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1 Introduction

The aim of this note is to present a joint work in collaboration with Frank Merle concerning the blow up of the critical Sobolev norm for the nonlinear Schrödinger equation

$$(NLS)\quad \begin{cases} iu_t = -\Delta u - |u|^{p-1}u, & (t,x) \in [0,T) \times \mathbf{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbf{R}^N \to \mathbf{C} \end{cases} \tag{1.1}$$

in dimension $N \geq 3$ with

$$1 < p < \frac{N+2}{N-2}.$$

From a result of Ginibre and Velo [4], (1.1) is locally well-posed in $H^1 = H^1(\mathbf{R}^N)$ and thus, for $u_0 \in H^1$, there exists $0 < T \leq +\infty$ and a unique solution $u(t) \in \mathcal{C}([0,T), H^1)$ to (1.1) and either $T = +\infty$, we say the solution is global, or $T < +\infty$ and then $\lim_{t \uparrow T} |\nabla u(t)|_{L^2} = +\infty$, we say the solution blows up in finite time.
(1.1) admits the following conservation laws in the energy space H^1:

$$\begin{array}{ll} L^2 - \text{norm}: & \int |u(t,x)|^2 = \int |u_0(x)|^2; \\ \text{Energy}: & E(u(t,x)) = \frac{1}{2}\int |\nabla u(t,x)|^2 - \frac{1}{p+1}\int |u(t,x)|^{p+1} = E(u_0). \end{array}$$

The scaling symmetry $\lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$ leaves the homogeneous Sobolev space $\dot{H}^{s_c}$ invariant with

$$s_c = \frac{N}{2} - \frac{2}{p-1}. \tag{1.2}$$

It is classical from the conservation of the energy and the L^2 norm that for $s_c < 0$, the equation is subcritical and all H^1 solutions are global and bounded in H^1. The smallest power for which blow up may occur is $p = 1 + \frac{4}{N}$ which corresponds to $s_c = 0$ and is referred to as the L^2 critical case. The case $0 < s_c < 1$ is the L^2 super critical and H^1 subcritical case.

We focus from now on onto the case $0 \leq s_c < 1$. The existence of finite time blow up solutions is a consequence of the virial identity, [14]: let an initial condition $u_0 \in \Sigma =$

$H^1 \cap \{xu \in L^2\}$ with $E(u_0) < 0$, then the corresponding solution $u(t)$ to (1.1) satisfies $u(t) \in \Sigma$ with:

$$\frac{d^2}{dt^2}\int |x|^2|u(t,x)|^2 = 4N(p-1)E(u_0) - \frac{16s_c}{N-2s_c}\int |\nabla u|^2 \leq 16E(u_0) \tag{1.3}$$

and thus the positive quantity $\int |x|^2|u(t,x)|^2$ cannot exist for whole times and u blows up in finite time.

Recall now from Cazenave and Weissler [2] that given $u_0 \in \dot{H}^{s_c}$, there exists a maximum time $T(u_0) > 0$ and a unique maximal solution $u(t) \in \mathcal{C}([0,T(u_0)),\dot{H}^{s_c})$ to (1.1). More generally and following the same procedure, given $s_c < s \leq 1$ and $u_0 \in \dot{H}^s$, there exists $T(s,u_0) > 0$ and a unique maximal solution $u(t) \in \mathcal{C}([0,T(s,u_0)),\dot{H}^{s_c})$ to (1.1), and as the problem is now subcritical with respect to $\dot{H}^s$, $T(s,u_0) < +\infty$ iff $\lim_{t\to T(s,u_0)} |u(t)|_{\dot{H}^s} = +\infty$.

Let now $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$, then there exists a maximum time $T > 0$ and a unique maximal solution $u(t) \in \mathcal{C}([0,T),\dot{H}^{s_c})$ to (1.1). Indeed, the life times given by the local Cauchy theory in $\dot{H}^{s_c}$ and $\dot{H}^{s_c} \cap \dot{H}^1$ are the same from a standard argument. Moreover, if $u(t)$ blows up in finite time $0 < T < +\infty$, then there holds the scaling lower bound:

$$\forall s_c < s \leq 1, \quad |u(t)|_{\dot{H}^s} \geq \frac{C(N,p,s)}{(T-t)^{\frac{s-s_c}{2}}}. \tag{1.4}$$

Indeed, let $s_c < s \leq 1$, $t \in [0,T)$ and consider $v_t(\tau,x) = \lambda^{\frac{2}{p-1}}(t)u(t+\lambda^2(t)\tau,\lambda(t)x)$ with $\lambda^{s-s_c}(t)|u(t)|_{\dot{H}^s} = 1$ so that $|v_t(0)|_{\dot{H}^s} = 1$, then from the local Cauchy theory in $\dot{H}^s$ which is subcritical, there exists $\tau_0(s) > 0$ such that v is defined on $[0,\tau_0(s)]$ from which $t+\lambda^2(t)\tau_0(s) < T$, this is (1.4).

Let us remark that this argument does not apply for the critical $\dot{H}^{s_c}$ norm. Numerics suggest at least in the radial case that finite time blow implies:

$$\lim_{t\to T} |u(t)|_{\dot{H}^{s_c}} = +\infty. \tag{1.5}$$

We conjecture that given $u_0 \in \dot{H}^{s_c}$, $0 < s_c < 1$, if the corresponding solution to (1.1) blows up in finite time, then (1.5) holds true. Note that this is in sharp contrast to the L^2 critical case where the L^2 norm is conserved and thus (1.5) breaks down.

Note that such kind of critical problems and behavior of the critical norms have been adressed in other settings, see for example [3] for the 3D Navier-Stockes problem.

1.1 A general strategy: reduction to a Liouville theorem

Let us present a general and robust strategy to attack the proof of (1.5) which is inspired form the works in Martel, Merle [8] and Merle, Raphaël [9]. This idea is to first argue by contradiction and use compactness arguments to extract from a renormalized version of the solution an asymptotic object which generates a global in time nonpositive energy solution to (1.1). Here the arguments are quite general and could be extended to a wider class of solutions and problems. In a second step, one concludes using a Liouville type blow up result for nonpositive energy solutions.

More precisely, let $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ with radial symmetry and assume that the corresponding solution $u(t)$ to (1.1) blows up in finite time $0 < T < \infty$ or equivalently from [2]:

$$\lim_{t \to T} |\nabla u(t)|_{L^2} < +\infty.$$

Let a sequence $t_n \to T$ such that

$$\lim_{t_n \to T} |\nabla u(t_n)|_{L^2} = +\infty$$

and

$$\forall n \geq 1, \quad \forall t \in [0, t_n], \quad |\nabla u(t)|_{L^2} \leq C |\nabla u(t_n)|_{L^2} \tag{1.6}$$

for some universal constant $C > 0$. We now assume that:

$$\forall n \geq 1, \quad |u(t_n)|_{\dot{H}^{s_c}} < +\infty \tag{1.7}$$

and look for a contradiction.
Let the sequence of rescaled initial data

$$u_n(0, x) = \lambda_u(t_n)^{\frac{2}{p-1}} u(t_n, \lambda_u(t_n) x)$$

with

$$\lambda_u(t) = \left(\frac{1}{|\nabla u(t)|_{L^2}} \right)^{\frac{1}{1-s_c}} \quad \text{so that} \quad |\nabla u_n(0)|_{L^2} = 1. \tag{1.8}$$

From the scaling invariance, (1.7) and the conservation of the energy, we have:

$$|u_n(0)|_{\dot{H}^{s_c}} = |u(t_n)|_{\dot{H}^{s_c}} \leq C \quad \text{and} \quad E(u_n(0)) = \lambda_u(t_n)^{2(1-s_c)} E(u_0) \to 0 \tag{1.9}$$

as $n \to +\infty$ and in particular

$$u_n(0) \rightharpoonup v(0) \quad \text{in} \quad \dot{H}^{s_c} \cap \dot{H}^1 \quad \text{as} \quad n \to +\infty \tag{1.10}$$

up to a subsequence. From the compact radial embedding $\dot{H}^{s_c} \cap \dot{H}^1 \hookrightarrow L^{p+1}$, we have up to a subsequence

$$u_n(0) \to v(0) \quad \text{in} \quad L^{p+1} \quad \text{as} \quad n \to +\infty$$

and thus

$$E(v(0)) \leq 0.$$

Observe now that the solution $u_n(\tau)$ to (1.1) with initial data $u_n(0)$ is explicitly

$$u_n(\tau, x) = \lambda_u(t_n)^{\frac{2}{p-1}} u(t_n + \lambda_u(t_n)^2\tau, \lambda_u(t_n)x), \tag{1.11}$$

and thus (1.6) and (1.8) imply:

$$\forall \tau \in (-\frac{t_n}{\lambda_u^2(t_n)}, 0], \quad |\nabla u_n(\tau)|_{L^2} = \frac{|\nabla u(t_n + \lambda_u(t_n)^2\tau)|_{L^2}}{|\nabla u(t_n)|_{L^2}} \leq C. \tag{1.12}$$

Let now $v(t)$ be the solution to (1.1) with initial data $v(0)$ and $(-T_v, 0]$ its maximum time interval existence on the left in time in $\dot{H}^{s_c} \cap \dot{H}^1$, then one may adapt the Lemma of stability of weak convergence in H^1, see Glangetas, Merle [5] and also Lemma 3 in [9], to conclude:

$$\forall \tau \leq 0, \quad u_n(\tau) \rightharpoonup v(\tau) \text{ in } \dot{H}^{s_c} \cap \dot{H}^1 \text{ as } n \to +\infty$$

and thus from (1.12):

$$|\nabla v(\tau)|_{L^2} \leq C \text{ and } -T_v = -\infty.$$

In other words, (1.7) implies the existence of a global in time radially symmetric nonpositive energy solution to (1.1) in $\dot{H}^{s_c} \cap \dot{H}^1$.

Our main result is that the existence of such an object may be ruled out in some cases from the following Liouville type result:

Theorem 1 (Finite time blow up for non positive energy solutions in $\dot{H}^{s_c} \cap \dot{H}^1$) *Assume*

$$N \geq 3 \quad \textit{and} \quad \frac{1}{2} \leq s_c < 1.$$

Let $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ with radial symmetry and

$$E(u_0) \leq 0,$$

then the corresponding solution $u(t)$ to (1.1) blows up in finite time $0 < T < +\infty$.

Comments on Theorem 1

1. On the assumption $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$: Let $u_0 \in \Sigma$ with $E_0 < 0$, then finite time blow up follows from the virial identity (1.3). If $u_0 \in H^1$ radial with $E_0 < 0$, a simple localization argument allows one to conclude also, see Ogawa, Tsutsumi [12]. Now if $u_0 \in H^1$ radial with $E_0 = 0$, then finite time blow up also follows. The key here is first the conservation of the energy and a Gagliardo-Nirenberg inequality:

$$|\nabla u(t)|_{L^2}^2 = \frac{2}{p+1}|u|_{L^{p+1}}^{p+1} \leq C|\nabla u(t)|_{L^2}^{2+s_c(p-1)}|u(t)|_{L^2}^{(1-s_c)(p-1)}$$

which implies from *the conservation of the L^2 norm* the *uniform lower bound*:

$$|\nabla u(t)|_{L^2} \geq \frac{C}{|u_0|_{L^2}^{\frac{1-s_c}{s_c}}}.$$

This together with a space localization of the virial identity (1.3) yields the claim.
Let us insist onto the fact that our need to work with low regularity $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ comes from the renormalization procedure before the extraction of the asymptotic object and a major difficulty is thus that we may no longer used the L^2 conservation law. Arguing by contradiction, we in fact need to rule out the possibility of a non linear self similar vanishing $|\nabla u(t)|_{L^2} \to 0$ as $t \to +\infty$ in the case when $E_0 = 0$. This difficulty already occured in [9], [10]. Our main tool is that for radial functions and $s_c \geq \frac{1}{2}$, we may replace the role of the L^2 norm by a suitable scaling invariant Morrey-Campanatto norm, see the definition (2.16), for which *uniform bounds in time are derived which somehow mimic the L^2 conservation law*. The key here is *a new kind of monotonicity statement* based on a localized virial identity. Here the techniques are thus restricted to radial solutions. The assumption $s_c \geq \frac{1}{2}$ arises to control some momentum terms in the time integration of the virial identity, see in particular (2.19).

2. On the sharpness of the result: We expect the assumptions on the initial data to be sharp in the following sense. One may obtain exact self similar blow up solutions by looking for solutions of the form

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} P\left(\frac{x}{\lambda(t)}\right) e^{i\log(T-t)} \quad \text{with} \quad \lambda(t) = \sqrt{2b(T-t)}$$

for some parameter $b > 0$ and some stationary profile P satisfying the non linear elliptic equation:

$$\Delta P - P + ib\left(\frac{2}{p-1}P + y \cdot \nabla P\right) + P|P|^{p-1} = 0. \tag{1.13}$$

Rigorous existence results of finite energy radially symmetric solutions to (1.13) are known only for p close to the L^2 critical value, see Kopell and Landman [6]. The obtained profiles are in $\dot{H}^1$ and have zero energy but always miss $\dot{H}^{s_c}$ due to a logarithmic growth at infinity. Such solutions, when they exist, thus provide explicit examples of zero energy solutions which blow up on the right in time but are global on the left.

3. On the range of parameters: The proof of Theorem 1 is complete only for $N \geq 3$ and $\frac{1}{2} \leq s_c < 1$. This covers in particular the physically relevant case $N = 3$, $p = 3$ which is $\dot{H}^{\frac{1}{2}}$ critical. We expect the result to in fact hold true in the whole super critical range $0 < s_c < 1$. Here we encounter a technical difficulaty related to the estimate of some localized momentum terms, see (2.19). For $N = 1, 2$, another difficulty would arise for $p \geq 5$ -while $p < 5$ is implied by the assumptions of Theorem 1-. Indeed, the proof of

Theorem 1 in particular requires that the singularity formation takes place at the origin for the radial solution. This can be proved to be necessary for $p < 5$ -because $p = 5$ is the one dimensional L^2 critical exponent-, but one can construct for $p = 5$, $N = 2$, a radial finite time blow up solution blowing up on a sphere, see Raphaël [13]. The intuition developped for the proof of Theorem 1 would not be well adapted for such an object.

The reduction procedure together with the Liouville Theorem 1 thus imply the following blow up result of the critical norm:

Theorem 2 (Blow up of the critical Sobolev norm) *Assume*

$$N \geq 3 \quad and \quad \frac{1}{2} \leq s_c < 1.$$

Let $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ with radial symmetry and assume that the corresponding solution to (1.1) blows up in finite or infinite time $0 < T \leq +\infty$, then

$$\limsup_{t \to T} |u(t)|_{\dot{H}^{s_c}} = +\infty. \tag{1.14}$$

In fact, by pushing further the arguments developped here and in particular for the proof of Theorem 1, one can obtain a much sharper understanding of the structure in space of the rescaled sequence $u_n(\tau)$ given by (1.11) which eventually implies a logarithmic lower bound on the blow up rate of the critical norm -which is optimal in some sense-, see Merle, Raphaël [11]:

Theorem 3 (Lower bound for the critical Sobolev norm) *Assume*

$$N \geq 3 \quad and \quad \frac{1}{2} \leq s_c < 1.$$

There exists a constant $\gamma = \gamma(N, p) > 0$ such that the following holds true. Let $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ with radial symmetry and assume that the corresponding solution to (1.1) blows up in finite time $0 < T < +\infty$, then

$$|u(t)|_{\dot{H}^{s_c}} \geq |\log(T - t)|^{\gamma} \tag{1.15}$$

for t close enough to T.

2 Proof of the Liouville Theorem 1

Our aim for the rest of this note is to present a mostly self contained presentation of the Liouville Theorem 1.

2.1 Some technical tools

Before detailing the proof of the Liouville Theorem 1, we recall two technical tools which proofs are standard. We refer to [11] for the details.

Let the scaling invariant Morrey-Campanatto norm

$$\rho(u,R)=\sup_{R'\geq R}\frac{1}{(R')^{2s_c}}\int_{R'\leq|x|\leq 2R'}|u|^2. \tag{2.16}$$

We claim the following elementary estimates.

Lemma 1 *There exists a universal constant $C>0$ such that for all $u\in\dot{H}^{s_c}$,*

$$\forall R>0,\quad \rho(u,R)\leq C|u|^2_{\dot{H}^{s_c}} \tag{2.17}$$

and

$$\lim_{R\to+\infty}\frac{1}{R^{2s_c}}\int_{|x|\leq R}|u|^2\to 0\quad as\quad R\to+\infty. \tag{2.18}$$

Moreover, let ψ be a smooth radially symmetric cut-off function supported in $|x|\leq 2$, then there exists a constant $C_\psi>0$ such that for all $u\in\dot{H}^{s_c}$:

$$\left|\left(\int\nabla\psi\left(\frac{x}{R}\right)\cdot\nabla u\overline{u}\right)\right|\leq C_\psi|u|^2_{\dot{H}^{s_c}}R^{2s_c-1}. \tag{2.19}$$

Note that (2.19) relies on our assumption $\frac{1}{2}\leq s_c\leq 1$.

Next, recall the standard Gagliardo-Nirenberg inequality :

$$\int|u|^{p+1}\leq C|u|^{p-1}_{\dot{H}^{s_c}}\left(\int|\nabla u|^2\right).$$

For radially symmetric distributions, we may sharpen this inequality by using the fact that for $N\geq 3$,

$$p<\frac{N+2}{N-2}<5,$$

and thus the nonlinearity is L^2 subcritical away from zero. We claim:

Proposition 1 (Radial Gagliardo-Nirenberg inequality) *For all $\eta>0$, there exists a constant $C_\eta>0$ such that for all $u\in\dot{H}^{s_c}$ with radial symmetry, for all $R>0$,*

$$\int_{|x|\geq R}|u|^{p+1}\leq\eta|\nabla u|^2_{L^2(|x|\geq R)}+\frac{C_\eta}{R^{2(1-s_c)}}\left[(\rho(u,R))^{\frac{2(p+3)}{5-p}}+(\rho(u,R))^{\frac{p+1}{2}}\right]. \tag{2.20}$$

2.2 Proof of the Liouville Theorem 1

We now prove the Liouville Theorem 1.

Proof of Theorem 1.

Let $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ with radial symmetry and $E(u_0) \leq 0$, we argue by contradiction and assume that the corresponding solution $u(t)$ to (1.1) is globally defined on $[0, +\infty)$ in $\dot{H}^{s_c} \cap \dot{H}^1$.

step 1 Localized virial control.

Let a smooth radially symmetric cut-off function ψ with $\psi(r) = \frac{r^2}{2}$ for $r \leq 2$, $\psi''(r) \leq 1$ and $\psi(r) = 0$ for $r \geq 3$. For a given $R > 0$, let

$$\psi_R(r) = R^2 \psi_R(r).$$

We claim the following localized virial control:

Lemma 2 (Localized virial estimate) *There holds for some constant $C = C(N, p, \psi)$: $\forall R > 0$, $\forall t \geq 0$,*

$$\int |\nabla u|^2 \leq C \left[-\frac{d}{dt} Im\left(\nabla \psi_R \cdot \nabla u \overline{u}\right) + \int_{|x| \geq R} |u|^{p+1} + \frac{1}{R^2} \int_{2R \leq |x| \leq 3R} |u|^2 \right]. \tag{2.21}$$

Proof of Lemma 2

Let χ be a smooth radially symmetric compactly supported cut-off function. We recall the following standard localized virial identities which up to standard regularization arguments are obtained by integration by parts on (1.1):

$$\frac{1}{2} \frac{d}{dt} \int \chi |u|^2 = Im\left(\nabla \chi \cdot \nabla u \overline{u}\right), \tag{2.22}$$

$$\frac{1}{2} \frac{d}{dt} Im\left(\nabla \chi \cdot \nabla u \overline{u}\right) = \int \chi'' |\nabla u|^2 - \frac{1}{4} \int \Delta^2 \chi |u|^2 - \left(\frac{1}{2} - \frac{1}{p+1}\right) \int \Delta \chi |u|^{p+1}. \tag{2.23}$$

Note that we used here that u has radial symmetry. Applying (2.23) with $\chi = \psi_R$, we get:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} Im\left(\nabla \psi_R \cdot \nabla u \overline{u}\right) \\
= & \int \psi''\left(\frac{x}{R}\right) |\nabla u|^2 - \frac{1}{4R^2} \int \Delta^2 \psi\left(\frac{x}{R}\right) |u|^2 - \left(\frac{1}{2} - \frac{1}{p+1}\right) \int \Delta \psi\left(\frac{x}{R}\right) |u|^{p+1} \\
\leq & \int |\nabla u|^2 + \frac{C}{R^2} \int_{2R \leq |x| \leq 3R} |u|^2 - N\left(\frac{1}{2} - \frac{1}{p+1}\right) \int |u|^{p+1} + C \int_{|x| \geq R} |u|^{p+1}.
\end{aligned}$$

Now from the conservation of the energy:

$$\int |u|^{p+1} = \frac{p+1}{2}\int |\nabla u|^2 - (p+1)E_0$$

from which

$$\int |\nabla u|^2 - N\left(\frac{1}{2} - \frac{1}{p+1}\right)\int |u|^{p+1} = \frac{N(p-1)}{2}E_0 - \frac{2s_c}{N-2s_c}\int |\nabla u|^2.$$

(2.21) follows from $0 < s_c < 1$ and $E_0 \leq 0$. This concludes the proof of Lemma 2.

step 2 A monotonicity statement.

We now come to the crux of the proof which is a monotonicity type of result derived from (2.21). The outcome is a uniform control of the scaling invariant Morrey-Campanatto norm ρ.
Consider $\varepsilon > 0$ small to be fixed later. From (2.18), there exists $A(\varepsilon)$ such that:

$$\forall A \geq A(\varepsilon), \quad \frac{1}{A^{2s_c}}\int_{|x|\leq 3A} |u(0)|^2 < \frac{\varepsilon}{10}. \tag{2.24}$$

We claim:

Lemma 3 (Uniform control of the scaling invariant ρ norm) *There exists $B(\varepsilon) > A(\varepsilon)$ such that*

$$\forall t \geq 0, \quad \rho(u(t), B(\varepsilon)\sqrt{1+t}) < \varepsilon. \tag{2.25}$$

Moreover, we have the dispersive estimate:

$$\forall t \geq 0, \quad \int_0^t \tau|\nabla u(\tau)|_{L^2} d\tau \leq C(1+t)^{1+s_c} \tag{2.26}$$

for some constant $C = C(u_0, N, p) > 0$.

Proof of Lemma 3

Proof of (2.25): From $u \in \mathcal{C}([0,+\infty), \dot{H}^{s_c} \cap \dot{H}^1)$, there exists a time $t_1(\varepsilon) > 0$ such that

$$\forall t \in [0, t_1(\varepsilon)], \quad \forall A \geq A(\varepsilon), \quad \rho(u(t), A\sqrt{1+t}) < 10\varepsilon. \tag{2.27}$$

We claim that there exists $B(\varepsilon) > A(\varepsilon)$ large enough such that:

$$\forall t \in [0, t_1(\varepsilon)], \quad \rho(u(t), A(\varepsilon)\sqrt{1+t}) < \frac{\varepsilon}{2} \tag{2.28}$$

what proves (2.25).
Proof of (2.28): Let $t_0 \in [0, t_1(\varepsilon)]$, $A \geq B(\varepsilon)$ to be chosen and set

$$R = R(A, t_0) = A\sqrt{1+t_0}. \tag{2.29}$$

We claim that:

$$\frac{1}{R^{2s_c}} \int_{R \leq |x| \leq 2R} |u(t_0)|^2 < \frac{\varepsilon}{10} + \frac{C(1+|u_0|^2_{\dot{H}^{s_c}})}{B^2(\varepsilon)}. \tag{2.30}$$

This estimate being uniform with respect to $A \geq B(\varepsilon)$ and $t_0 \in [0, t_1(\varepsilon)]$, (2.28) follows for $B(\varepsilon)$ large enough.
To prove (2.30), consider the localization (2.21) for R given by (2.29) and estimate the terms of the right hand side. First observe from the monotonicity of ρ and (2.27) that:

$$\forall t \in [0, t_0], \quad \rho(u(t), R) \leq \rho(u(t), B(\varepsilon)\sqrt{1+t}) \leq 10\varepsilon. \tag{2.31}$$

Thus:

$$\frac{1}{R^2} \int_{2R \leq |x| \leq 3R} |u(t)|^2 \leq \frac{1}{R^{2(1-s_c)}} \rho(u(t), R) \leq \frac{1}{R^{2(1-s_c)}}. \tag{2.32}$$

The non linear term in (2.21) is estimated from the refined Gagliardo-Nirenberg estimate (2.20) provided $\eta > 0$ has been chosen small enough and (2.31): $\forall t \in [0, t_0]$,

$$\begin{aligned} C\int_{|x|\geq R} |u|^{p+1} &\leq \frac{1}{2}\int |\nabla u(t)|^2 + \frac{C}{R^{2(1-s_c)}}\left[(\rho(u(t),R))^{\frac{2(p+3)}{5-p}} + (\rho(u(t),R))^{\frac{p+1}{2}}\right] \\ &\leq \frac{1}{2}\int |\nabla u(t)|^2 + \frac{1}{R^{2(1-s_c)}} \end{aligned} \tag{2.33}$$

provided $\varepsilon > 0$ is small enough. We now inject (2.32) and (2.33) into (2.21) and integrate in time, we get: $\forall t \in [0, t_0]$,

$$\begin{aligned} CIm\left(\int \nabla\psi_R \cdot \nabla u(t)\overline{u(t)}\right) + \int_0^t |\nabla u(\tau)|^2_{L^2} d\tau &\leq CIm\left(\int \nabla\psi_R \cdot \nabla u(0)\overline{u(0)}\right) + \frac{Ct_0}{R^{2(1-s_c)}} \\ &\leq C|u_0|^2_{\dot{H}^{s_c}} R^{2s_c} + \frac{Ct_0}{R^{2(1-s_c)}} \end{aligned}$$

where we used (2.19). We integrate once more in time from (2.22) and get:

$$\begin{aligned} \int \psi_R |u(t_0)|^2 + \int_0^{t_0} t|\nabla u(t)|^2_{L^2} dt &\leq C\int \psi_R |u(0)|^2 \\ &+ C|u_0|^2_{\dot{H}^{s_c}} t_0 R^{2s_c} + \frac{Ct_0^2}{R^{2(1-s_c)}}. \end{aligned} \tag{2.34}$$

Now observe from the definition of ψ that

$$\int \psi_R |u(t_0)|^2 \geq \frac{R^2}{2} \int_{R \leq |x| \leq 2R} |u(t_0)|^2.$$

We thus divide (2.34) by $R^{2(1+s_c)}$ and get : $\forall t_0 \in [0, t_0]$,

$$\begin{aligned}\frac{1}{R^{2s_c}}\int_{R\leq|x|\leq 2R}|u(t_0)|^2 &\leq \frac{1}{R^{2s_c}}\int_{|x|\leq 3R}|u(0)|^2 + C|u_0|^2_{\dot{H}^{s_c}}\frac{t_0}{R^2} + \frac{Ct_0^2}{R^4}\\ &\leq \frac{\varepsilon}{10} + \frac{C(1+|u_0|^2_{\dot{H}^{s_c}})}{B^2(\varepsilon)}\end{aligned}$$

where we used (2.24) and (2.29) in the last step. This is (2.30).
Proof of (2.26): We come back to (2.34) with $\varepsilon > 0$ fixed and $A = B(\varepsilon)$ which implies: $\forall t_0 \geq 0$,

$$\int_0^{t_0} t|\nabla u(t)|^2_{L^2}dt \leq C(u_0, N, p)\left[t_0 R^{2s_c}(t_0) + \frac{t_0^2}{R^{2(1-s_c)}(t_0)}\right] \leq C(u_0, N, p)(1+t_0)^{1+s_c}$$

from (2.29). This concludes the proof of (2.26) and Lemma 3.

step 3 Control of the local L^2 norm.

Let

$$\lambda(t) = \left(\frac{1}{|\nabla u(t)|_{L^2}}\right)^{\frac{1}{1-s_c}}, \tag{2.35}$$

then from (2.26), there exists a sequence $t_n \to +\infty$ such that:

$$\forall n \geq 0, \quad |\nabla u(t_n)|_{L^2} \leq \frac{C}{(1+t_n)^{\frac{1-s_c}{2}}} \quad \text{or equivalently} \quad \frac{1+t_n}{\lambda^2(t_n)} \leq C \tag{2.36}$$

for some constant $C = C(u_0, N, p) > 0$. We now claim a scaling invariant control of the local L^2 norm of u on the sequence t_n:

Lemma 4 (Scaling invariant control of the local L^2 norm of $u(t_n)$) *There exists $B_0 \geq 1$ large enough such that: $\forall B \geq B_0$, $\forall n \geq 0$,*

$$\frac{1}{\lambda^{2s_c}(t_n)}\int_{|x|\leq B\lambda(t_n)}|u(t_n)|^2 \leq \frac{1}{\lambda^{2s_c}(t_n)}\int_{|x|\leq 2B\lambda(t_n)}|u(0)|^2 + \frac{C}{B^{2(1-s_c)}} \tag{2.37}$$

for some constant $C = C(u_0, N, p)$.

Proof of Lemma 4

Let $n \geq 0$, $D \geq D_0$ large enough and set

$$R_n = D\lambda(t_n). \tag{2.38}$$

Fix $\varepsilon > 0$ and $B(\varepsilon)$ such that (2.25) holds true. Observe from (2.36) that:

$$\forall t \in [0, t_n], \quad R_n = D\lambda(t_n) \geq \frac{D_0}{C}\sqrt{1+t_n} \geq B(\varepsilon)\sqrt{1+t}$$

for D_0 large enough. We conclude from (2.25):

$$\forall n \geq 0, \quad \forall t \in [0, t_n], \quad \rho(u(t), R_n) \leq \rho(u(t), B(\varepsilon)\sqrt{1+t}) < \varepsilon. \tag{2.39}$$

We now apply (2.22) and (2.23) for some smooth nonnegative radially symmetric cut-off function $\chi_{R_n}(r) = \chi(\frac{x}{R_n})$ where $\chi(r) = 1$ for $r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$. We first get from (2.23): $\forall t \in [0, t_n]$,

$$\begin{aligned}
\left|\frac{d}{dt} Im\left(\nabla\chi_{R_n} \cdot \nabla u\overline{u}\right)\right| &\leq \frac{C}{R_n^2}\left[\int |\nabla u(t)|^2 + \frac{1}{R_n^2}\int_{R_n \leq |x| \leq 2R_n} |u(t)|^2 + \int_{|x| \geq R_n} |u(t)|^{p+1}\right] \\
&\leq \frac{C}{R_n^2}\left[\int |\nabla u(t)|^2 + \frac{1}{R_n^{2(1-s_c)}}\right]
\end{aligned}$$

where we used (2.20) and (2.39). Integrating twice in time from (2.22) and using (2.19), (2.26) and (2.38), we estimate:

$$\begin{aligned}
&\int \chi_{R_n}|u(t_n)|^2 \leq \int \chi_{R_n}|u(0)|^2 + Ct_n\left|Im\left(\nabla\chi_{R_n} \cdot \nabla u(0)\overline{u(0)}\right)\right| + \frac{C}{R_n^2}\int_0^{t_n} t|\nabla u(t)|^2_{L^2} \\
+ &\quad \frac{Ct_n^2}{R_n^{2(1-s_c)+2}} \leq \int_{|x| \leq 2R_n} |u(0)|^2 + \frac{C|u_0|^2_{\dot{H}^{s_c}} t_n}{R_n^{2(1-s_c)}} + \frac{C(1+t_n)^{1+s_c}}{R_n^2} + \frac{Ct_n^2}{R_n^{2(1-s_c)+2}} \\
\leq &\quad \int_{|x| \leq 2R_n} |u(0)|^2 + C(u_0, N, p)\left[\frac{t_n}{D^{2(1-s_c)}\lambda^{2(1-s_c)}(t_n)} + \frac{(1+t_n)^{1+s_c}}{D^2\lambda^2(t_n)} + \frac{t_n^2}{D^{4-2s_c}\lambda^{4-2s_c}(t_n)}\right].
\end{aligned}$$

We divide by $\lambda^{2s_c}(t_n)$ and have from the choice of χ:

$$\begin{aligned}
&\frac{1}{\lambda^{2s_c}(t_n)}\int_{|x| \leq R_n} |u(t_n)|^2 \leq \frac{1}{\lambda^{2s_c}(t_n)}\int \chi_{R_n}|u(t_n)|^2 \\
\leq &\quad \frac{1}{\lambda^{2s_c}(t_n)}\int_{|x| \leq 2R_n} |u(0)|^2 + \frac{C(u_0, N, p)}{D^{2(1-s_c)}}\left[\frac{1+t_n}{\lambda^2(t_n)} + \left(\frac{1+t_n}{\lambda^2(t_n)}\right)^{1+s_c} + \left(\frac{1+t_n}{\lambda^2(t_n)}\right)^2\right] \\
\leq &\quad \frac{1}{\lambda^{2s_c}(t_n)}\int_{|x| \leq 2R_n} |u(0)|^2 + \frac{C(u_0, N, p)}{D^{2(1-s_c)}}
\end{aligned}$$

from (2.36). This concludes the proof of (2.37) and Lemma 4.

step 4 Contradiction from the conservation of the energy.

We now are in position to obtain a contradiction to the global existence of $u(t)$ which is based on (2.37).
Let

$$v_n(y) = \lambda^{\frac{2}{p-1}}(t_n)u(t_n\lambda(t_n)y),$$

then from (2.35) and the conservation of the energy:

$$E(v_n) = \lambda^{2(1-s_c)}(t_n)E(u(t_n)) = \lambda^{2(1-s_c)}(t_n)E(u_0) \leq 0, \quad \text{and} \quad |\nabla v_n|_{L^2} = 1 \tag{2.40}$$

and thus

$$\int |v_n|^{p+1} \geq \frac{p+1}{2}. \tag{2.41}$$

Pick an $\varepsilon > 0$ to be fixed later. We claim on the one hand that there exists $D_0(\varepsilon)$ large enough such that for all $D \geq D_0(\varepsilon)$, $\forall n \geq 0$,

$$\int_{|y|\geq D} |v_n|^{p+1} \leq 2\varepsilon \tag{2.42}$$

Indeed, from (2.39):

$$\rho(v_n, D_0) = \rho(u(t_n), \lambda(t_n)D_0) < \varepsilon$$

and thus using (2.20) and (2.40):

$$\int_{|y|\geq D} |v_n|^{p+1} \leq \varepsilon + \frac{C(\varepsilon)}{D^{2(1-s_c)}} < 2\varepsilon$$

for $D \geq D_0(\varepsilon)$ large enough, this is (2.42).
On the other hand, from the localization of the L^2 mass of v_n, we claim that there exists $D_1(\varepsilon)$ such that for all $D \geq D_1(\varepsilon)$, there exists $N(\varepsilon)$ such that:

$$\forall n \geq N(\varepsilon), \quad \int_{|y|\leq D} |v_n|^{p+1} \leq C\varepsilon^{(p-1)(1-s_c)}. \tag{2.43}$$

Taking $\varepsilon > 0$ small enough and n large enough, (2.42) and (2.43) now contradict (2.41) and conclude the proof of Theorem 1.
Proof of (2.43): We have from (2.37):

$$\begin{aligned}
\int_{|y|\leq 2D} |v_n|^2 &= \frac{1}{\lambda^{2s_c}(t_n)} \int_{|x|\leq 2D\lambda(t_n)} |u(t_n)|^2 \leq \frac{1}{\lambda^{2s_c}(t_n)} \int_{|x|\leq 4D\lambda(t_n)} |u(0)|^2 + \frac{C}{D^{2(1-s_c)}} \\
&\leq \frac{1}{\lambda^{2s_c}(t_n)} \int_{|x|\leq 4D\lambda(t_n)} |u(0)|^2 + \varepsilon
\end{aligned}$$

for $D \geq D_1(\varepsilon)$ large enough. Next $\lambda(t_n) \to +\infty$ as $n \to \infty$ from (2.36) and we thus conclude from (2.18):

$$\int_{|y|\leq 2D} |v_n|^2 \leq 2\varepsilon \quad \text{for} \quad n \geq N(\varepsilon) \quad \text{large enough.} \tag{2.44}$$

Recall now the Gagliardo-Nirenberg inequality

$$\int |w|^{p+1} \leq C|w|_{L^2}^{(p-1)(1-s_c)} |\nabla w|_{L^2}^{2+(p-1)s_c}$$

which we may localize to get:

$$\int_{|y|\leq D} |v_n|^{p+1} \leq C|w_n|_{L^2(|y|\leq 2D)}^{(p-1)(1-s_c)} |\nabla w_n|_{L^2}^{2+(p-1)s_c} + C|w_n|_{L^2(|y|\leq 2D)}^{p+1} \leq C\varepsilon^{(p-1)(1-s_c)}$$

from (2.44) and (2.40). This is (2.43).

This concludes the proof of the Liouville Theorem 1.

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