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ON THE SOLVABILITY OF PSEUDODIFFERENTIAL OPERATORS

NILS DENCKER

1. Introduction

In this paper we shall study the question of local solvability of a classical pseudodifferential operator $P \in \Psi^m_{cl}(M)$ on a C^∞ manifold M. Thus, we assume that the symbol of P is an asymptotic sum of homogeneous terms, and that $p = \sigma(P)$ is the homogeneous principal symbol of P. We shall also assume that P is of principal type, which means that the Hamilton vector field H_p and the radial vector field are linearly independent when $p = 0$, thus $dp \neq 0$ when $p = 0$.

Local solvability of P at a compact set $K \subseteq M$ means that the equation

$$Pu = v \tag{1.1}$$

has a local solution $u \in \mathcal{D}'(M)$ in a neighborhood of K for any $v \in C^\infty(M)$ in a set of finite codimension. One can also define microlocal solvability at any compactly based cone $K \subset T^*M$, see [11, Definition 26.4.3]. Hans Lewy's famous counterexample [21] from 1957 showed that not all smooth linear differential operators are solvable. It was conjectured by Nirenberg and Treves [23] in 1970 that local solvability of principal type pseudodifferential operators is equivalent to condition (Ψ), which means that

$$\begin{aligned}&\operatorname{Im}(ap) \text{ does not change sign from } - \text{ to } + \\ &\qquad\qquad\qquad \text{along the oriented bicharacteristics of } \operatorname{Re}(ap)\end{aligned} \tag{1.2}$$

for any $0 \neq a \in C^\infty(T^*M)$. The oriented bicharacteristics are the positive flow-outs of the Hamilton vector field $H_{\operatorname{Re}(ap)} \neq 0$ on $\operatorname{Re}(ap) = 0$ (also called semi-bicharacteristics). Condition (1.2) is invariant under multiplication of p with non-vanishing factors, and conjugation of P with elliptic Fourier integral operators, see [11, Lemma 26.4.10].

The necessity of (Ψ) for local solvability of pseudodifferential operators was proved by Moyer [22] in 1978 for the two dimensional case, and by Hörmander [10] in 1981 for the general case. In the analytic category, the sufficiency of condition (Ψ) for solvability of microdifferential operators acting on microfunctions was proved by Trépreau [24] in 1984 (see also [12, Chapter VII]). The sufficiency of condition (Ψ) for solvability of pseudodifferential operators in two dimensions was proved by Lerner [15] in 1988, leaving the higher dimensional case open.

For differential operators, condition (Ψ) is equivalent to condition (P), which rules out any sign changes of $\operatorname{Im}(ap)$ along the bicharacteristics of $\operatorname{Re}(ap)$ for non-vanishing $a \in C^\infty(T^*M)$. The sufficiency of (P) for local solvability of pseudodifferential operators was proved in 1970 by Nirenberg and Treves [23] in the case when the principal symbol is real analytic. Beals and Fefferman [1] proved the general case in 1973, by using a new calculus that was later developed by Hörmander into the Weyl calculus.

In all these solvability results, one obtains *a priori* estimates for the adjoint operator with loss of one derivative (compared with the elliptic case). In 1994 Lerner [16] constructed counterexamples to the sufficiency of (Ψ) for local solvability with loss of one derivative in dimensions greater than two, raising doubts on whether the condition really was sufficient for solvability. But the author proved in 1996 [4] that Lerner's counterexamples are locally solvable with loss of at most two derivatives (compared with the elliptic case). There are several other results giving local solvability under conditions stronger than (Ψ), see [5], [13] and [17]. The Nirenberg–Treves conjecture was finally resolved by the author [8], proving solvability with a loss of two derivatives (compared with the elliptic case). This has been improved to a loss of arbitrarily more than 3/2 derivatives by the author [7]. Recently Lerner [20] has improved the result to a loss of exactly 3/2 derivatives.

In this paper we shall show how the proof of [8] can be adapted to give solvability with a loss of 3/2 derivatives, using some ideas of Lerner [20]. We shall rely on the results of [8] and only emphasize the changes to the proofs. To get local solvability at a point x_0 we shall also assume a strong form of the non-trapping condition at x_0:

$$p = 0 \implies \partial_\xi p \neq 0. \tag{1.3}$$

This means that all semi-bicharacteristics are transversal to the fiber $T^*_{x_0}M$, which originally was the condition for principal type of Nirenberg and Treves [23]. Microlocally, we can always obtain (1.3) after a canonical transformation.

Theorem 1.1. *If $P \in \Psi^m_{cl}(M)$ is of principal type and satisfies condition (Ψ) given by (1.2) microlocally near $(x_0, \xi_0) \in T^*M$, then we obtain*

$$\|u\| \le C(\|P^*u\|_{(3/2-m)} + \|Ru\| + \|u\|_{(-1)}) \qquad u \in C_0^\infty(M). \tag{1.4}$$

Here $R \in \Psi^1_{1,0}(M)$ such that $(x_0, \xi_0) \notin \mathrm{WF}\, R$, which gives microlocal solvability of P at (x_0, ξ_0) with a loss of at most 3/2 derivatives. If P satisfies conditions (Ψ) and (1.3) locally near $x_0 \in M$, then we obtain (1.4) with $x \neq x_0$ in $\mathrm{WF}\, R$, which gives local solvability of P at x_0 with a loss of at most 3/2 derivatives.

Observe that there are no counterexamples showing a loss of more that $1+\varepsilon$ derivatives, for arbitrarily small ε. The method of proof is essentially the same as in [8], but we shall also use some improvements of Lerner [20] and Hörmander [14].

2. The multiplier estimate

Next, we shall microlocalize and reduce the proof of Theorem 1.1 to the semiclassical multiplier estimate of Proposition 2.5 for a microlocal normal form of the adjoint operator. We shall consider operators

$$P_0 = D_t + iF(t, x, D_x) \tag{2.1}$$

where $F \in C^\infty(\mathbf{R}, \Psi^1_{1,0}(\mathbf{R}^n))$ has real principal symbol $\sigma(F) = f$. In the following, we shall assume that P_0 satisfies condition $(\overline{\Psi})$:

$$f(t, x, \xi) > 0 \quad \text{and } s > t \implies f(s, x, \xi) \ge 0 \tag{2.2}$$

for any t, $s \in \mathbf{R}$ and $(x, \xi) \in T^*\mathbf{R}^n$. This means that the L^2 adjoint P_0^* satisfies condition (Ψ). Observe that if $\chi \ge 0$ then χf also satisfies (2.2), thus the condition can be localized.

Remark 2.1. *We shall also consider symbols $f \in L^\infty(\mathbf{R}, S^1_{1,0}(\mathbf{R}^n))$, that is, $f(t, x, \xi) \in L^\infty(\mathbf{R} \times T^*\mathbf{R}^n)$ is bounded in $S^1_{1,0}(\mathbf{R}^n)$ for almost all t. Then we say that P_0 satisfies condition $(\overline{\Psi})$ if for every (x, ξ) condition* (2.2) *holds for almost all s, $t \in \mathbf{R}$. We find that f has a representative satisfying* (2.2) *for any t, s and (x, ξ) after putting $f(t, x, \xi) \equiv 0$ for t in a countable union of null sets.*

In fact, since $(x, \xi) \mapsto f(t, x, \xi)$ is continuous for almost all t it suffices to check (2.2) for (x, ξ) in a countable dense subset of $T^*\mathbf{R}^n$.

In order to prove Theorem 1.1 we shall make a second microlocalization using the specialized symbol classes of the Weyl calculus, and the Weyl quantization of symbols $a \in \mathcal{S}'(T^*\mathbf{R}^n)$ defined by:

$$(a^w u, v) = (2\pi)^{-n} \iint \exp\left(i\langle x - y, \xi\rangle\right) a\left(\tfrac{x+y}{2}, \xi\right) u(y)\overline{v}(x)\, dx dy d\xi \qquad u, v \in \mathcal{S}(\mathbf{R}^n).$$

Observe that $\operatorname{Re} a^w = (\operatorname{Re} a)^w$ is the symmetric part and $i \operatorname{Im} a^w = (i \operatorname{Im} a)^w$ the antisymmetric part of the operator a^w. Also, if $a \in S^m_{1,0}(\mathbf{R}^n)$ then $a^w(x, D_x) = a(x, D_x)$ modulo $\Psi^{m-1}_{1,0}(\mathbf{R}^n)$ by [11, Theorem 18.5.10].

We recall the definitions of the Weyl calculus: let g_w be a Riemannean metric on $T^*\mathbf{R}^n$, $w = (x, \xi)$, then we say that g is slowly varying if there exists $c > 0$ so that $g_{w_0}(w - w_0) < c$ implies $g_w \cong g_{w_0}$, i.e., $1/C \le g_w / g_{w_0} \le C$. Let σ be the standard symplectic form on

$T^*\mathbf{R}^n$, and assume $g^\sigma(w) \ge g(w)$ where g^σ is the dual metric of $w \mapsto g(\sigma(w))$. We say that g is σ temperate if it is slowly varying and

$$g_w \le C g_{w_0}(1 + g_w^\sigma(w - w_0))^N \qquad w,\, w_0 \in T^*\mathbf{R}^n.$$

A positive real valued function $m(w)$ on $T^*\mathbf{R}^n$ is g continuous if there exists $c > 0$ so that $g_{w_0}(w - w_0) < c$ implies $m(w) \cong m(w_0)$. We say that m is σ, g temperate if it is g continuous and

$$m(w) \le C m(w_0)(1 + g_w^\sigma(w - w_0))^N \qquad w,\, w_0 \in T^*\mathbf{R}^n.$$

If m is σ, g temperate, then m is a weight for g and we can define the symbol classes: $a \in S(m, g)$ if $a \in C^\infty(T^*\mathbf{R}^n)$ and

$$|a|_j^g(w) = \sup_{T_i \ne 0} \frac{|a^{(j)}(w, T_1, \dots, T_j)|}{\prod_1^j g_w(T_i)^{1/2}} \le C_j m(w) \qquad w \in T^*\mathbf{R}^n \qquad \text{for } j \ge 0, \tag{2.3}$$

which gives the seminorms of $S(m, g)$. If $a \in S(m, g)$ then we say that the corresponding Weyl operator $a^w \in \operatorname{Op} S(m, g)$. For more on the Weyl calculus, see [11, Section 18.5].

Definition 2.2. Let m be a weight for the metric g. We say that $a \in S^+(m, g)$ if $a \in C^\infty(T^*\mathbf{R}^n)$ and $|a|_j^g \le C_j m$ for $j \ge 1$.

Observe that by Taylor's formula we find that

$$|a(w) - a(w_0)| \le C_1 \sup_{\theta \in [0,1]} g_{w_\theta}(w - w_0)^{1/2} m(w_\theta) \le C' m(w_0)(1 + g_{w_0}^\sigma(w - w_0))^{(3N+1)/2} \tag{2.4}$$

where $w_\theta = \theta w + (1 - \theta) w_0$, which implies that $m + |a|$ is a weight for g. Clearly, $a \in S(m + |a|, g)$, so the operator a^w is well-defined.

Lemma 2.3. *Assume that m_j is a weight for $g_j = h_j g^\sharp \le g^\sharp = (g^\sharp)^\sigma$ and $a_j \in S^+(m_j, g_j)$, $j = 1$, 2. Let $g = g_1 + g_2$ and $h^2 = \sup g_1/g_2^\sigma = \sup g_2/g_1^\sigma = h_1 h_2$, then*

$$a_1^w a_2^w - (a_1 a_2)^w \in \operatorname{Op} S(m_1 m_2 h, g). \tag{2.5}$$

We also obtain the usual expansion of (2.5) with terms in $S(m_1 m_2 h^k, g)$, $k \ge 1$. We also have that

$$\operatorname{Re} a_1^w a_2^w - (a_1 a_2)^w \in \operatorname{Op} S(m_1 m_2 h^2, g). \tag{2.6}$$

if $a_j \in C^\infty(T^*\mathbf{R}^n)$ is real and $|a_j|_k^{g_j} \le C_k m_j$, $k \ge 2$, for $j = 1$, 2. In that case we have $a_j \in S(m_j + |a_j| + |a_j|_1^{g_j}, g_j)$.

Proof. As shown after Definition 2.2 we have that $m_j + |a_j|$ is a weight for g_j and $a_j \in S(m_j + |a_j|, g_j)$, $j = 1$, 2. Thus $a_1^w a_2^w \in \operatorname{Op} S((m_1 + |a_1|)(m_2 + |a_2|), g)$ is given by Proposition 18.5.5 in [11]. We find that $a_1^w a_2^w - (a_1 a_2)^w = a^w$ with

$$a(w) = E(\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2}))\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2}) a_1(w_1) a_2(w_2)\big|_{w_1 = w_2 = w}$$

where $E(z) = (e^z - 1)/z = \int_0^1 e^{\theta z}\, d\theta$. Here $\sigma(D_{w_1}, D_{w_2}) a_1(w_1) a_2(w_2) \in S(MH, G)$ where $M(w_1, w_2) = m_1(w_1) m_2(w_2)$, $G_{w_1,w_2}(z_1, z_2) = g_{1,w_1}(z_1) + g_{2,w_2}(z_2)$ and $H^2 = \sup G/G^\sigma$ so that $H(w, w) = h(w)$. Now the proof of Theorem 18.5.5 in [11] works when $\sigma(D_{w_1}, D_{w_2})$ is replaced by $\theta\sigma(D_{w_1}, D_{w_2})$, uniformly in $0 \le \theta \le 1$. By integrating over $\theta \in [0, 1]$ we obtain that $a(w)$ has an asymptotic expansion in $S(m_1 m_2 h^k, g)$, which proves (2.5). If $|a_j|_k^{g_j} \le C_k m_j$, $k \ge 2$, then we have by Taylor's formula as in (2.4) that

$$
\begin{aligned}
|a(w) - a(w_0)| &\le g_{w_0}(w - w_0)^{1/2} |a_j|_1^g(w_0) + C_1 \sup_{\theta \in [0,1]} g_{w_\theta}(w - w_0) m(w_\theta) \\
&\le C'(|a_j|_1^g(w_0) + m(w_0))(1 + g_{w_0}^\sigma(w - w_0))^{2N+1}
\end{aligned}
$$

$$
\begin{aligned}
|\langle T, \partial_w a_j(w)\rangle - \langle T, \partial_w a_j(w_0)\rangle| &\le C_2 \sup_{\theta \in [0,1]} g_{w_\theta}(T)^{1/2} g_{w_\theta}(w - w_0)^{1/2} m(w_\theta) \\
&\le C_3 g_{w_0}(T)^{1/2} m(w_0)(1 + g_{w_0}^\sigma(w - w_0))^{(4N+1)/2}
\end{aligned}
$$

thus $m_j + |a_j| + |a_j|_1^{g_j}$ is a weight for g_j and clearly $a_j \in S(m_j + |a_j| + |a_j|_1^{g_j}, g_j)$. Now if a_1 and a_2 are real, then $\operatorname{Re} a_1^w a_2^w - (a_1 a_2)^w = a^w$ with

$$
a(w) = \operatorname{Re} E(\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2}))(\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2}))^2 a_1(w_1) a_2(w_2)/2\big|_{w_1 = w_2 = w}
$$

where $\sigma(D_{w_1}, D_{w_2})^2 a_1(w_1) a_2(w_2) \in S(MH^2, G)$, with the same E, M, G and H as before. The proof of (2.6) follows in the same way as the proof of (2.5). □

Remark 2.4. *The conclusions of Lemma 2.3 also hold if a_1 has values in $\mathcal{L}(B_1, B_2)$ and a_2 in B_1 where B_1 and B_2 are Banach spaces, then $a_1^w a_2^w$ has values in B_2.*

Let $\|u\|$ be the L^2 norm on $\mathbf{R}^{n+1}$, and (u, v) the corresponding sesquilinear inner product. As before, we say that $f \in L^\infty(\mathbf{R}, S(m, g))$ if $f(t, x, \xi)$ is measurable and bounded in $S(m, g)$ for almost all t. The following is the semiclassical estimate that we shall prove in this note.

Proposition 2.5. *Assume that $P_0 = D_t + i f^w(t, x, D_x)$, with real $f \in L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$ satisfying condition $(\overline{\Psi})$ given by (2.2), here $0 < h \le 1$ and $g^\sharp = (g^\sharp)^\sigma$ are constant. Then there exists $T_0 > 0$ and real valued symbols $b_T(t, x, \xi) \in L^\infty(\mathbf{R}, S(h^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp))$ uniformly for $0 < T \le T_0$, so that*

$$
h^{1/2}\left(\|b_T^w u\|^2 + \|u\|^2\right) \le C_0 T \operatorname{Im}(P_0 u, b_T^w u) \tag{2.7}
$$

for $u(t, x) \in \mathcal{S}(\mathbf{R} \times \mathbf{R}^n)$ having support where $|t| \le T$. The constants C_0, T_0 and the seminorms of b_T only depend on the seminorms of f in $L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$.

Observe that it follows from (2.7) by the Cauchy-Schwarz inequality that

$$
\|u\| \le CTh^{-1/2}\|P_0 u\|,
$$

which will give a loss of 3/2 derivatives after microlocalization. In fact, by microlocalizing near (x_0, ξ_0), letting $h^{-1} = \langle \xi_0 \rangle = 1 + |\xi_0|$ and doing a symplectic dilation: $(x, \xi) \mapsto (h^{-1/2}x, h^{1/2}\xi)$, we find that $S^k_{1,0} = S(h^{-k}, hg^\sharp)$ and $S^k_{1/2,1/2} = S(h^{-k}, g^\sharp)$, $(g^\sharp)^\sigma = g^\sharp$, $k \in \mathbf{R}$. Proposition 2.5 will be proved at the end of Section 6.

There are two difficulties present in estimates of the type (2.7). The first is that b_T is not C^∞ in the t variables, therefore one has to be careful not to involve b^w_T in the calculus with symbols in all the variables. We shall avoid this problem by using tensor products of operators and the Cauchy-Schwarz inequality. The second difficulty lies in the fact that $|b_T| \gg h^{1/2}$, so it is not obvious that lower order terms and cut-off errors can be controlled. To resolve this difficulty, we recall Lemma 2.6 from [8].

Lemma 2.6. *The estimate* (2.7) *can be perturbed with terms in* $L^\infty(\mathbf{R}, S(1, hg^\sharp))$ *in the symbol of* P_0 *for small enough* T*, by changing* b_T *(satisfying the same conditions). Thus it can be microlocalized: if* $\phi(w) \in S(1, hg^\sharp)$ *is real valued and independent of* t*, then we have*

$$\operatorname{Im}(P_0\phi^w u, b^w_T \phi^w u) \le \operatorname{Im}(P_0 u, \phi^w b^w_T \phi^w u) + Ch^{1/2}\|u\|^2 \tag{2.8}$$

where $\phi^w b^w_T \phi^w$ *satisfies the same conditions as* b^w_T*.*

In the following, we shall use the norms:

$$\|u\|_s = \|\langle D_x \rangle^s u\|, \tag{2.9}$$

and we shall prove an estimate for the microlocal normal form of the adjoint operator.

Corollary 2.7. *Assume that* $P_0 = D_t + iF^w(t, x, D_x)$*, with* $F^w \in L^\infty(\mathbf{R}, \Psi^1_{1,0}(\mathbf{R}^n))$ *having real principal symbol* f *satisfying condition* $(\overline{\Psi})$ *given by* (2.2)*. Then there exists* $T_0 > 0$ *and real valued symbols* $b_T(t, x, \xi) \in L^\infty(\mathbf{R}, S^1_{1/2,1/2}(\mathbf{R}^n))$ *with homogeneous gradient* $\nabla b_T = (\partial_x b_T, |\xi|\partial_\xi b_T) \in L^\infty(\mathbf{R}, S^1_{1/2,1/2}(\mathbf{R}^n))$ *uniformly for* $0 < T \le T_0$*, such that*

$$\|b^w_T u\|^2_{-1/2} + \|u\|^2 \le C_0(T \operatorname{Im}(P_0 u, b^w_T u) + \|u\|^2_{-1}) \tag{2.10}$$

for $u \in \mathcal{S}(\mathbf{R}^{n+1})$ *having support where* $|t| \le T$*. The constants* T_0*,* C_0 *and the seminorms of* b_T *only depend on the seminorms of* F *in* $L^\infty(\mathbf{R}, S^1_{1,0}(\mathbf{R}^n))$*.*

Since $\nabla b_T \in L^\infty(\mathbf{R}, S^1_{1/2,1/2})$ we find that the commutators of b^w_T with operators in $L^\infty(\mathbf{R}, \Psi^0_{1,0})$ are in $L^\infty(\mathbf{R}, \Psi^0_{1/2,1/2})$. This will make it possible to localize the estimate. The idea to use the first term in (2.7) and (2.10) is due to Lerner [20].

Proof of that Proposition 2.5 gives Corollary 2.7. Choose real symbols $\{\phi_j(x, \xi)\}_j$ and $\{\psi_j(x, \xi)\}_j \in S^0_{1,0}(\mathbf{R}^n)$ having values in ℓ^2, such that $\sum_j \phi_j^2 = 1$, $\psi_j \phi_j = \phi_j$ and $\psi_j \ge 0$. We may assume that the supports are small enough so that $\langle \xi \rangle \cong \langle \xi_j \rangle$ in $\operatorname{supp} \psi_j$ for some ξ_j, and that there is a fixed bound on number of overlapping supports. Then, after doing a symplectic dilation $(y,\eta) = (x\langle\xi_j\rangle^{1/2}, \xi/\langle\xi_j\rangle^{1/2})$ we obtain that $S^m_{1,0}(\mathbf{R}^n) = S(h_j^{-m}, h_j g^\sharp)$ and $S^m_{1/2,1/2}(\mathbf{R}^n) = S(h_j^{-m}, g^\sharp)$ in $\operatorname{supp}\psi_j$, $m \in \mathbf{R}$, where $h_j = \langle\xi_j\rangle^{-1} \le 1$ and $g^\sharp(dy,d\eta) = |dy|^2 + |d\eta|^2$.

By using the calculus in the y variables we find $\phi_j^w P_0 = \phi_j^w P_{0j}$ modulo $\operatorname{Op} S(h_j, h_j g^\sharp)$, where

$$P_{0j} = D_t + i(\psi_j F)^w(t,y,D_y) = D_t + if_j^w(t,y,D_y) + r_j^w(t,y,D_y)$$

with $f_j = \psi_j f \in L^\infty(\mathbf{R}, S(h_j^{-1}, h_j g^\sharp))$ satisfying (2.2), and $r_j \in L^\infty(\mathbf{R}, S(1, h_j g^\sharp))$ uniformly in j. Then, by using Proposition 2.5 and Lemma 2.6 for P_{0j} we obtain real valued symbols $b_{j,T}(t,y,\eta) \in L^\infty(\mathbf{R}, S(h_j^{-1/2}, g^\sharp) \bigcap S^+(1,g^\sharp))$ uniformly for $0 < T \ll 1$, such that

$$\|b_{j,T}^w \phi_j^w u\|^2 + \|\phi_j^w u\|^2 \le C_0 T(h_j^{-1/2} \operatorname{Im}\left(P_0 u, \phi_j^w b_{j,T}^w \phi_j^w u\right) + \|u\|^2) \qquad \forall\, j \tag{2.11}$$

for $u \in \mathcal{S}$ having support where $|t| \le T$. By substituting $\psi_j^w u$ in (2.11) we obtain that

$$\|b_{j,T}^w \phi_j^w \psi_j^w u\|^2 + \|\phi_j^w \psi_j^w u\|^2 \le C_0 T(h_j^{-1/2} \operatorname{Im}\left(P_0 \psi_j^w u, \phi_j^w b_{j,T}^w \phi_j^w \psi_j^w u\right) + \|\psi_j^w u\|^2) \tag{2.12}$$

for $u \in \mathcal{S}$ having support where $|t| \le T$. Here

$$h_j^{-1/2} \operatorname{Im}\left(P_0 \psi_j^w u, \phi_j^w b_{j,T}^w \phi_j^w \psi_j^w u\right) = h_j^{-1/2} \langle [P_0, \psi_j^w] u, \phi_j^w b_{j,T}^w \phi_j^w \psi_j^w u\rangle + \langle P_0 u, B_{j,T}^w u\rangle$$

where $B_{j,T}^w = h_j^{-1/2} \psi_j^w \phi_j^w b_{j,T}^w \phi_j^w \psi_j^w \in \operatorname{Op} S(h^{-1}, g^\sharp)$ is symmetric. Now $[P_0, \psi_j^w] = [F^w, \psi_j^w]$ and the calculus give that $\left\{ h_j^{-1/2} b_{j,T}^w \phi_j^w [F^w, \psi_j^w] \right\}_j \in \Psi^0_{1,0}(\mathbf{R}^n)$ with values in ℓ^2 for almost all t, which gives

$$\sum_j h_j^{-1/2} \langle [P_0, \psi_j^w] u, \phi_j^w b_{j,T}^w \phi_j^w \psi_j^w u\rangle \le C\|u\|^2.$$

Now, $\sum_j \phi_j^2 = 1$ and $\phi_j \psi_j = \phi_j$ so the calculus gives

$$\|u\|^2 \le \sum_j \|\phi_j^w \psi_j^w u\|^2 + C\|u\|^2_{-1}.$$

Let $b_T^w = \sum_j B_{j,T}^w \in L^\infty(\mathbf{R}, \Psi^1_{1/2,1/2})$, then we find by the finite bound on the overlap of the supports that

$$b_T^w \langle D_x\rangle^{-1} b_T^w = \sum_{|j-k|\le N} B_{j,T}^w \langle D_x\rangle^{-1} B_{k,T}^w \qquad \text{modulo } \Psi^0(\mathbf{R}^n)$$

for some N, thus

$$\|b_T^w u\|^2_{-1/2} = \|\langle D_x\rangle^{-1/2} b_T^w u\|^2 \le C_N \left(\sum_j \|B_{j,T}^w u\|^2_{-1/2} + \|u\|^2 \right).$$

We also have $\langle D_x\rangle^{-1/2} h_j^{-1/2} \psi_j^w \phi_j^w \in \Psi^0(\mathbf{R}^n)$ uniformly which gives

$$\|B_{j,T}^w u\|_{-1/2} \le C\|b_{j,T}^w \psi_j^w \phi_j^w u\|.$$

Thus, by summing up we obtain

$$\|b_T^w u\|^2_{-1/2} + \|u\|^2 \le C_1 \left(T(\operatorname{Im}(P_0 u, b_T^w u) + \|u\|^2) + \|u\|^2_{-1}\right) \tag{2.13}$$

for $u \in \mathcal{S}$ having support where $|t| \le T$. The homogeneous gradient $\nabla b_T \in S^1_{1/2,1/2}$ since $b_T = \sum_j h_j^{-1/2} b_{j,T}\phi_j^2 \in S^1_{1/2,1/2}$ modulo $S^0_{1/2,1/2}$, where $\phi_j \in S(1, h_j g^\sharp)$ is supported where $\langle \xi \rangle \simeq h_j^{-1}$ and $b_{j,T} \in S^+(1, g^\sharp)$ for almost all t. For small enough T we obtain (2.10) and the corollary. $\square$

Proof that Corollary 2.7 gives Theorem 1.1. We shall prove that there exists ϕ and $\psi \in S^0_{1,0}(T^*M)$ such that $\phi = 1$ in a conical neighborhood of (x_0, ξ_0), $\psi = 1$ on $\operatorname{supp}\phi$, and for any $T > 0$ there exists $R_T \in S^1_{1,0}(M)$ with the property that $\mathrm{WF}\, R_T^w \bigcap T^*_{x_0}M = \emptyset$ and

$$\|\phi^w u\| \le C_1 \left(\|\psi^w P^* u\|_{(3/2-m)} + T\|u\| \right) + \|R_T^w u\| + C_0 \|u\|_{(-1)} \qquad u \in C_0^\infty(M). \tag{2.14}$$

Here $\|u\|_{(s)}$ is the Sobolev norm and the constants are independent of T. Then for small enough T we obtain (1.4) and microlocal solvability, since $(x_0, \xi_0) \notin \mathrm{WF}(1-\phi)^w$. In the case that P satisfies condition (Ψ) and $\partial_\xi p \ne 0$ near x_0 we may choose finitely many $\phi_j \in S^0_{1,0}(M)$ such that $\sum \phi_j \ge 1$ near x_0 and $\|\phi_j^w u\|$ can be estimated by the right hand side of (2.14) for some suitable ψ and R_T. By elliptic regularity, we then obtain the estimate (1.4) for small enough T.

By multiplying with an elliptic pseudodifferential operator, we may assume that $m = 1$. Let $p = \sigma(P)$, then it is clear that it suffices to consider $w_0 = (x_0, \xi_0) \in p^{-1}(0)$, otherwise $P^* \in \Psi^1_{cl}(M)$ is elliptic near w_0 and we easily obtain the estimate (2.14). It is clear that we may assume that $\partial_\xi \operatorname{Re} p(w_0) \ne 0$, in the microlocal case after a conical transformation. Then, we may use Darboux' theorem and the Malgrange preparation theorem to obtain microlocal coordinates $(t, y; \tau, \eta) \in T^*\mathbf{R}^{n+1}$ so that $w_0 = (0, 0; 0, \eta_0)$, $t = 0$ on $T^*_{x_0}M$ and $p = q(\tau - if)$ in a conical neighborhood of w_0, where $f \in C^\infty(\mathbf{R}, S^1_{1,0})$ is real and homogeneous satisfying condition (2.2), and $0 \ne q \in S^0_{1,0}$, see Theorem 21.3.6 in [11]. By conjugation with elliptic Fourier integral operators and using the Malgrange preparation theorem successively on lower order terms, we obtain that

$$P^* = Q^w (D_t + i\,(\chi F)^w) + R^w \tag{2.15}$$

microlocally in a conical neighborhood Γ of w_0 (see the proof of Theorem 26.4.7$'$ in [11]). Here $Q \in S^0_{1,0}(\mathbf{R}^{n+1})$ and $R \in S^1_{1,0}(\mathbf{R}^{n+1})$, such that Q^w has principal symbol $q \ne 0$ in Γ and $\Gamma \bigcap \mathrm{WF}\, R^w = \emptyset$. Moreover, $\chi(\tau, \eta) \in S^0_{1,0}(\mathbf{R}^{n+1})$ is equal to 1 in Γ, $|\tau| \le C|\eta|$ in $\operatorname{supp}\chi(\tau, \eta)$, and $F^w \in C^\infty(\mathbf{R}, \Psi^1_{1,0}(\mathbf{R}^n))$ has real principal symbol f satisfying (2.2). By cutting off in the t variable we may assume that $f \in L^\infty(\mathbf{R}, S^1_{1,0}(\mathbf{R}^n))$. We shall choose ϕ and ψ so that $\operatorname{supp}\phi \subset \operatorname{supp}\psi \subset \Gamma$ and

$$\phi(t, y; \tau, \eta) = \chi_0(t, \tau, \eta)\phi_0(y, \eta)$$

where $\chi_0(t, \tau, \eta) \in S^0_{1,0}(\mathbf{R}^{n+1})$, $\phi_0(y, \eta) \in S^0_{1,0}(\mathbf{R}^n)$, $t \ne 0$ in $\operatorname{supp}\partial_t \chi_0$, $|\tau| \le C|\eta|$ in $\operatorname{supp}\chi_0$ and $|\tau| \cong |\eta|$ in $\operatorname{supp}\partial_{\tau,\eta}\chi_0$.

Since $q \neq 0$ and $R = 0$ on $\operatorname{supp}\psi$ it is no restriction to assume that $Q \equiv 1$ and $R \equiv 0$ when proving the estimate (2.14). Now, by Theorem 18.1.35 in [11] we may compose $C^\infty(\mathbf{R}, \Psi^m_{1,0}(\mathbf{R}^n))$ with operators in $\Psi^k_{1,0}(\mathbf{R}^{n+1})$ having symbols vanishing when $|\tau| \geq c(1 + |\eta|)$, and we obtain the usual asymptotic expansion in $\Psi^{m+k-j}_{1,0}(\mathbf{R}^{n+1})$ for $j \geq 0$. Since $|\tau| \leq C|\eta|$ in $\operatorname{supp}\phi$ and $\chi = 1$ on $\operatorname{supp}\psi$, it suffices to prove (2.14) for $P^* = P_0 = D_t + iF^w$.

By using Corollary 2.7 on $\phi^w u$, we obtain that

$$
\begin{aligned}
(2.16)\quad & \|b^w_T \phi^w u\|^2_{-1/2} + \|\phi^w u\|^2 \\
& \leq C_0 T \left(\operatorname{Im}\left(\phi^w P_0 u, b^w_T \phi^w u\right) + \operatorname{Im}\left([P_0, \phi^w]u, b^w_T \phi^w u\right)\right) + C_1 \|u\|^2_{(-1)}
\end{aligned}
$$

where $b^w_T \in L^\infty(\mathbf{R}, \Psi^1_{1/2,1/2}(\mathbf{R}^n))$ is symmetric with $\nabla b_T \in L^\infty(\mathbf{R}, S^1_{1/2,1/2}(\mathbf{R}^n))$. We find $[P_0, \phi^w] = -i\partial_t \phi^w + \{\, f, \phi \,\}^w \in \Psi^0_{1,0}(\mathbf{R}^{n+1})$ modulo $\Psi^{-1}_{1,0}(\mathbf{R}^{n+1})$ by the expansion. For any $u,\ v \in \mathcal{S}(\mathbf{R}^n)$ we have that

$$
(2.17)\qquad |\,(v, b^w_T u)\,| = |\,\big(\langle D_y\rangle^{1/2} v, \langle D_y\rangle^{-1/2} b^w_T u\big)\,| \leq C(\|v\|^2_{(1/2)} + \|b^w_T u\|^2_{-1/2})
$$

since $\|\langle D_y\rangle^{1/2} v\| \leq \|v\|_{(1/2)}$, $\langle D_y\rangle = 1 + |D_y|$. Now $\phi^w = \phi^w \psi^w$ modulo $\Psi^{-2}_{1,0}(\mathbf{R}^{n+1})$, thus we find from (2.17) that

$$
(2.18)\qquad |\,(\phi^w P_0 u, b^w_T \phi^w u)\,| \leq C(\|\psi^w P_0 u\|^2_{(1/2)} + \|b^w_T \phi^w u\|^2_{-1/2})
$$

where the last term can be cancelled for small enough T in (2.16). We also have to estimate the commutator term $\operatorname{Im}\left([P_0, \phi^w]u, b^w_T \phi^w u\right)$ in (2.16). Since $\phi = \chi_0 \phi_0$ we find that $\{\, f, \phi \,\} = \phi_0 \{\, f, \chi_0 \,\} + \chi_0 \{\, f, \phi_0 \,\}$, where $\phi_0 \{\, f, \chi_0 \,\} = R_0 \in S^0_{1,0}(\mathbf{R}^{n+1})$ is supported when $|\tau| \cong |\eta|$ and $\psi = 1$. Now $(\tau + if)^{-1} \in S^{-1}_{1,0}(\mathbf{R}^{n+1})$ when $|\tau| \cong |\eta|$, thus by [11, Theorem 18.1.35] we find that $R^w_0 = A^w_1 \psi^w P_0$ modulo $\Psi^{-1}_{1,0}(\mathbf{R}^{n+1})$ where $A_1 = R_0(\tau + if)^{-1} \in S^{-1}_{1,0}(\mathbf{R}^{n+1})$. As before, we find from (2.17) that

$$
\begin{aligned}
(2.19)\quad |\,(R^w_0 u, b^w_T \phi^w u)\,| & \leq C(\|R^w_0 u\|^2_{(1/2)} + \|b^w_T \phi^w u\|^2_{-1/2}) \\
& \leq C_0(\|\psi^w P_0 u\|^2_{(-1/2)} + \|b^w_T \phi^w u\|^2_{-1/2})
\end{aligned}
$$

and $|\,(\partial_t \phi^w u, b^w_T \phi^w u)\,| \leq \|R^w_1 u\|^2 + \|b^w_T \phi^w u\|^2_{-1/2}$ by (2.17), where $R^w_1 = \langle D_y\rangle^{1/2} \partial_t \phi^w \in \Psi^{1/2}_{1,0}(\mathbf{R}^{n+1})$, thus $t \neq 0$ in $\operatorname{WF} R^w_1$.

It remains to estimate the term $\operatorname{Im}\left((\{\, f, \phi_0 \,\}\, \chi_0)^w u, b^w_T \phi^w u\right)$, where $(\{\, f, \phi_0 \,\}\, \chi_0)^w = \{\, f, \phi_0 \,\}^w \chi^w_0$ and $\phi^w = \phi^w_0 \chi^w_0$ modulo $\Psi^{-1}_{1,0}(\mathbf{R}^{n+1})$. As in (2.17) we find

$$
|\,(R^w u, b^w_T v)\,| = |\,\big(\langle D_y\rangle R^w u, \langle D_y\rangle^{-1} b^w_T v\big)\,| \leq C(\|u\|^2 + \|v\|^2)
$$

for $R \in S^{-1}_{1,0}(\mathbf{R}^{n+1})$, thus we find

$$
|\operatorname{Im}\left((\{\, f, \phi_0 \,\}\, \chi_0)^w u, b^w_T \phi^w u\right)| \leq |\operatorname{Im}\left(\{\, f, \phi_0 \,\}^w \chi^w_0 u, b^w_T \phi^w_0 \chi^w_0 u\right)| + C\|u\|^2.
$$

The calculus gives $b_T^w\phi_0^w = (b_T\phi_0)^w$ and $2i\,\mathrm{Im}\,((b_T\phi_0)^w\,\{\,f,\phi_0\,\}^w) = \{\,b_T\phi_0,\{\,f,\phi_0\,\}\,\}^w = 0$ modulo $L^\infty(\mathbf{R},\Psi^0_{1/2,1/2}(\mathbf{R}^n))$ since $\nabla(b_T\phi_0) \in L^\infty(\mathbf{R},S^1_{1/2,1/2}(\mathbf{R}^n))$. We obtain

$$|\,\mathrm{Im}\,(\{\,f,\phi_0\,\}^w\,\chi_0^w u, b_T^w\phi_0^w\chi_0^w u)\,| \le C\|\chi_0^w u\|^2 \le C'\|u\|^2 \tag{2.20}$$

and the estimate (2.14) for small enough T, which completes the proof of Theorem 1.1. $\square$

It remains to prove Proposition 2.5, which will be done at the end of Section 6. The proof involves the construction of a multiplier b_T^w, and it will occupy most of the remaining part of the paper.

In the following, we let $\|u\|(t)$ be the L^2 norm of $x \mapsto u(t,x)$ in $\mathbf{R}^n$ for fixed t, and $(u,v)\,(t)$ the corresponding sesquilinear inner product. Let $\mathcal{B} = \mathcal{B}(L^2(\mathbf{R}^n))$ be the set of bounded operators $L^2(\mathbf{R}^n) \mapsto L^2(\mathbf{R}^n)$. We shall use operators which depend measurably on t.

Definition 2.8. We say that $t \mapsto A(t)$ is weakly measurable if $A(t) \in \mathcal{B}$ for all t and $t \mapsto A(t)u$ is weakly measurable for every $u \in L^2(\mathbf{R}^n)$, i.e., $t \mapsto (A(t)u,v)$ is measurable for any u, $v \in L^2(\mathbf{R}^n)$. We say that $A(t) \in L^\infty_{loc}(\mathbf{R},\mathcal{B})$ if $t \mapsto A(t)$ is weakly measurable and locally bounded in $\mathcal{B}$.

If $A(t) \in L^\infty_{loc}(\mathbf{R},\mathcal{B})$, then we find that the function $t \mapsto (A(t)u,v) \in L^\infty_{loc}(\mathbf{R})$ has weak derivative $\frac{d}{dt}\,(Au,v) \in \mathcal{D}'(\mathbf{R})$ for any u, $v \in \mathcal{S}(\mathbf{R}^n)$ given by

$$\tfrac{d}{dt}\,(Au,v)\,(\phi) = -\int (A(t)u,v)\,\phi'(t)\,dt \qquad \phi(t) \in C_0^\infty(\mathbf{R}).$$

If $u(t)$, $v(t) \in L^\infty_{loc}(\mathbf{R},L^2(\mathbf{R}^n))$ and $A(t) \in L^\infty_{loc}(\mathbf{R},\mathcal{B})$, then we find $t \mapsto (A(t)u(t),v(t)) \in L^\infty_{loc}(\mathbf{R})$ is measurable. We shall use the following multiplier estimate, which is given by Proposition 2.9 in [8] (see also [15] and [17] for similar estimates).

Proposition 2.9. *Let $P_0 = D_t + iF(t)$ with $F(t) \in L^\infty_{loc}(\mathbf{R},\mathcal{B})$. Assume that $B(t) = B^*(t) \in L^\infty_{loc}(\mathbf{R},\mathcal{B})$, such that*

$$\tfrac{d}{dt}\,(Bu,u) + 2\,\mathrm{Re}\,(Bu,Fu) \ge (mu,u) \quad \text{in } \mathcal{D}'(I) \qquad \forall\, u \in \mathcal{S}(\mathbf{R}^n) \tag{2.21}$$

where $m(t) = m^*(t) \in L^\infty_{loc}(\mathbf{R},\mathcal{B})$ and $I \subseteq \mathbf{R}$ is open. Then we have

$$\int (mu,u)\,dt \le 2\int \mathrm{Im}\,(Pu,Bu)\,dt \tag{2.22}$$

for $u \in C_0^1(I,\mathcal{S}(\mathbf{R}^n))$.

3. The symbol classes

In this section we shall define the symbol classes we shall use. Assume that $f \in L^\infty(\mathbf{R},S(h^{-1},hg^\sharp))$ satisfies (2.2), here $0 < h \le 1$ and $g^\sharp = (g^\sharp)^\sigma$ are constant. By changing h we obtain that $|\partial_w f| \le h^{-1/2}$ which we assume in what follows. The results are uniform in the usual sense, they only depend on the seminorms of f in $L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$. Let

$$X_+(t) = \{\, w \in T^*\mathbf{R}^n : \exists\, s \le t,\ \ f(s,w) > 0 \,\} \tag{3.1}$$

$$X_-(t) = \{\, w \in T^*\mathbf{R}^n : \exists\, s \ge t,\ \ f(s,w) < 0 \,\}\,. \tag{3.2}$$

Clearly, $X_\pm(t)$ are open in $T^*\mathbf{R}^n$, $X_+(s) \subseteq X_+(t)$ and $X_-(s) \supseteq X_-(t)$ when $s \le t$. By condition $(\overline{\Psi})$ we obtain that $X_-(t) \bigcap X_+(t) = \emptyset$ and $\pm f(t,w) \ge 0$ when $w \in X_\pm(t)$, $\forall\, t$. Let $X_0(t) = T^*\mathbf{R}^n \setminus (X_+(t) \bigcup X_-(t))$ which is closed in $T^*\mathbf{R}^n$. By the definition of $X_\pm(t)$ we have $f(t,w) = 0$ when $w \in X_0(t)$. Let

$$d_0(t_0, w_0) = \inf \left\{\, g^\sharp(w_0 - z)^{1/2} :\ \ z \in X_0(t_0) \,\right\} \tag{3.3}$$

be is the $g^\sharp$ distance in $T^*\mathbf{R}^n$ to $X_0(t_0)$ for fixed t_0, it is equal to $+\infty$ in the case that $X_0(t_0) = \emptyset$.

Definition 3.1. We define the signed distance function $\delta_0(t,w)$ by

$$\delta_0 = \operatorname{sgn}(f) \min(d_0, h^{-1/2}), \tag{3.4}$$

where d_0 is given by (3.3) and

$$\operatorname{sgn}(f)(t,w) = \begin{cases} \pm 1, & w \in X_\pm(t) \\ 0, & w \in X_0(t) \end{cases} \tag{3.5}$$

so that $\operatorname{sgn}(f) f \ge 0$.

Definition 3.2. We say that $w \mapsto a(w)$ is Lipschitz continuous on $T^*\mathbf{R}^n$ with respect to the metric $g^\sharp$ if $|a(w) - a(z)| \le C g^\sharp(w - z)^{1/2}$ for any z, w.

It is clear that the signed distance function $w \mapsto \delta_0(t,w)$ given by Definition 3.1 is Lipschitz continuous with respect to the metric $g^\sharp$, $\forall\, t$, with Lipschitz constant equal to 1, see Proposition 3.3 in [8]. We also find that $t \mapsto \delta_0(t,w)$ is non-decreasing, $0 \le \delta_0 f$, $|\delta_0| \le h^{-1/2}$ and $|\delta_0| = d_0$ when $|\delta_0| < h^{-1/2}$.

In the following, we shall treat t as a parameter which we shall suppress, and we shall denote $f' = \partial_w f$ and $f'' = \partial_w^2 f$. We shall also in the following assume that we have choosen $g^\sharp$ orthonormal coordinates so that $g^\sharp(w) = |w|^2$.

Definition 3.3. Let $G_1 = H_1 g^\sharp$ where

$$H_1^{-1/2} = 1 + |\delta_0| + \frac{|f'|}{|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}}. \tag{3.6}$$

We have that

$$1 \le H_1^{-1/2} \le 1 + |\delta_0| + h^{-1/4}|f'|^{1/2} \le 3h^{-1/2} \tag{3.7}$$

since $|f'| \le h^{-1/2}$ and $|\delta_0| \le h^{-1/2}$. Moreover, $|f'| \le H_1^{-1/2}(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2})$ so by the Cauchy-Schwarz inequality we obtain

$$|f'| \le 2|f''|H_1^{-1/2} + 3h^{1/2}H_1^{-1} \le C_2 H_1^{-1/2}. \tag{3.8}$$

Definition 3.4. Let

$$M = |f| + |f'|H_1^{-1/2} + |f''|H_1^{-1} + h^{1/2}H_1^{-3/2} \tag{3.9}$$

then we have that $h^{1/2} \le M \le C_3 h^{-1}$.

Proposition 3.5. *We find that $H_1^{-1/2}$ is Lipschitz continuous, G_1 is σ temperate such that $G_1 = H_1^2 G_1^\sigma$ and*

$$H_1(w) \le C_0 H_1(w_0)(1 + H_1(w)g^\sharp(w - w_0)). \tag{3.10}$$

We have that M is a weight for G_1 such that $f \in S(M, G_1)$ and

$$M(w) \le C_1 M(w_0)(1 + H_1(w_0)g^\sharp(w - w_0))^{3/2}. \tag{3.11}$$

In the case when $1 + |\delta_0(w_0)| \le H_1^{-1/2}(w_0)/2$ we have $|f'(w_0)| \ge h^{1/2}$,

$$|f^{(k)}(w_0)| \le C_k |f'(w_0)| H_1^{\frac{k-1}{2}}(w_0) \qquad k \ge 1, \tag{3.12}$$

and $1/C \le |f'(w)|/|f'(w_0| \le C$ when $|w - w_0| \le cH_1^{-1/2}(w_0)$ for some $c > 0$.

Proof. The Proposition follows from [8, Proposition 3.7] except for the Lipschitz continuity of $H_1^{-1/2}$. Since the first terms of (3.6) are Lipschitz continuous, we only have to prove that

$$|f'|/(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}) = E/D$$

is Lipschitz. Since this is a local property, it suffices to prove this when $|\Delta w| = |w - w_0| \le 1$. Then we have that $D(w) \cong D(w_0)$, in fact $D^2 \cong h + h^{1/2}|f'| + |f''|^2$ so

$$D^2(w) \le C(D^2(w_0) + |f''(w_0)|h^{1/2} + h) \le C' D^2(w_0)$$

when $|\Delta w| \le 1$. We find that

$$\left|\Delta \frac{E}{D}\right| = \left|\frac{E(w)}{D(w)} - \frac{E(w_0)}{D(w_0)}\right| \le \frac{|\Delta E|}{D(w)} + \frac{E(w_0)|\Delta D|}{D(w)D(w_0)}.$$

Taylor's formula gives that

$$|\Delta E| \le (|f''(w)| + Ch^{1/2})|\Delta w| \le CD(w) \tag{3.13}$$

when $|\Delta w| \le 1$. We shall show that $E(w_0)|\Delta D| \le CD(w)D(w_0)|\Delta w|$, which is trivial if $E(w_0) = 0$. Else, we have

$$|\Delta|f''|| \le Ch^{1/2}|\Delta w| \le CD^2(w_0)|\Delta w|/E(w_0) \le C' D(w_0)D(w)||\Delta w|/E(w_0)$$

when $|\Delta w| \le 1$ since $h^{1/2} \le D^2/E$ and $D(w_0) \le CD(w)$. Finally, we have

$$\begin{aligned} h^{1/4}|\Delta|f'|^{1/2}| &\le h^{1/4}|\Delta E|/(|f'(w_0)|^{1/2} + |f'(w)|^{1/2}) \\ &\le Ch^{1/4}|f'(w_0)|^{1/2} D(w)|\Delta w|/|f'(w_0)| \le CD(w_0)D(w)|\Delta w|/E(w_0) \end{aligned}$$

when $|\Delta w| \le 1$ by (3.13). This completes the proof of Proposition 3.5. $\square$

We obtain the following result from Propositions 3.9 and.10 in [8].

Proposition 3.6. *We have that $M \le CH_1^{-1}$, which gives that $f \in S(H_1^{-1}, G_1)$. We also obtain that*

$$1/C \le M/(|f''|H_1^{-1} + h^{1/2}H_1^{-3/2}) \le C. \tag{3.14}$$

When $|\delta_0| \le \kappa_0 H_1^{-1/2}$ and $H_1^{1/2} \le \kappa_0$ for $0 < \kappa_0$ sufficiently small, we find

$$1/C_1 \le M/|f'|H_1^{-1/2} \le C_1. \tag{3.15}$$

There exists $\kappa_1 > 0$ so that if $\langle \delta_0 \rangle = 1 + |\delta_0| \le \kappa_1 H_1^{-1/2}$ then

$$f = \alpha_0 \delta_0 \tag{3.16}$$

where $\kappa_1 M H^{1/2} \le \alpha_0 \in S(MH_1^{1/2}, G_1)$, which implies that $\delta_0 = f/\alpha_0 \in S(H_1^{-1/2}, G_1)$.

4. The Weight function

In this section, we shall define the weight m_1 we shall use. Let $\delta_0(t,w)$ and $H_1^{-1/2}(t,w)$ be given by Definitions 3.1 and 3.3 for $f \in L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$ satisfying condition $(\overline{\Psi})$ given by (2.2) such that $|f'| \le h^{-1/2}$. The weight m_1 will essentially measure how much $t \mapsto \delta_0(t,w)$ changes between the minima of $t \mapsto H_1^{1/2}(t,w)\langle \delta_0(t,w)\rangle^2$, which will give restrictions on the sign changes of the symbol. As before, we assume that we have choosen $g^\sharp$ orthonormal coordinates so that $g^\sharp(w) = |w|^2$, and the results will only depend on the seminorms of f.

Definition 4.1. For $(t,w) \in \mathbf{R} \times T^*\mathbf{R}^n$ we let

$$\begin{aligned} m_1(t,w) = \inf_{t_1 \le t \le t_2} \big\{ &|\delta_0(t_1,w) - \delta_0(t_2,w)| \\ &+ \max\big(H_1^{1/2}(t_1,w)\langle \delta_0(t_1,w)\rangle^2, H_1^{1/2}(t_2,w)\langle \delta_0(t_2,w)\rangle^2\big)/2 \,\big\} \end{aligned} \tag{4.1}$$

where $\langle \delta_0 \rangle = 1 + |\delta_0|$.

Remark 4.2. *When $t \mapsto \delta_0(t,w)$ is constant for fixed w, we find that $t \mapsto m_1(t,w)$ is equal to the largest quasi-convex minorant of $t \mapsto H_1^{1/2}(t,w)\langle \delta_0(t,w)\rangle^2/2$, i.e., $\sup_I m_1 = \sup_{\partial I} m_1$ for compact intervals $I \subset \mathbf{R}$, see* [12, Definition 1.6.3].

The main difference between the present note and [8] is the use of $H_1^{1/2}\langle \delta_0 \rangle^2$ in the definition of m_1 instead of $H_1^{1/2}\langle \delta_0 \rangle$.

Proposition 4.3. *We have that* $m_1 \in L^\infty(\mathbf{R} \times T^*\mathbf{R}^n)$*, such that* $w \mapsto m_1(t,w)$ *is uniformly Lipschitz continous,* $\forall\, t$*, and*

$$h^{1/2}\langle\delta_0\rangle^2/6 \le m_1 \le H_1^{1/2}\langle\delta_0\rangle^2/2 \le \langle\delta_0\rangle/2. \tag{4.2}$$

We may choose $t_1 \le t_0 \le t_2$ *so that*

$$\max_{j=0,1,2}\langle\delta_0(t_j,w_0)\rangle \le 2 \min_{j=0,1,2}\langle\delta_0(t_j,w_0)\rangle. \tag{4.3}$$

and

$$H_0^{1/2} = \max(H_1^{1/2}(t_1,w_0),\ H_1^{1/2}(t_2,w_0)) \tag{4.4}$$

satisfies

$$H_0^{1/2} < 16 m_1(t_0,w_0)/\langle\delta_0(t_j,w_0)\rangle^2 \qquad \text{for } j = 0,\ 1,\ 2. \tag{4.5}$$

If $m_1(t_0,w_0) \le \varrho\langle\delta_0(t_0,w_0)\rangle$ *for* $\varrho \ll 1$*, then we may choose* $g^\sharp$ *orthonormal coordinates so that* $w_0 = (x_1, 0)$*,* $|x_1| < 2\langle\delta_0(t_0,w_0)\rangle < 32\varrho H_0^{-1/2}$*, and*

$$\operatorname{sgn}(w_1) f(t_0,w) \ge 0 \quad \text{when } |w_1| \ge (m_1(t_0,w_0) + H_0^{1/2}|w'|^2)/c_0 \tag{4.6}$$

$$|\delta_0(t_1,w) - \delta_0(t_2,w)| \le (m_1(t_0,w_0) + H_0^{1/2}|w-w_0|^2)/c_0 \tag{4.7}$$

when $|w| \le c_0 H_0^{-1/2}$*. The constant* c_0 *only depends on the seminorms of* f*.*

Observe that condition (4.6) is not empty when $m_1(t_0,w_0) \le \varrho\langle\delta_0(t_0,w_0)\rangle$, for ϱ sufficiently small, because of (4.5).

Proof. If we let

$$F(s,t,w) = |\delta_0(s,w) - \delta_0(t,w)| + \max(H_1^{1/2}(s,w)\langle\delta_0(s,w)\rangle^2, H_1^{1/2}(t,w)\langle\delta_0(t,w)\rangle^2)/2$$

then we find that $w \mapsto F(s,t,w)$ is uniformly Lipschitz continuous. In fact, it suffices to show this when $|\Delta w| = |w - w_0| \ll 1$, and then $H_1^{-1/2}$ and $\langle\delta_0\rangle$ only vary with a fixed factor. The first term $|\delta_0(s,w) - \delta_0(t,w)|$ is obviously uniformly Lipschitz continuous. We have for fixed t that

$$\left|\Delta(H_1^{1/2}\langle\delta_0\rangle^2)\right| \le C(\langle\delta_0\rangle^2|\Delta H_1^{1/2}| + H_1^{1/2}\langle\delta_0\rangle|\Delta\delta_0|)$$

where $H_1^{1/2}\langle\delta_0\rangle \le 1$ and $|\Delta H_1^{1/2}| \le C H_1 |\Delta H_1^{-1/2}| \le C' H_1 |\Delta w|$ by Proposition 3.5, which gives the uniform Lipschitz continuity of $F(s,t,w)$. By taking the infimum, we obtain (4.2) and the uniform Lipschitz continuity of m_1. In fact, $h^{1/2}/3 \le H_1^{1/2}$ by (3.7) and since $t \mapsto \delta_0(t,w)$ is monotone, we find that $t \mapsto \langle\delta_0(t,w)\rangle$ is quasi-convex. Thus $h^{1/2}\langle\delta_0(t_0,w_0)\rangle/6 \le F(s,t,w_0)$ when $s \le t_0 \le t$.

By approximating the infimum, we may choose $t_1 \le t_0 \le t_2$ so that $F(t_1, t_2, w_0) < m_1(t_0, w_0) + h^{1/2}/6$. Since $h^{1/2}/6 \le m_1 \le H_1^{1/2}\langle\delta_0\rangle^2/2$ by (4.2), we find that

$$|\delta_0(t_1, w_0) - \delta_0(t_2, w_0)| < m_1(t_0, w_0) \le \langle\delta_0(t_0, w_0)\rangle/2 \qquad \text{and} \tag{4.8}$$

$$H_1^{1/2}(t_j, w_0)\langle\delta_0(t_j, w_0)\rangle^2/2 < 2m_1(t_0, w_0) \qquad \text{for } j = 1 \text{ and } 2. \tag{4.9}$$

Since $t \mapsto \delta_0(t, w_0)$ is monotone, we obtain (4.3) from (4.8), and (4.5) from (4.9) and (4.3).

Next assume that $m_1(t_0, w_0) \le \varrho\langle\delta_0(t_0, w_0)\rangle$ for some $0 < \varrho \le 1$. Then we find from (4.5) that

$$1 + |\delta_0(t_j, w_0)| < 16\varrho H_0^{-1/2} \qquad \text{for } j = 0,\ 1,\ 2. \tag{4.10}$$

Choose $g^\sharp$ orthonormal coordinates so that $w_0 = 0$. Since $\langle\delta_0(t_j, 0)\rangle < 16\varrho H_1^{-1/2}(t_j, 0)$ by (4.10), we find from Proposition 3.5 that

$$h^{1/2} \le |\partial_w f(t_j, 0)| \cong |\partial_w f(t_j, w)| \quad \text{for } |w| \le cH_0^{-1/2} \le cH_1^{-1/2}(t_j, 0),\ j = 1,\ 2$$

when $\varrho \ll 1$. Since $H_0^{-1/2} \le 3h^{-1/2}$ we find that $f(t_j, \widetilde{w}_j) = 0$ for some $|\widetilde{w}_j| < 16\varrho H_0^{-1/2}$ by (4.10) when $\varrho < 1/48$ and $j = 1,\ 2$. Thus, when $16\varrho \le c$ we obtain that

$$|f(t_j, w)| \le C|\partial_w f(t_j, 0)|H_0^{-1/2} \qquad \text{when } |w| < cH_0^{-1/2}$$

and then (3.12) gives $f(t_j, w) \in S(|\partial_w f(t_j, 0)|H_0^{-1/2}, H_0 g^\sharp)$ since $H_1^{1/2}(t_j, 0) \le H_0^{1/2}$, $j = 1,\ 2$. Choose coordinates $z = H_0^{1/2}w$, we shall use Proposition 4.3 in [8] with

$$f_j(z) = H_0^{1/2} f(t_j, H_0^{-1/2}z)/|\partial_w f(t_j, 0)| \in C^\infty \qquad \text{for } j = 1,\ 2.$$

Let $\delta_j(z) = H_0^{1/2}\delta_0(t_j, H_0^{-1/2}z)$ be the signed distance functions to $f_j^{-1}(0)$, then $|f_j'(0)| = 1$, $|\delta_j(0)| < 16\varrho$ and

$$|\delta_1(0) - \delta_2(0)| = \varepsilon < H_0^{1/2} m_1(t_0, 0) \le H_0^{1/2}\langle\delta_0(t_0, 0)\rangle/2 < 8\varrho$$

by (4.8) and (4.10). Thus, for sufficiently small ϱ we may use [8, Proposition 4.3] to obtain $g^\sharp$ orthogonal coordinates (z_1, z') so that $w_0 = z_0 = (y_1, 0)$, $|y_1| = |\delta_1(0)|$ and

$$\begin{cases} \operatorname{sgn}(z_1) f_j(z) \ge 0 & \text{when } |z_1| \ge (\varepsilon + |z'|^2)/c_0 \\ |\delta_1(z) - \delta_2(z)| \le (\varepsilon + |z - z_0|^2)/c_0 \end{cases}$$

when $|z| \le c_0$. Let $x_1 = H_0^{-1/2} y_1$ then $|x_1| < 2\langle\delta_0(t_0, 0)\rangle < 32\varrho H_0^{-1/2}$ by (4.3) and (4.10). We then obtain (4.6)–(4.7) by condition $(\overline{\Psi})$, since $H_0^{-1/2}\varepsilon < m_1(t_0, 0)$. ∎

Proposition 4.4. *There exists $C > 0$ such that*

$$m_1(t_0, w) \le C m_1(t_0, w_0)(1 + |w - w_0|/\langle\delta_0(t_0, w_0)\rangle)^3 \tag{4.11}$$

thus m_1 is a weight for $g^\sharp$.

Proof. Since $m_1 \le \langle\delta_0\rangle/2$ we only have to consider the case when

$$m_1(t_0, w_0) \le \varrho\langle\delta_0(t_0, w_0)\rangle \tag{4.12}$$

for some $\varrho > 0$. In fact, otherwise we have

$$m_1(t_0, w) \le \langle \delta_0(t_0, w)\rangle/2 < m_1(t_0, w_0)(1 + |w - w_0|/\langle \delta_0(t_0, w_0)\rangle)/2\varrho$$

since the Lipschitz continuity of $w \mapsto \delta_0(t_0, w)$ gives

$$\langle \delta_0(t, w)\rangle \le \langle \delta_0(t, w_0)\rangle(1 + |w - w_0|/\langle \delta_0(t, w_0)\rangle) \qquad \forall\, t. \tag{4.13}$$

If (4.12) is satisfied for $\varrho \ll 1$, we may use Proposition 4.3 to obtain $t_1 \le t_0 \le t_2$ such that (4.3), (4.5) and (4.7) hold with $H_0^{1/2} = \max(H_1^{1/2}(t_1, w_0),\ H_1^{1/2}(t_2, w_0))$.

Now, for fixed w_0 it suffices to prove (4.11) when

$$|w - w_0| \le \varrho H_0^{-1/2} \tag{4.14}$$

for some $\varrho > 0$. In fact, when $|w - w_0| > \varrho H_0^{-1/2}$ we obtain from (4.12) that

$$\begin{aligned}|w - w_0|^2/\langle \delta_0(t_0, w_0)\rangle^2 > \varrho^2 H_0^{-1}/\langle \delta_0(t_0, w_0)\rangle^2 > \varrho^2\langle \delta_0(t_0, w_0)\rangle^2/256 m_1^2(t_0, w_0)\\ \ge \varrho^2\langle \delta_0(t_0, w_0)\rangle m_1(t_0, w)/64\langle \delta_0(t_0, w)\rangle m_1(t_0, w_0)\end{aligned}$$

since $\langle \delta_0\rangle \ge 2m_1$. By (4.13) we obtain that (4.11) is satisfied with $C = 64/\varrho^2$. Thus in the following we shall only consider w such that (4.14) is satisfied for $\varrho \ll 1$. We find by (4.5) and (4.7) that

$$\begin{aligned}|\delta_0(t_1, w) - \delta_0(t_2, w)| &\le (m_1(t_0, w_0) + H_0^{1/2}|w - w_0|^2)/c_0\\ &< 16 m_1(t_0, w_0)(1 + |w - w_0|^2/\langle \delta_0(t_0, w_0)\rangle^2)/c_0\end{aligned} \tag{4.15}$$

when $|w - w_0| \le c_0 H_0^{-1/2}$. Now G_1 is slowly varying, uniformly in t, thus we find for small enough $\varrho > 0$ that

$$H_1^{1/2}(t_j, w) \le C H_1^{1/2}(t_j, w_0) \qquad \text{when } |w - w_0| \le \varrho H_0^{-1/2} \le \varrho H_1^{-1/2}(t_j, w_0)$$

for $j = 1,\ 2$. By (4.13) and (4.3) we obtain that

$$H_1^{1/2}(t_j, w)\langle \delta_0(t_j, w)\rangle^2 \le 4C H_1^{1/2}(t_j, w_0)\langle \delta_0(t_j, w_0)\rangle^2(1 + |w - w_0|/\langle \delta_0(t_0, w_0)\rangle)^2 \tag{4.16}$$

when $j = 1,\ 2$, and $|w - w_0| \le c_0 H_0^{-1/2}$. Now $H_1^{1/2}(t_j, w_0)\langle \delta_0(t_j, w_0)\rangle^2 < 16 m_1(t_0, w_0)$ by (4.5) for $j = 1,\ 2$. Thus, by using (4.15), (4.16) and taking the infimum we obtain

$$m_1(t_0, w) \le C_0 m_1(t_0, w_0)(1 + |w - w_0|/\langle \delta_0(t_0, w_0)\rangle)^2$$

when $|w - w_0| \le \varrho H_0^{-1/2}$ for $\varrho \ll 1$. □

The following result will be important for the proof of Proposition 2.5 in Section 6.

Proposition 4.5. *Let M be given by Definition 3.4 and m_1 by Definition 4.1. Then there exists $C_0 > 0$ such that*

$$M H_1^{3/2} \le C_0 m_1/\langle \delta_0\rangle^2. \tag{4.17}$$

Proof of Proposition 4.5. We shall omit the dependence on t in the proof. Observe that since $h^{1/2}\langle\delta_0\rangle^2/6 \le m_1$ we find that (4.17) is equivalent to

$$|f''|H_1^{1/2} \le Cm_1/\langle\delta_0\rangle^2 \tag{4.18}$$

by Proposition 3.6. First we note that if $m_1 \ge c\langle\delta_0\rangle > 0$, then $MH_1^{3/2}\langle\delta_0\rangle^2 \le C\langle\delta_0\rangle \le Cm_1/c$ since $\langle\delta_0\rangle \le H_1^{-1/2}$ and $M \le CH_1^{-1}$ by Proposition 3.6.

Thus, we only have to consider the case $m_1 \le \varrho\langle\delta_0\rangle$ at w_0 for some $\varrho > 0$ to be chosen later. Then we may use Proposition 4.3 for $\varrho \ll 1$ to choose $g^\sharp$ orthonormal coordinates so that $|w_0| < 2\langle\delta_0(w_0)\rangle < 32\varrho H_0^{-1/2}$ and f satisfies (4.6) with

$$h^{1/2}/3 \le H_0^{1/2} < 16m_1(w_0)/\langle\delta_0(w_0)\rangle^2 \le 8H_1^{1/2}(w_0) \tag{4.19}$$

by (4.5) and (4.2). Thus it suffices to prove the estimate

$$|f''|H_1^{1/2} \le CH_0^{1/2} \tag{4.20}$$

at w_0. Now it actually suffices to prove (4.20) at $w = 0$. In fact, (3.10) gives

$$H_1(w_0) \le C_0H_1(0)(1 + H_1(w_0)|w_0|^2) \le 5C_0H_1(0)$$

since $|w_0| < 2\langle\delta_0(w_0)\rangle \le 2H_1^{-1/2}(w_0)$. Thus Taylor's formula gives

$$|f''(w_0)|H_1^{1/2}(w_0) \le \left(|f''(0)| + C_3h^{1/2}|w_0|\right) H_1^{1/2}(w_0) \le C_1(|f''(0)|H_1^{1/2}(w_0) + h^{1/2}) \tag{4.21}$$

since $|f^{(3)}| \le C_3h^{1/2}$. By Definition 3.3 we find that

$$\begin{aligned} H_1^{-1/2} &\ge 1 + |f'|/(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}) \\ &\ge (|f'| + |f''| + h^{1/2})/(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}), \end{aligned}$$

thus (4.20) follows if we prove

$$|f''|(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}) \le C\left(|f'| + |f''| + h^{1/2}\right) H_0^{1/2} \qquad \text{at } 0. \tag{4.22}$$

Since $h^{1/2}/3 \le H_0^{1/2}$ we obtain (4.22) by the Cauchy-Schwarz inequality if we prove that

$$|f''(0)| \le C(H_0^{1/4}|f'(0)|^{1/2} + h^{1/2}). \tag{4.23}$$

Let $F(z) = H_0f(H_0^{-1/2}z)$, then (4.6) gives

$$\operatorname{sgn}(z_1)F(z) \ge 0 \quad \text{when } |z_1| \ge \varepsilon + |z'|^2/r \text{ and } |z| \le r$$

where $r = c_0$ and

$$\varepsilon = H_0^{1/2}m_1(w_0)/c_0 \le 16m_1^2(w_0)/c_0\langle\delta_0(w_0)\rangle^2 \le 16\varrho^2/c_0 \le c_0/5$$

by (4.19) when $\varrho \le c_0/4\sqrt{5}$ which we shall assume. Proposition 4.2 in [8] then gives that

$$|F''(0)| \le C_1\left(|F'(0)|/\varrho_0 + H_0^{-1/2}h^{1/2}\varrho_0\right) \qquad \varepsilon \le \varrho_0 \le c_0/\sqrt{10}$$

since $\|F^{(3)}\|_\infty \le C_3 H_0^{-1/2} h^{1/2}$. Observe that $|F'(0)| \le C_2$ since $H_0^{1/2} \le 8H_1^{1/2}(w_0) \le CH_1^{1/2}(0)$ and $|f'(0)| \le CH_1^{-1/2}(0)$. Choose

$$\varrho_0 = \varepsilon + \lambda |F'(0)|^{1/2} \le c_0/\sqrt{10}$$

with $\lambda = c_0(\sqrt{10}-2)/10\sqrt{C_2}$, then we obtain that

$$|F''(0)| \le C_2(|F'(0)|^{1/2} + h^{1/2} m_1(w_0))$$

since $H_0^{-1/2} \le 3h^{-1/2}$ and $\varepsilon = H_0^{1/2} m_1(w_0)/c_0$. If $h^{1/2} m_1(w_0) \le |F'(0)|^{1/2}$ then we obtain (4.23) since $F' = H_0^{1/2} f'$ and $F'' = f''$. If $|F'(0)|^{1/2} \le h^{1/2} m_1(w_0)$, then we find

$$|f''(0)| \le 2C_2 h^{1/2} m_1(w_0) \le 4C_2 m_1(w_0)/\langle \delta_0(w_0)\rangle.$$

Thus (4.18) follows from (4.21) since $H_1^{1/2}(w_0) \le \langle \delta_0(w_0)\rangle^{-1}$, which completes the proof of the proposition. $\square$

The following convexity property of $t \mapsto m_1(t,w)$ will be essential for the proof. For a proof, see the proof of Proposition 5.7 in [8].

Proposition 4.6. *Let m_1 be given by Definition 4.1. Then*

$$\sup_{t_1 \le t \le t_2} m_1(t,w) \le \delta_0(t_2,w) - \delta_0(t_1,w) + m_1(t_1,w) + m_1(t_2,w) \qquad \forall\, w. \tag{4.24}$$

Next, we shall construct the pseudo-sign $B = \delta_0 + \varrho_0$, which we shall use in Section 6 to prove Proposition 2.5 with the multiplier $b^w = B^{Wick}$.

Proposition 4.7. *Assume that δ_0 is given by Definition 3.1 and m_1 is given by Definition 4.1. Then for $T > 0$ there exists real valued $\varrho_T(t,w) \in L^\infty(\mathbf{R} \times T^*\mathbf{R}^n)$ with the property that $w \mapsto \varrho_T(t,w)$ is uniformly Lipschitz continuous, and*

$$|\varrho_T| \le m_1 \tag{4.25}$$

$$T\partial_t(\delta_0 + \varrho_T) \ge m_1/2 \qquad \text{in } \mathcal{D}'(\mathbf{R}) \tag{4.26}$$

when $|t| < T$.

This follows from Proposition 5.8 in [8]. Since

$$\varrho_T(t,w) = \sup_{-T \le s \le t} \left(\delta_0(s,w) - \delta_0(t,w) + \frac{1}{2T}\int_s^t m_1(r,w)\, dr - m_1(s,w) \right) \tag{4.27}$$

the uniformly Lipschitz continuity $w \mapsto \varrho_T(t,w)$ is clear.

5. The Wick quantization

In order to define the multiplier we shall use the Wick quantization. As before, we shall assume that $g^\sharp = (g^\sharp)^\sigma$ and the coordinates chosen so that $g^\sharp(w) = |w|^2$. For $a \in L^\infty(T^*\mathbf{R}^n)$ we define the Wick quantization:

$$a^{Wick}(x,D_x)u(x) = \int_{T^*\mathbf{R}^n} a(y,\eta)\Sigma^w_{y,\eta}(x,D_x)u(x)\, dy d\eta \qquad u \in \mathcal{S}(\mathbf{R}^n)$$

using the orthonormal projections $\Sigma^w_{y,\eta}(x,D_x)$ with Weyl symbol

$$\Sigma_{y,\eta}(x,\xi) = \pi^{-n}\exp(-g^\sharp(x-y,\xi-\eta))$$

(see [5, Appendix B] or [17, Section 4]). We find that a^{Wick}: $\mathcal{S}(\mathbf{R}^n) \mapsto \mathcal{S}'(\mathbf{R}^n)$ so that

$$a \ge 0 \implies \big(a^{Wick}(x,D_x)u,u\big) \ge 0 \qquad u \in \mathcal{S}(\mathbf{R}^n) \tag{5.1}$$

$(a^{Wick})^* = (\overline{a})^{Wick}$ and $\|a^{Wick}(x,D_x)\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le \|a\|_{L^\infty(T^*\mathbf{R}^n)}$, which is the main advantage with the Wick quantization (see [17, Proposition 4.2]). If $a_t(x,\xi) \in L^\infty(\mathbf{R}\times T^*\mathbf{R}^n)$ depends on a parameter t, then we find that

$$\int_{\mathbf{R}} \big(a_t^{Wick}u,u\big)\,\phi(t)\,dt = \big(A_\phi^{Wick}u,u\big) \qquad u \in \mathcal{S}(\mathbf{R}^n) \tag{5.2}$$

where $A_\phi(x,\xi) = \int_{\mathbf{R}} a_t(x,\xi)\phi(t)\,dt$. We obtain from the definition that $a^{Wick} = a_0^w$ where

$$a_0(w) = \pi^{-n}\int_{T^*\mathbf{R}^n} a(z)\exp(-|w-z|^2)\,dz \tag{5.3}$$

is the Gaussian regularization, thus Wick operators with real symbols have real Weyl symbols.

In the following, we shall assume that $G = Hg^\sharp \le g^\sharp$ is a slowly varying metric satisfying

$$H(w) \le C_0 H(w_0)(1+|w-w_0|)^{N_0} \tag{5.4}$$

and m is a weight for G satisfying (5.4) with H replaced by m. This means that G and m are strongly σ temperate in the sense of [2, Definition 7.1]. Recall the symbol class $S^+(1,g^\sharp)$ given by Definition 2.2. The following result follows from Proposition 6.1 and Lemma 6.2 in [8].

Proposition 5.1. *Assume that $a \in L^\infty(T^*\mathbf{R}^n)$ such that $|a| \le Cm$, then $a^{Wick} = a_0^w$ where $a_0 \in S(m,g^\sharp)$ is given by (5.3). If $a \ge m$ we obtain that $a_0 \ge c_0 m$ for a fixed constant $c_0 > 0$. If $a \in S(m,G)$ then $a_0 = a$ modulo symbols in $S(mH,G)$. If $|a| \le Cm$ and $a = 0$ in a fixed G ball with center w, then $a \in S(mH^N,G)$ at w for any N. If a is Lipschitz continuous then we have $a_0 \in S^+(1,g^\sharp)$. If $a(t,w)$ and $\mu(t,w) \in L^\infty(\mathbf{R}\times T^*\mathbf{R}^n)$ and $\partial_t a(t,w) \ge \mu(t,w)$ in $\mathcal{D}'(\mathbf{R})$ for almost all $w \in T^*\mathbf{R}^n$, then we find $\big(\partial_t(a^{Wick})u,u\big) \ge \big(\mu^{Wick}u,u\big)$ in $\mathcal{D}'(\mathbf{R})$ when $u \in \mathcal{S}(\mathbf{R}^n)$.*

By localization we find, for example, that if $|a| \le Cm$ and $a \in S(m,G)$ in a G neighborhood of w_0, then $a_0 = a$ modulo $S(mH,G)$ in a smaller G neighborhood of w_0. Observe that the results are uniform in the metrics and weights. We also have the following result about the composition of Wick operators.

Proposition 5.2. *Assume that a and $b \in L^\infty(T^*\mathbf{R}^n)$. If $|a| \le m_1$ and $|b'| \le m_2$, where m_j are weights for $g^\sharp$ satisfying* (5.4), *then*

$$a^{Wick} b^{Wick} = (ab)^{Wick} + r^w \tag{5.5}$$

with $r \in S(m_1 m_2, g^\sharp)$. If a and b are real such that $|a| \le m_1$ and $|b''| \le m_2$, then

$$\operatorname{Re} a^{Wick} b^{Wick} = \left(ab - \frac{1}{2} a' \cdot b' \right)^{Wick} + R^w \tag{5.6}$$

with $R \in S(m_1 m_2, g^\sharp)$.

Observe that since b' is Lipschitz continuous, $a' \cdot b'$ is well defined. Proposition 5.2 essentially follows from Proposition 3.4 in [19] and Lemma A.1.5 in [20] but we shall for completeness give a proof.

Proof. By Proposition 5.1 we have $a^{Wick} b^{Wick} = a_0^w b_0^w$ in (5.5) where $a_0 \in S(m_1, g^\sharp)$ and $b_0 \in S^+(m_2, g^\sharp)$. By Lemma 2.3 we find $a^{Wick} b^{Wick} = (a_0 b_0)^w$ modulo $\operatorname{Op} S(m_1 m_2, g^\sharp)$, where

$$a_0(w) b_0(w) = \pi^{-2n} \iint a(w + z_1) b(w + z_2) e^{-|z_1|^2 - |z_2|^2} \, dz_1 dz_2. \tag{5.7}$$

By using the Taylor formula we find that $b(w + z_2) = b(w + z_1) + r_1(w, z_1, z_2)$ where $|r_1(w, z_1, z_2)| \le C m_2(w)(1 + |z_1| + |z_2|)^N$ by (5.4). Integration in z_2 then gives (5.5).

For the proof of (5.6) we use that $\operatorname{Re} a_0^w b_0^w = (a_0 b_0)^w$ modulo $\operatorname{Op} S(m_1 m_2, g^\sharp)$ by Lemma 2.3, since a_0 and b_0 are real and $b_0'' \in S(m_2, g^\sharp)$. We use the Taylor formula again:

$$b(w + z_2) = b(w + z_1) + b'(w + z_1) \cdot (z_2 - z_1) + r_2(w, z_1, z_2)$$

where $|r_2(w, z_1, z_2)| \le C m_2(w)(1 + |z_1| + |z_2|)^N$. The term with z_2 is odd and gives a vanishing contribution in (5.7). Since $\partial_{z_1} e^{-|z_1|^2 - |z_2|^2} = -2 z_1 e^{-|z_1|^2}$ we obtain (5.6) after an integration by parts, since $|ab''| \le m_1 m_2$. □

Example 5.3. If $a \in S(H_1^{-1/2}, g^\sharp)$, $a' \in S(1, g^\sharp)$ and $b \in S(M, G_1)$, then $\operatorname{Re} a^{Wick} b^{Wick} = (ab)^{Wick}$ modulo $\operatorname{Op} S(M H_1^{1/2}, g^\sharp)$.

We shall compute the Weyl symbol for the Wick operator $(\delta_0 + \varrho_T)^{Wick}$, where ϱ_T is given by Proposition 4.7. In the following we shall suppress the t variable.

Proposition 5.4. *Let $B = \delta_0 + \varrho_0$, where δ_0 is given by Definition 3.1 and ϱ_0 is real valued and Lipschitz continuous, satisfying $|\varrho_0| \le m_1$, with $m_1 \le \langle \delta_0 \rangle / 2$ given by Definition 4.1. Then we find*

$$B^{Wick} = b^w$$

where $b = \delta_1 + \varrho_1$ is real, $\delta_1 \in S(H_1^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp)$, and $\varrho_1 \in S(m_1, g^\sharp) \bigcap S^+(1, g^\sharp)$. Also, there exists $\kappa_2 > 0$ so that $\delta_1 = \delta_0$ modulo $S(H_1^{1/2}, G_1)$ when $\langle \delta_0 \rangle \le \kappa_2 H_1^{-1/2}$. For any $\lambda > 0$ we find that $|\delta_0| \ge \lambda H_1^{-1/2}$ and $H_1^{1/2} \le \lambda/3$ imply that $|B| \ge \lambda H_1^{-1/2}/3$.

Proof. Let $\delta_0^{Wick} = \delta_1^w$ and $\varrho_0^{Wick} = \varrho_1^w$. Since $|\delta_0| \le H_1^{-1/2}$, $|\varrho_0| \le m_1$ and the symbols are real valued, we obtain from Proposition 5.1 that $\delta_1 \in S(H_1^{-1/2}, g^\sharp)$ and $\varrho_1 \in S(m_1, g^\sharp)$ are real valued. Since δ_0 and ϱ_0 are uniformly Lipschitz continuous, we find that δ_1 and $\varrho_1 \in S^+(1, g^\sharp)$ by Proposition 5.1.

If $\langle \delta_0 \rangle \le \kappa H_1^{-1/2}$ at w_0 for sufficiently small $\kappa > 0$, then we find by the Lipschitz continuity of δ_0 and the slow variation of G_1 that $\langle \delta_0 \rangle \le C_0 \kappa H_1^{-1/2}$ in a fixed G_1 neighborhood ω_κ of w_0 (depending on κ). For $\kappa \ll 1$ we find that $\delta_0 \in S(H_1^{-1/2}, G_1)$ in ω_κ by Proposition 3.6, which implies that $\delta_1 = \delta_0$ modulo $S(H_1^{1/2}, G_1)$ near w_0 by Proposition 5.1 after localization.

When $|\delta_0| \ge \lambda H_1^{-1/2} \ge \lambda > 0$ at w_0, then we find that

$$|\varrho_0| \le m_1 \le \langle \delta_0 \rangle/2 \le (1 + H_1^{1/2}/\lambda)|\delta_0|/2.$$

We obtain that $|\varrho_0| \le 2|\delta_0|/3$ and $|B| \ge |\delta_0|/3 \ge \lambda H_1^{-1/2}/3$ when $H_1^{1/2} \le \lambda/3$, which completes the proof. $\square$

Let m_1 be given by Definition 4.1, then m_1 is a weight for $g^\sharp$ according to Proposition 4.4. We are going to use the symbol classes $S(m_1^k, g^\sharp)$, $k \in \mathbf{R}$. The following proposition shows that the operator m_1^{Wick} dominates all operators in $\operatorname{Op} S(m_1, g^\sharp)$.

Proposition 5.5. *If $c \in S(m_1, g^\sharp)$ then there exists a positive constant C_0 such that*

$$|\langle c^w u, u \rangle| \le C_0 \left(m_1^{Wick} u, u \right) \qquad u \in \mathcal{S}(\mathbf{R}^n). \tag{5.8}$$

Here C_0 only depends on the seminorms of $c \in S(m_1, g^\sharp)$ and $f \in L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$.

Proof. We shall use an argument by Hörmander [14]. Let $0 < \varrho \le 1$

$$M_\varrho(w_0) = \sup_w m_1(w)/(1 + \varrho|w - w_0|)^3 \tag{5.9}$$

then $m_1 \le M_\varrho \le C m_1/\varrho^3$ and

$$M_\varrho(w) \le C M_\varrho(w_0)(1 + \varrho|w - w_0|)^3 \qquad \text{uniformly in } 0 < \varrho \le 1 \tag{5.10}$$

by (4.11) and the triangle inequality. Thus, M_ϱ is a weight for $g_\varrho = \varrho^2 g^\sharp$, uniformly in ϱ. Take $0 \le \chi \in C_0^\infty(T^*\mathbf{R}^n)$ such that $\int_{T^*\mathbf{R}^n} \chi(w)\, dw > 0$ and let

$$m_\varrho(w) = \varrho^{-2n} \int \chi(\varrho(w - z)) M_\varrho(z)\, dz.$$

Then by (5.10) we find $1/C_0 \le m_\varrho/M_\varrho \le C_0$, and $|\partial^\alpha m_\varrho| \le C_\alpha \varrho^{|\alpha|} m_\varrho$ thus $m_\varrho \in S(m_\varrho, g_\varrho)$ uniformly in $0 < \varrho \le 1$. Let $m_\varrho^{Wick} = \mu_\varrho^w$ then Proposition 5.1 gives $m_\varrho/c \le \mu_\varrho \in$

$S(m_\varrho, g_\varrho)$ uniformly in $0 < \varrho \le 1$ (in fact, this follows directly from (5.3)). Since $m_1 \cong m_\varrho$, we may replace m_1^w with $m_\varrho^{Wick} = \mu_\varrho^w$ in (5.8) for any fixed $\varrho > 0$.

Let $a_\varrho = \mu_\varrho^{-1/2} \in S(m_\varrho^{-1/2}, g_\varrho^\sharp)$ with $0 < \varrho \le 1$ to be chosen later. Since g_ϱ is uniformly σ temperate, $g_\varrho/g_\varrho^\sigma = \varrho^4$, m_ϱ is uniformly σ, g_ϱ temperate, and $\mu_\varrho^{\pm 1/2} \in S(m_\varrho^{\pm 1/2}, g_\varrho)$ uniformly, the calculus gives that $a_\varrho^w (a_\varrho^{-1})^w = 1 + r_\varrho^w$ where $r_\varrho/\varrho^2 \in S(1, g^\sharp)$ uniformly for $0 < \varrho \le 1$. Similarly, we find that $a_\varrho^w \mu_\varrho^w a_\varrho^w = 1 + s_\varrho^w$ where $s_\varrho/\varrho^2 \in S(1, g^\sharp)$ uniformly. We obtain that the L^2 operator norms

$$\|r_\varrho^w\|_{\mathcal{L}(L^2)} + \|s_\varrho^w\|_{\mathcal{L}(L^2)} \le C\varrho^2 \le 1/2$$

for sufficiently small ϱ. By fixing such a value of ϱ we find that $1/2 \le a_\varrho^w \mu_\varrho^w a_\varrho^w \le 2$ and

$$\frac{1}{2}\|u\| \le \|a_\varrho^w (a_\varrho^{-1})^w u\| \le 2\|u\| \tag{5.11}$$

thus $u \mapsto a_\varrho^w (a_\varrho^{-1})^w u$ is an homeomorphism on L^2. The estimate (5.8) then follows from

$$|\langle c^w a_\varrho^w (a_\varrho^{-1})^w u, a_\varrho^w (a_\varrho^{-1})^w u\rangle| \le C\langle \mu_\varrho^w a_\varrho^w (a_\varrho^{-1})^w u, a_\varrho^w (a_\varrho^{-1})^w u\rangle$$

which holds since $a_\varrho^w c^w a_\varrho^w \in \operatorname{Op} S(1, g^\sharp)$ is bounded in L^2. Observe that the bounds only depend on the seminorms of c in $S(m_1, g^\sharp)$, since ϱ and a_ϱ are fixed. □

6. The lower bounds

In this section we shall obtain a proof of Proposition 2.5 by giving lower bounds on $\operatorname{Re} b_T^w f^w$, where $b_T^w = B_T^{Wick}$ is given by Proposition 5.4. In the following, we shall omit the t variable and assume the coordinates chosen so that $g^\sharp(w) = |w|^2$. The results will hold for almost all $|t| \le T$ and only depend on the seminorms of f in $L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$.

Proposition 6.1. *Let $B = \delta_0 + \varrho_0$, where δ_0 is given by Definition 3.1 and ϱ_0 is real valued and Lipschitz continuous, satisfying $|\varrho_0| \le m_1$, with $m_1 \le \langle \delta_0 \rangle/2$ given by Definition 4.1. Then we have*

$$\operatorname{Re}\left(f^w B^{Wick} u, u\right) \ge (C^w u, u) \qquad \forall\, u \in \mathcal{S}(\mathbf{R}^n) \tag{6.1}$$

where $C \in S(m_1, g^\sharp)$.

Proof. We shall localize in $T^*\mathbf{R}^n$ with respect to the metric $G_1 = H_1 g^\sharp$, and estimate the localized operators. We shall use the neighborhoods

$$\omega_{w_0}(\varepsilon) = \left\{ w :\ |w - w_0| < \varepsilon H_1^{-1/2}(w_0) \right\} \qquad \text{for } w_0 \in T^*\mathbf{R}^n. \tag{6.2}$$

We may in the following assume that ε is small enough so that $w \mapsto H_1(w)$ and $w \mapsto M(w)$ only vary with a fixed factor in $\omega_{w_0}(\varepsilon)$. Then by the uniform Lipschitz continuity of $w \mapsto \delta_0(w)$ we can find $\kappa_0 > 0$ with the following property: for $0 < \kappa \le \kappa_0$ there exist positive constants c_κ and ε_κ so that for any $w_0 \in T^*\mathbf{R}^n$ we have

$$|\delta_0(w)| \le \kappa H_1^{-1/2}(w) \qquad w \in \omega_{w_0}(\varepsilon_\kappa) \qquad \text{or} \tag{6.3}$$

$$|\delta_0(w)| \ge c_\kappa H_1^{-1/2}(w) \qquad w \in \omega_{w_0}(\varepsilon_\kappa). \tag{6.4}$$

In fact, we have by the Lipschitz continuity that $|\delta_0(w) - \delta_0(w_0)| \le \varepsilon_\kappa H_1^{-1/2}(w_0)$ when $w \in \omega_{w_0}(\varepsilon_\kappa)$. Thus, if $\varepsilon_\kappa \ll \kappa$ we obtain that (6.3) holds when $|\delta_0(w_0)| \ll \kappa H_1^{-1/2}(w_0)$ and (6.4) holds when $|\delta_0(w_0)| \ge c\kappa H_1^{-1/2}(w_0)$.

By shrinking κ_0 we may assume that $M \cong |f'|H_1^{-1/2}$ when $|\delta_0| \le \kappa_0 H_1^{-1/2}$ and $H_1^{1/2} \le \kappa_0$ according to Proposition 3.6. Let κ_1 be given by Proposition 3.6, κ_2 by Proposition 5.4, and let ε_κ and c_κ be given by (6.3)–(6.4) for $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$. Using Proposition 5.4 with $\lambda = c_\kappa$ we find that

$$|B| \ge c_\kappa H_1^{-1/2}/3 \qquad \text{in } \omega_{w_0}(\varepsilon_\kappa) \tag{6.5}$$

if $H_1^{1/2} \le c_\kappa/3$ and (6.4) holds in $\omega_{w_0}(\varepsilon_\kappa)$.

Choose real symbols $\{\,\psi_j(w)\,\}_j$ and $\{\,\Psi_j(w)\,\}_j \in S(1, G_1)$ with values in ℓ^2, such that $\sum_k \psi_j^2 \equiv 1$, $\psi_j\Psi_j = \psi_j$, $\Psi_j = \phi_j^2 \ge 0$ for some $\{\,\phi_j(w)\,\}_j \in S(1, G_1)$ with values in ℓ^2 so that

$$\operatorname{supp}\phi_j \subseteq \omega_j = \omega_{w_j}(\varepsilon_\kappa).$$

We have that $B^{Wick} = b^w$ where $b = \delta_1 + \varrho_1$ is given by Proposition 5.4.

Lemma 6.2. *We find that* $A_j = \Psi_j f b \in S(MH_1^{-1/2}, g^\sharp) \bigcap S^+(M, g^\sharp)$ *uniformly in* j*, and*

$$\operatorname{Re}(f^w b^w) = \sum_j \psi_j^w A_j^w \psi_j^w \qquad \textit{modulo } \operatorname{Op} S(m_1, g^\sharp). \tag{6.6}$$

We have $A_j^w = \operatorname{Re} f_j^w b^w$ *modulo* $\operatorname{Op} S(m_1, g^\sharp)$ *uniformly in* j*, where* $f_j = \Psi_j f$.

Proof. Since $b \in S(H_1^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp)$ we obtain that $A_j \in S(MH_1^{-1/2}, g^\sharp) \bigcap S^+(M, g^\sharp)$ uniformly in j. Proposition 4.5 gives that

$$MH_1^{3/2}\langle\delta_0\rangle^2 \le Cm_1 \tag{6.7}$$

thus we may ignore terms in $\operatorname{Op} S(MH_1^{3/2}\langle\delta_0\rangle^2, g^\sharp)$. Now, since $b \in S(H_1^{-1/2}, g^\sharp)$, $\{\,\psi_k\,\}_k \in S(1, G_1)$ has values in ℓ^2 and $A_k \in S(MH_1^{-1/2}, g^\sharp)$ uniformly, we find by Lemma 2.3 and Remark 2.4 that the symbols of $f^w b^w$, $f_j^w b^w$ and $\sum_k \psi_k^w A_k^w \psi_k^w$ have expansions in $S(MH_1^{j/2}, g^\sharp)$. Observe that in the domains ω_j where $H_1^{1/2} \ge c > 0$, we find that the symbols of $\sum_k \psi_k^w A_k^w \psi_k^w$, $f_j^w b^w$ and $b^w f^w$ are in $S(MH_1^{3/2}, g^\sharp)$ giving the result in this case. Thus we may assume $H_1^{1/2} \le \kappa_2/2$ in what follows. We shall consider the neighborhoods where (6.3) or (6.4) holds.

If (6.4) holds then we find that $\langle\delta_0\rangle \cong H_1^{-1/2}$ so $S(MH_1^{1/2}, g^\sharp) \subseteq S(m_1, g^\sharp)$ in ω_j by (6.7). Since $b \in S^+(1, g^\sharp)$ and $A_j \in S^+(M, g^\sharp)$ we find that the symbols of both $f^w b^w$ and $\sum_k \psi_k^w A_k^w \psi_k^w$ are equal to $\sum_k \psi_k^2 A_k = fb$ modulo $S(MH_1^{1/2}, g^\sharp)$ in ω_j. We also find that the symbol of $f_j^w b^w$ is equal to A_j modulo $S(MH_1^{1/2}, g^\sharp)$, which proves the result in this case.

Next, we consider the case when (6.3) holds with $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$ and $H_1^{1/2} \le \kappa_2/2$ in ω_j. Then $\langle \delta_0 \rangle \le \kappa_2 H_1^{-1/2}$ so $b = \delta_1 + \varrho_1 \in S(H_1^{-1/2}, G_1) + S(m_1, g^\sharp)$ in ω_j by Proposition 5.4. We obtain from Lemma 2.3 that the symbol of $\mathrm{Re}(f^w b^w - (fb)^w)$ is in $S(MH_1^{3/2}, G_1) + S(MH_1 m_1, g^\sharp) \subseteq S(m_1, g^\sharp)$ in ω_j since $M \le CH_1^{-1}$. Similarly, we find that $A_j^w = \mathrm{Re}\, f_j^w b^w$ modulo $S(m_1, g^\sharp)$. Since $A_j \in S(MH_1^{-1/2}, G_1) + S(Mm_1, g^\sharp)$ uniformly, we find that the symbol of $\sum_k \psi_k^w A_k^w \psi_k^w$ is equal to bf modulo $S(m_1, g^\sharp)$ in ω_j, which proves (6.6) and Lemma 6.2. $\square$

In order to estimate the localized operator we shall use the following

Lemma 6.3. *If $A_j = \Psi_j f b$ then there exists $C_j \in S(m_1, g^\sharp)$ uniformly, such that*

$$\big(A_j^w u, u\big) \ge \big(C_j^w u, u\big) \qquad u \in \mathcal{S}(\mathbf{R}^n). \tag{6.8}$$

We obtain from (6.6) and (6.8) that

$$\mathrm{Re}\,(f^w b^w u, u) \ge \sum_j \big(\psi_j^w C_j^w \psi_j^w u, u\big) + (R^w u, u) \qquad u \in \mathcal{S}(\mathbf{R}^n)$$

where $\sum_j \psi_j^w C_j^w \psi_j^w$ and $R^w \in \mathrm{Op}\, S(m_1, g^\sharp)$, which gives Proposition 6.1. $\square$

Proof of Lemma 6.3. As before we are going to consider the cases when $H_1^{1/2} \cong 1$ or $H_1^{1/2} \ll 1$, and when (6.3) or (6.4) holds in $\omega_j = \omega_{w_j}(\varepsilon_\kappa)$ for $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$. When $H_1^{1/2} \ge c > 0$ we find that $A_j \in S(MH_1^{3/2}, g^\sharp) \subseteq S(m_1, g^\sharp)$ uniformly by (6.7) which gives the lemma with $C_j = A_j$ in this case. Thus, we may assume that

$$H_1^{1/2} \le \kappa_4 = \min(\kappa_0, \kappa_1, \kappa_2, \kappa_3)/2 \qquad \text{in } \omega_j \tag{6.9}$$

with $\kappa_3 = 2c_\kappa/3$ so that (6.5) follows from (6.4).

First, we consider the case when (6.3) holds with $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$ and $H_1^{1/2} \le \kappa_4 \le \kappa$ in ω_j. Then $\langle \delta_0 \rangle \le 2\kappa H_1^{-1/2}$ so we obtain from Proposition 3.6 that $M \cong |f'| H_1^{-1/2}$ and $\delta_0 \in S(H_1^{-1/2}, G_1)$ in ω_j. We shall use an argument of Lerner [20]. We have that $b^w = (\delta_0 + \varrho_0)^{Wick} = B^{Wick}$, where $|\varrho_0| \le m_1 \le H^{1/2} \langle \delta_0 \rangle^2 /2$ by (4.2). Also, Lemma 6.2 gives $A_j = \mathrm{Re}\, f_j^w B^{Wick}$ modulo $\mathrm{Op}\, S(m_1, g^\sharp)$.

Take $\chi(t) \in C^\infty(\mathbf{R})$ such that $0 \le \chi(t) \le 1$, $|t| \ge 2$ in $\operatorname{supp} \chi(t)$ and $\chi(t) = 1$ for $|t| \ge 3$. Let $\chi_0 = \chi(\delta_0)$, then $2 \le |\delta_0|$ and $\langle \delta_0 \rangle / |\delta_0| \le 3/2$ in $\operatorname{supp} \chi_0$, thus

$$1 + \chi_0 \varrho_0 / \delta_0 \ge 1 - \chi_0 \langle \delta_0 \rangle / 2|\delta_0| \ge 1/4. \tag{6.10}$$

Since $|\delta_0| \le 3$ in $\operatorname{supp}(1 - \chi_0)$ we find by Proposition 5.4 that

$$B^{Wick} = (\delta_0 + \chi_0 \varrho_0)^{Wick}$$

modulo $\operatorname{Op} S(m_1/\langle\delta_0\rangle, g^\sharp) \subseteq \operatorname{Op} S(H^{1/2}\langle\delta_0\rangle, g^\sharp)$ by (4.2). Since $|\chi_0\varrho_0/\delta_0| \le 3H_1^{1/2}\langle\delta_0\rangle/4$ we find from (5.5) that

$$B^{Wick} = \delta_0^{Wick} B_0^{Wick} \qquad \text{modulo } \operatorname{Op} S(H_1^{1/2}\langle\delta_0\rangle, g^\sharp).$$

where $B_0 = 1 + \chi_0\varrho_0/\delta_0$. Proposition 5.1 gives $(\chi_0\varrho_0/\delta_0)^{Wick} \in \operatorname{Op} S(H_1^{1/2}\langle\delta_0\rangle, g^\sharp)$ and $\delta_0^{Wick} = \delta_1^w$ where $\delta_1 = \delta_0 + \gamma$ with $\gamma \in S(H_1^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp)$ such that $\gamma \in S(H_1^{1/2}, G_1)$ in ω_j. Thus Lemma 2.3 gives

$$B^{Wick} = \delta_0^{Wick} B_0^{Wick} = \delta_0^w B_0^{Wick} + c^w \qquad \text{modulo } \operatorname{Op} S(H_1^{1/2}\langle\delta_0\rangle, g^\sharp) \tag{6.11}$$

where $c \in S(H_1^{-1/2}, g^\sharp)$ such that $\operatorname{supp} c \bigcap \omega_j = \emptyset$.

We find from Proposition 3.6 that $f = \alpha_0\delta_0$, where $\kappa_1 M H_1^{1/2} \le \alpha_0 \in S(MH_1^{1/2}, G_1)$, so $\alpha_0^{1/2} \in S(M^{1/2}H_1^{1/4}, G_1)$. Let

$$a_j = \alpha_0^{1/2}\phi_j\delta_0 \in S(M^{1/2}H_1^{-1/4}, G_1).$$

Since $f_j = \Psi_j f = \phi_j^2 f$ the calculus gives

$$a_j^w(\alpha_0^{1/2}\phi_j)^w = f_j^w \qquad \text{modulo } \operatorname{Op} S(MH_1, G_1). \tag{6.12}$$

Similarly, we find that $f_j^w c^w \in \operatorname{Op} S(MH_1^{3/2}, g^\sharp)$ and

$$\operatorname{Re} f_j^w \delta_0^w = a_j^w a_j^w \qquad \text{modulo } \operatorname{Op} S(MH_1^{3/2}, G_1) \tag{6.13}$$

with imaginary part in $\operatorname{Op} S(MH_1^{1/2}, G_1)$. We obtain from (6.11) and (6.12) that

$$f_j^w B^{Wick} = f_j^w(\delta_0^w B_0^{Wick} + c^w + r^w) = f_j^w\delta_0^w B_0^{Wick} + a_j^w R_j^w \quad \text{modulo } \operatorname{Op} S(m_1, g^\sharp) \tag{6.14}$$

where $r \in S(H_1^{1/2}\langle\delta_0\rangle, g^\sharp)$ which gives $R_j = (\alpha_0^{1/2}\phi_j)^w r^w \in S(M^{1/2}H^{3/4}\langle\delta_0\rangle, g^\sharp)$. Since

$$\operatorname{Re} FB = \operatorname{Re}(\operatorname{Re} F)B + i[\operatorname{Im} F, B]$$

when $B^* = B$, we find from (6.13) that

$$\operatorname{Re} f_j^w\delta_0^w B_0^{Wick} = \operatorname{Re}(a_j^w a_j^w B_0^{Wick}) \qquad \text{modulo } \operatorname{Op} S(m_1, g^\sharp). \tag{6.15}$$

In fact, $B_0 = 1 + \chi_0\varrho_0/\delta_0$ and $(\chi_0\varrho_0/\delta_0)^{Wick} \in \operatorname{Op} S(H_1^{1/2}\langle\delta_0\rangle, g^\sharp)$, thus

$$[a^w, B_0^{Wick}] = [a^w, (\chi_0\varrho_0/\delta_0)^{Wick}] \in \operatorname{Op} S(MH_1^{3/2}\langle\delta_0\rangle, g^\sharp)$$

when $a \in S(MH_1^{1/2}, G_1)$. Similarly, since $a_j \in S(M^{1/2}H_1^{-1/4}, G_1)$ we find that

$$a_j^w a_j^w B_0^{Wick} = a_j^w(B_0^{Wick}a_j^w + s_j^w) \qquad \text{modulo } \operatorname{Op} S(m_1, g^\sharp) \tag{6.16}$$

where $s_j \in S(M^{1/2}H^{3/4}\langle\delta_0\rangle, g^\sharp)$. Since $B_0 \ge 1/4$ we find from (6.14)–(6.16) that

$$\operatorname{Re} f_j^w B^{Wick} \ge \frac{1}{4} a_j^w a_j^w + \operatorname{Re} a_j^w S_j^w \qquad \text{modulo } \operatorname{Op} S(m_1, g^\sharp)$$

where $S_j \in S(M^{1/2}H^{3/4}\langle\delta_0\rangle, g^\sharp)$. Completing the square, we find

$$A_j^w = \operatorname{Re} f_j^w B^{Wick} \ge \frac{1}{4}\left(a_j^w + 2S_j^w\right)^*\left(a_j^w + 2S_j^w\right) \ge 0 \qquad \text{modulo } \operatorname{Op} S(m_1, g^\sharp)$$

since $(S_j^w)^* S_j^w \in \operatorname{Op} S(MH_1^{3/2}\langle\delta_0\rangle^2, g^\sharp)$. This gives (6.8) and the lemma in this case.

Finally, we consider the case when $H_1^{1/2} \le \kappa_4$ and (6.4) holds in ω_j. Since $|\delta_0(w)| \ge c_\kappa H_1^{-1/2}(w)$, we find $\langle \delta_0 \rangle \cong H_1^{-1/2}$ in ω_j. As before we may ignore terms in $S(MH_1^{1/2}, g^\sharp) \subseteq S(MH_1^{3/2}\langle\delta_0\rangle^2, g^\sharp)$ in ω_j by (6.7). We find from (6.5) that $\operatorname{sgn}(f)B \ge 0$ in ω_j, thus $f_j B \ge 0$. Since $f_j \in S(M, G_1)$, we find $f_j^w = f_j^{Wick}$ modulo $\operatorname{Op} S(MH_1, G_1)$ by Proposition 5.1, thus we may replace f_j^w with f_j^{Wick}. We find from Example 5.3 that

$$A_j^w = \operatorname{Re} f_j^{Wick} B^{Wick} = (f_j B)^{Wick} \ge 0 \qquad \text{modulo } \operatorname{Op} S(MH_1^{1/2}, g^\sharp).$$

This completes the proof of Lemma 6.3. □

We shall finish the paper by giving a proof of Proposition 2.5.

Proof of Proposition 2.5. Let $f \in L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$ be real valued satisfying condition $(\overline{\Psi})$ given by (2.2). By changing h, we may assume that $|\partial_w f| \le h^{-1/2}$. Let $B_T = \delta_0 + \varrho_T$, where $\delta_0 + \varrho_T$ is the Lipschitz continuous pseudo-sign for f given by Proposition 4.7 for $0 < T \le 1$, so that $|\varrho_T| \le m_1 \le \langle\delta_0\rangle/2$ and

$$\partial_t(\delta_0 + \varrho_T) \ge m_1/2T \qquad \text{in } \mathcal{D}'\big(]-T,T[\big). \tag{6.17}$$

We put $B_T \equiv 0$ when $|t| > T$, then that $B_T^{Wick} = b_T^w$ where $b_T(t,w) \in L^\infty(\mathbf{R}, S(H_1^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp))$ uniformly by Proposition 5.4. We find by Proposition 5.1 and (6.17) that

$$\left(\partial_t B_T^{Wick} u, u\right) \ge \left(m_1^{Wick} u, u\right)/2T \qquad \text{in } \mathcal{D}'\big(]-T,T[\big) \tag{6.18}$$

when $u \in \mathcal{S}(\mathbf{R}^n)$. By Proposition 6.1, we find for almost all $t \in [-T,T]$ that

$$\operatorname{Re}\left((f^w B_T^{Wick})\big|_t u, u\right) = (C^w(t)u, u) \quad u \in \mathcal{S}(\mathbf{R}^n) \tag{6.19}$$

with $C(t) \in S(m_1, g^\sharp)$ uniformly. Proposition 5.5 gives $C_0 > 0$ so that

$$|\left(C^w(t)u, u\right)| \le C_0 \left(m_1^{Wick} u, u\right) \tag{6.20}$$

for $u \in \mathcal{S}(\mathbf{R}^n)$ and $|t| \le T$. We find from (6.18)–(6.20) that

$$(\partial_t b_T^w u, u) + 2\operatorname{Re}(b_T^w u, f^w u) \ge (1/2T - 2C_0)\left(m_1^{Wick} u, u\right) \qquad \text{in } \mathcal{D}'\big(]-T,T[\big)$$

for $u \in \mathcal{S}(\mathbf{R}^n)$.

Since $|B_T| \le |\delta_0| + m_1 \le 3\langle\delta_0\rangle/2$ and $h^{1/2}\langle\delta_0\rangle^2/6 \le m_1$ by (4.2), we find that $b_T \in S(h^{-1/4}{m_1}^{1/2}, g^\sharp)$ so $h^{1/2}((b_T^w)^2 + 1) \in \operatorname{Op} S(m_1, g^\sharp)$ and Proposition 5.5 gives

$$h^{1/2}(\|b_T^w u\|^2 + \|u\|^2) \le C_1\left(m_1^{Wick} u, u\right) \qquad u \in \mathcal{S}(\mathbf{R}^n). \tag{6.21}$$

Finally, using Proposition 2.9 with $P_0 = D_t + if^w(t,x,D_x)$, $B = B_T^{Wick} = b_T^w$ and $m = m_1^{Wick}/4T$ we obtain that

$$C_1^{-1}h^{1/2}\int \|b_T^w u\|^2 + \|u\|^2\,dt \le \int \left(m_1^{Wick} u, u\right)\,dt \le 8T\int \operatorname{Im}(P_0 u, b_T^w u)\,dt$$

if $u \in \mathcal{S}(\mathbf{R}\times\mathbf{R}^n)$ has support where $|t| < T \le 1/8C_0$. This completes the proof of Proposition 2.5. □

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