for $\Re(\lambda) > \frac{n}{2} - CN$ where $C < 1$ is a constant. By Fredholm theorem we obtain the meromorphic extension of $R(\lambda)$ on weighted spaces and a standard application of determinant methods shows the bound on the counting function of resonances claimed in Theorem 1.1. Of course many estimates on singular values of $K_N(\lambda)$ in term of λ, N are required and a bit technical, see [8]. The non-optimal bound comes from similar reasons than in [12], first the poles coming from the model resolvent on $\mathbb{H}^{n+1}$ (used for the regular neighbourhood of infinity M_r) with rank which are delicate to bound optimaly: because of the cut-off functions, the cancellations that appear for these ranks in the model $\mathbb{H}^{n+1}$ do not hold here (at least we are not able to prove it). The other problem comes from the fact that it appears difficult to use the dimension reduction through the scattering operator as for surfaces [11] or compact perturbations of Euclidean space [27].

A detailed study of the mapping properties of $R(\lambda)$ using this construction in [9] gives

$$R(\lambda) : \dot{C}^\infty(\bar{X}) \to \rho^\lambda R_c^{-1} C^\infty_{\mathrm{acc}}(\bar{X}) \tag{3.10}$$

which can be compared to the convex co-compact case (with no cusps) $R(\lambda) : \dot{C}^\infty(\bar{X}) \to \rho^\lambda C^\infty(\bar{X})$.

4. Poisson problem and contruction of scattering operator

The construction of the scattering operator can be obtained by 2 methods, either by looking at asymptotic boundary values (or much "infinity values") of the resolvent or by solving a Poisson problem, both methods being equivalent.

The Poisson problem in the convex co-compact case is detailed by Graham-Zworski [6], it can be described by: for λ not resonance, $\Re(\lambda) \geq \frac{n}{2}$ and $\lambda \notin \frac{n}{2} + \mathbb{N}$, then for all $f \in C^\infty(\partial\bar{X})$ there exists a unique $\mathcal{P}(\lambda)f \in C^\infty(X)$ such that

$$\begin{cases} (\Delta_X - \lambda(n-\lambda))\mathcal{P}(\lambda)f = 0 \\ \mathcal{P}(\lambda)f = \rho^{n-\lambda}F(\lambda,f) + \rho^\lambda G(\lambda,f) \\ F(\lambda,f), G(\lambda,f) \in C^\infty(\bar{X}) \\ F(\lambda,f)|_{\rho=0} = f \end{cases}.$$

The scattering operator is the operator on the boundary

$$S(\lambda) : C^\infty(\partial\bar{X}) \to C^\infty(\partial\bar{X}), \quad S(\lambda)f := G(\lambda,f)|_{\partial\bar{X}}$$

it is proved in [14] to be a pseudo-differential operator of order $2\lambda - n$ on $\partial\bar{X}$ with principal symbol

$$\sigma_{\mathrm{pr}}(S(\lambda))(y,\xi) = 2^{n-2\lambda}\frac{\Gamma(\frac{n}{2}-\lambda)}{\Gamma(\lambda-\frac{n}{2})}|\xi|^{2\lambda-n}_{h_0(y)}$$

where $h_0 = \rho^2 g|_{T\partial\bar{X}}$.

In our case with cusp, the same problem can not be solved because of singularities of the objects at the cusps points. Instead a "natural" Poisson problem can be expressed as follows: for λ not resonance, $\Re(\lambda) \geq \frac{n}{2}$ and $\lambda \notin \frac{n}{2} + \mathbb{N}$, then for all $f \in R_c^{-1}C^\infty_{\mathrm{acc}}(\partial\bar{X})$ there exists a unique $\mathcal{P}(\lambda)f \in C^\infty(X)$ such that

$$\begin{cases} (\Delta_X - \lambda(n-\lambda))\mathcal{P}(\lambda)f = 0 \\ \mathcal{P}(\lambda)f = \rho^{n-\lambda}F(\lambda,f) + \rho^\lambda G(\lambda,f) \\ F(\lambda,f), G(\lambda,f) \in R_c^{-1}C^\infty_{\mathrm{acc}}(\bar{X}) \\ F(\lambda,f)|_{\rho=0} = f \end{cases}. \tag{4.1}$$

It can actually be solved using an indicial equation to solve the problem up to $\dot{C}^\infty(\bar{X})$ and then correct with the help of the resolvent. In the neighbourhood M_k of the cusp c_k the Laplacian looks like

$$\Delta_X = -(\rho\partial_\rho)^2 + n\rho\partial_\rho - 2k(\rho^2+|u|^2)^{-1}\rho^3\partial_\rho + \rho^2\Delta_{h(\rho)}$$

with $h(\rho) = du^2 + (\rho^2+|u|^2)^2dz^2$ a metric on $\{0<|u|<1\}\times T^k_z$; by elementary computation we obtain

$$R_c\Delta_X R_c^{-1} = -(\rho\partial_\rho)^2 + n\rho\partial_\rho + \rho^2(\Delta_u + (\rho^2+|u|^2)^{-2}\Delta_z)$$

where Δ_u, Δ_z are the flat Laplacians on $\mathbb{R}^{n-k}_u, T^k_z$. Then it is easy to check the indicial equation

$$(\Delta_X - \lambda(n-\lambda))\rho^{n-\lambda+j}R_c^{-1}f - j(2\lambda-n-j)\rho^{n-\lambda+j}R_c^{-1}f \in \rho^{n-\lambda+j+1}R_c^{-1}C^\infty_{\mathrm{acc}}(\bar{X}).$$

Here, the key fact is that the singular term $r_c^{-4}\Delta_z$ applied to $f \in C^\infty_{acc}(\partial\bar{X})$ gives a functions vanihing at all order at the cusps submanifold c by (2.8). Therefore for all $f \in R_c^{-1}C^\infty_{\mathrm{acc}}(\partial\bar{X})$

one can construct by induction and Borel lemma (see [6] for more details) a function $\Phi(\lambda)f \in \rho^{n-\lambda}R_c^{-1}C^\infty_{\rm acc}(\bar{X})$ for $\lambda \in \mathbb{C} \setminus \frac{1}{2}(n+\mathbb{N})$ such that

$$(\Delta_X - \lambda(n-\lambda))\Phi(\lambda)f \in \dot{C}^\infty(\bar{X}), \quad \rho^{\lambda-n}\Phi(\lambda)f|_{\rho=0} = f.$$

By construction, we have the formal Taylor expansion

$$\Phi(\lambda)f = \rho^{n-\lambda}\sum_{j=0}^{\infty}\rho^{2j}c_{j,\lambda}P_{j,\lambda}f, \quad \forall f \in C^\infty_{\rm acc}(\partial\bar{X}) \tag{4.2}$$

where $P_{j,\lambda}$ is a differential operator on B which is polynomial in λ and

$$c_{j,\lambda} := (-1)^j\frac{\Gamma(\lambda - \frac{n}{2} - j)}{2^{2j}j!\Gamma(\lambda - \frac{n}{2})}.$$

Now we can set for $\lambda \notin \frac{1}{2}(n+\mathbb{N})$ and λ not a resonance

$$\mathcal{P}(\lambda)f = \Phi(\lambda)f - R(\lambda)(\Delta_X - \lambda(n-\lambda))\Phi(\lambda)f \tag{4.3}$$

which satisfies (4.1), thanks to (3.10). To prove uniqueness of the solution of (4.1), we first note that if $\mathcal{P}_1(\lambda)f, \mathcal{P}_2(\lambda)f$ are two solutions, then the indicial equation shows that $\mathcal{P}_1(\lambda)f - \mathcal{P}_2(\lambda)f \in \rho^\lambda R_c^{-1}C^\infty_{\rm acc}(\bar{X})$ thus in $L^2(X)$ for $\Re(\lambda) > \frac{n}{2}$, it is then 0 if $\lambda(n-\lambda) \notin \sigma_{pp}(\Delta_X)$. The case $\Re(\lambda) = \frac{n}{2}$ can be proved by a Green formula exactly as in [6], we refer to [9] for details.

The Poisson operator is proved to have a Schwartz kernel given by weighted limit of the resolvent kernel, as in the convex co-compact case (see [14, 6]). Indeed if we define the Eisenstein function by $E(\lambda) := (\rho^{-\lambda}R(\lambda))|_{B\times X}$ then it is meromorphic in $\lambda \in \mathbb{C}$, satisfies

$$E(\lambda) \in {R_c}^{-1}C^\infty(\partial\bar{X} \times X)$$

and the Schwartz kernel of $\mathcal{P}(\lambda)$ is $(2\lambda - n)E(\lambda; w; b) \in C^\infty(X \times B)$. The Eisenstein functions can be expressed quite explicitly uging the resolvent parametrix construction, it can even be checked [9] that they define Hilbert-Schmidt operators

$$E(\lambda) : \rho^N L^2(X) \to L^2(B)$$

meromorphically in $|\Re(\lambda) - \frac{n}{2}| < CN$ for some constant C, as stated in the introduction. This gives a meromorphic extension of $\mathcal{P}(\lambda)$.

The Eisenstein functions are classically linked to the spectral projectors (via Stone's formula) of Δ_X in the following sense If $\Re(\lambda) = \frac{n}{2}$ and $\lambda \neq \frac{n}{2}$ then

$$R(\lambda; w; w') - R(n-\lambda; w; w') = (n-2\lambda)\int_B E(\lambda; b; w')E(n-\lambda; b; w)\ {\rm dvol}_{h_0}(b) \tag{4.4}$$

where $h_0 = (\rho^2 g)|_B$. Moreover there exists $C > 0$ such that for N large, we have

$$R(\lambda) - R(n-\lambda) = (2\lambda - n)^t E(n-\lambda)E(\lambda)$$

in the strip $|\Re(\lambda) - \frac{n}{2}| \leq CN$ as operators from $\rho^N L^2(X)$ to $\rho^{-N}L^2(X)$. This relation and the definition of $E(\lambda)$ prove that resolvent and Poisson operator (or Eisenstein function) have same poles, except possibly at the points λ such that $\lambda(n-\lambda) \in \sigma_{pp}(\Delta_X)$.

Using notations of (4.1), we can define the scattering operator as the linear operator

$$S(\lambda) : \left\{ \begin{array}{ccc} R_c^{-1}C^\infty_{\rm acc}(\partial\bar{X}) & \to & R_c^{-1}C^\infty_{\rm acc}(\partial\bar{X}) \\ f & \to & G(\lambda, f)|_B \end{array} \right. \tag{4.5}$$

for $\Re(\lambda) \geq \frac{n}{2}$, $\lambda \notin \frac{1}{2}(n+\mathbb{N})$ and λ not a resonance. With (4.3), one obtains a meromorphic continuation of $S(\lambda)$ to $\mathbb{C}$.

The Eisenstein function $E(\lambda; b, w')$ are actually continuous function on $(b, w') \in B \times (\bar{X} \setminus c)$ if $\Re(\lambda) < 0$. Then from (4.3), (4.5) and the fact that the Poisson operator has ${}^tE(\lambda)$ for kernel, we deduce that for $f \in C_0^\infty(B)$ and $\Re(\lambda) < 0$

$$S(\lambda)f = \lim_{\rho\to 0}[\rho^{-\lambda}((2\lambda - n){}^tE(\lambda)f - \Phi(\lambda)f)] = (2\lambda - n)\lim_{\rho\to 0}[\rho^{-\lambda}({}^tE(\lambda)f)]$$

which is well defined. As a consequence the distributional kernel of $S(\lambda)$ on B is

$$S(\lambda; b; b') = (2\lambda - n)\lim_{w'\to b'}(\rho(w')^{-\lambda}E(\lambda; b; w'))$$

which can be rewritten using the symmetry of the resolvent kernel as the restriction

$$S(\lambda) = (2\lambda - n)(\rho^{-\lambda}\rho'^{-\lambda}R(\lambda))|_{\rho=\rho'=0} \tag{4.6}$$

for $\Re(\lambda) < 0$ and λ not resonance. This last relation extends meromorhically to $\mathbb{C}$. If $\Re(\lambda) < 0$, we have by standard arguments, for $w \in X$, $b' \in B$,

$$E(\lambda; b'; w) = -\int_B S(\lambda; b'; b)E(n - \lambda; b; w)\ \mathrm{dvol}_{h_0}(b)$$

and there exists $C > 0$ such that for N large the meromorphic identity

$$E(\lambda) = -S(\lambda)E(n - \lambda) \tag{4.7}$$

holds true in the strip $\frac{n}{2} - CN < \Re(\lambda) \leq \frac{n}{2}$ as operators from $\rho^N L^2(X)$ to $L^2(B)$. Now, the identities (4.6) and (4.7) shows that the poles of $S(\lambda)$ in $\Re(\lambda) < \frac{n}{2}$ are exactly the poles of the resolvent, i.e. the resonances. However we are not able for the moment to obtain explicit relations between their multiplicities as in the convex co-compact case [7].

In $\Re(\lambda) > \frac{n}{2}$, additional poles occur at $\frac{n}{2} + \mathbb{N}$ for the scattering poles, exactly as in Graham-Zworski [6], these are infinite rank, first order, poles coming from the construction of $\Phi(\lambda)f$ in (4.2). We prove in [9] that the residue are differential operators on the boundary B, and actually they are exactly the GJMS conformal powers of the Laplacian defined in [5]. The proof goes through by mimicking the arguments of Graham-Zworski in this setting, in particular

$$\mathrm{Res}_{\lambda_0}S(\lambda) = \begin{cases} -\frac{(-1)^{j+1}2^{-2j}}{j!(j-1)!}P_j + \Pi_{\lambda_0} & \text{if } \lambda_0 = \frac{n}{2} + j, j \in \mathbb{N} \\ \Pi_{\lambda_0} & \text{if } \lambda_0 \notin \frac{n}{2} + \mathbb{N} \end{cases}$$

where P_j is the differential operator on (B, h_0) with principal symbol $\sigma_0(P_j) = |\xi|^{2j}_{h_0}$, defined by

$$[\mathrm{Res}_{\frac{n}{2}+j}\rho^{-\lambda}\Phi(\lambda)]|_{\rho=0} = \frac{(-1)^j 2^{-2j}}{j!(j-1)!}P_j \tag{4.8}$$

and Π_{λ_0} is a finite-rank operator with Schwartz kernel $2j\left((\rho\rho')^{-\lambda_0}\mathrm{Res}_{\lambda_0}R(\lambda)\right)|_{B\times B}$ satisfying $\mathrm{rank}\,\Pi_{\lambda_0} = \dim\ker_{L^2}(\Delta_X - \lambda_0(n - \lambda_0))$. The fact that P_j in (4.8) is the GJMS operator comes from the constant curvature, which make every derivative $\partial_\rho h(0)$ in (2.9) locally determined by $h(0)$ in the same way that for convex co-compact manifolds (which are particular cases of Poincaré-Einstein manifold).

5. The scattering operator as a Φ-Pdeudo-differential operator on the boundary

The manifold B compactifies into $\bar{B}$ and has ends diffomorphic to $[0, \epsilon) \times S^{n-k-1} \times T^k$, each end arising from cusp points. The compactification is done by radially compactifying $(\mathbb{R}^{n-k}_y \setminus \{|y| < 1\}) \times T^k_z$ in y. The metric h_0 is of the form

$$h_0 = |y|^{-4}(dy^2 + dz^2)$$

near the cusp c_k, in the compactifiction coordinates $v := |y|^{-1}, \omega := yv$ this gives

$$h_0 = dv^2 + v^2 d\omega^2 + v^4 dz^2.$$

The boundary of $\bar{B}$ is $S^{n-k-1} \times T^k$ and has a natural fibration over S^{n-k-1} (the projection), this gives $\bar{B}$ a structure of manifold with fibred boundary in the sense of Mazzeo-Melrose [19], let us call Φ the fibration. In [19], the authors define a general class of pseudo-differential operators by their Schwartz kernel lifted on some manifold with corners $\bar{B} \times_\Phi \bar{B}$ which is the blow-up of $\bar{B} \times \bar{B}$ at some submanifolds defined by the fibration Φ. In some sense, these class capure the different growth behaviour of the symbol in phase space, recall that here the manifold B is not compact.

The study of the Schwartz kernel of $S(\lambda)$ in compact parts of the manifold B (roughly coming from the regular neighbourhood of infinity of $\bar{X}$) are very similar to the convex co-compact case: there is a classical conormal singularity that make them local pseudo-differential operators of order $2\lambda - n$. The most interesting part comes from the cusps: we can deduce from the resolvent construction and (4.6) that all the main singularities of $S(\lambda)$ near c_k are contained in the term

$$c(\lambda)\psi_L^k |y|^{2\lambda-n} \Delta_{Y_k}^{\lambda-\frac{n}{2}} |y|^{2\lambda-n} \psi^k \text{ with } c(\lambda) := 2^{n-2\lambda} \frac{\Gamma(\frac{n}{2}-\lambda)}{\Gamma(\lambda-\frac{n}{2})}.$$

(viewed as acting on $L^2(Y_k, dydz)$) where ψ_L^k, ψ^k are cut-off functions equal to 1 near the cusp c_k and $Y_k := \mathbb{R}^{n-k} \times T^k$. The growth of its symbol in the phase space has two kind of behaviour since Δ_{Y_k} can be roughly decomposed (by Fourier transform) as a Laplacian on $\mathbb{R}^{n-k}$ corresponding to the constant mode in the torus T^k and a family of $\Delta_{\mathbb{R}^{n-k}} + c_j$ where $c_j > 0$ are the non-zero eigenvalues of Δ_{T^k}. For instance, the Schwartz kernel $\kappa_j(\lambda, y, y')$ of last term has fast vanishing in space at infinity in $|y|, |y'|$ variables outside the diagonal whereas the first term has kernel $\kappa_0(\lambda, y, y')$ with a growth $|y|^{k-n}|y'|^{k-n}$.

These different asymptotic regimes are analyzed in detail on the blow-up manifold $\bar{B} \times_\Phi \bar{B}$ in the paper [9], we refer the reader there for details.

We finally emphasize that in a model irrational cusp case $\Gamma_\infty \backslash \mathbb{H}^{n+1}$, a spectral decomposition of the Laplacian $\Delta_{\Gamma_\infty \backslash \mathbb{H}^{n+1}}$ leads to the study of a family complex powers of operators of the form $\Delta_{\mathbb{R}^{n-k}} + c_j$ as before, but the terms c_j accumulate at 0, contrary to the rational case where there is a spectral gap.

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Département de mathématiques J.Dieudonné, Université de Nice, Parc Valrose, Nice, France
E-mail address: `cguillar@math.unice.fr`