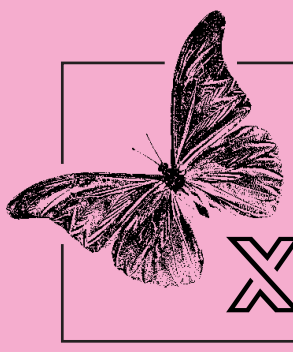

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UNCERTAINTY PRINCIPLES FOR ORTHONORMAL BASES

PHILIPPE JAMING

Abstract. In this survey, we present various forms of the uncertainty principle (Hardy, Heisenberg, Benedicks...). We further give a new interpretation of the uncertainty principles as a statement about the time-frequency localization of elements of an orthonormal basis, which improves previous unpublished results of H. Shapiro.

Finally, we reformulate some uncertainty principles in terms of properties of the free heat and shrödinger equations.

1. Introduction

The uncertainty principle is a "metamathematical" statement that asserts that

a function and its Fourier transfrom can not both be sharply localized.

There are various precise mathematical formulations of this general fact, the most well known being those of Hardy and of Heisenberg-Pauli-Weil. It is one of the aims of this survey to present various statements that can be interpreted as uncertainty principles. The reader may find many extensions of the results presented here, other manifestations of the uncertainty principle as well as more references to the vast literature in the surveys [FS, BD] and the book [HJ]. Part of this survey has serious overlaps with these texts.

We start with results around Heisenberg's Uncertainty Principle and show, following de Bruijn, how this can be proved using the spectral theory of the Hermite Operator. This further shows that Hermite Functions are "successive optimals" of Heisenberg's Uncertainty Principle. We complete this section with a new result joint with A. Powell [JP] that shows that the elements of an orthonormal basis and their Fourier Transforms have their means and dispersions that grow at least like those of the Hermite Basis. This gives a quantitative version of an unpublished result by H. Shapiro (see [Sh1, Sh2]).

In a second section, we consider the problem of the joint smallness of the support and spectrum (support of the Fourier Transform) of a function. For instance, it is easy to show that a function and its Fourier Transform can not both have compact support. It was shown by Benedicks [Be] that the same is true when one asks the support and spectrum to be of finite measure. Further, it was independently proved by Amrein and Berthier [AB] that the L^2-norm of a function is controled by the L^2-norm of the function outside a set of finite measure and the L^2-norm of its Fourier transform outside an other set of finite measure. We complete the section by mentioning the closely related problem of energy concentration in a

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compact set studied by Landau, Pollak and Slepian and conclude it with the local uncertainty principle of Faris, Price and Sitaram and show how this is related to the previous section.

We pursue the paper by measuring concentration by the speed of decay away from the mean. We first sketch a new proof of Hardy's Theorem due to B. Demange [De]. This roughly states that a function and its Fourier Transform can not both decrease faster than gaussians and, if they decrease like a gaussian, then they are actually gaussians. We conclude the section by asking whether all elements of an orthonormal sequence as well as their Fourier Transforms be bounded by a fixed L^2-function. We show that this implies that the sequence is finite and give bounds on the number of elements, thus improving a previous result of H. Shapiro [Sh1, Sh2].

We conclude the paper by some simple results in the spirit of [EKV, EKPV], reformulating some uncertainty principles in terms of solutions of the free heat and Shrödinger equations.

2. Heisenberg's uncertainty principle

In this section, we are going to prove Heisenberg's Uncertainty Principle and show that the Hermite Basis is the best concentrated in the time-frequency plane when concentration is measured by dispersion. This is done using the spectral theory of the Hermite Operator.

2.1. **Notations.**

For sake of simplicity, all results will be stated on $\mathbb{R}$. Generalizations to $\mathbb{R}^n$ are either strateforward using tensorization or available in the litterature.

For $f \in L^2(\mathbb{R})$, let

— $\mu(f) = \dfrac{1}{\|f\|_2^2} \displaystyle\int_{\mathbb{R}} t|f(t)|^2 \mathrm{d}t$ its *mean*;

— $\Delta^2(f) = \displaystyle\int_{\mathbb{R}} |t-\mu(f)|^2 |f(t)|^2 \mathrm{d}t$ its *variance* and $\Delta(f) = \sqrt{\Delta^2(f)}$ its *dispersion.*

For $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, the Fourier Transform is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2i\pi x\xi} \mathrm{d}x$$

and is then extended to all of $L^2(\mathbb{R})$ in the usual way. The Inverse Fourier Transform of f is denoted by $\check{f}$.

For $f \in \mathcal{C}^2(\mathbb{R})$ we define the Hermite operator as

$$Hf(t) = -\frac{1}{4\pi^2}\frac{\mathrm{d}}{\mathrm{d}x} + t^2 f(t).$$

The Hermite Functions are then defined as

$$h_k(t) = \frac{2^{1/4}}{\sqrt{k!}} \left(-\frac{1}{\sqrt{2\pi}}\right)^k e^{\pi t^2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k e^{-2\pi t^2}.$$

They have the following well known properties:

(i) they form an orthonormal basis in $L^2(\mathbb{R})$,

(ii) they are eigenfunctions of the Fourier transform $\widehat{h_k} = i^{-k} h_k$,

(iii) they are eigenfunctions of the Hermite operator $Hh_k = \dfrac{2k+1}{2\pi} h_k$.

This allows to extend the Hermite Operator as a positive self-adjoint operator on $L^2(\mathbb{R})$ via

$$Hf = \sum_{k=0}^{+\infty} \frac{2k+1}{2\pi} \langle f, h_k \rangle h_k.$$

Finally, an easy computation shows that

$$\langle Hf, f \rangle = \mu(f)^2 \|f\|^2 + \Delta^2(f) + \mu(\widehat{f})^2 \|\widehat{f}\|^2 + \Delta^2(\widehat{f}).$$

2.2. **Heisenberg's uncertainty principle.**
It is now easy to give a proof of the following:

Heisenberg-Pauli-Weil's Uncertainty principle. *For every $f \in L^2(\mathbb{R})$,*

$$\Delta(f)\Delta(\widehat{f}) \geq \frac{1}{4\pi} \|f\|_2^2.$$

Moreover equality occurs only if there exists $C \in \mathbb{C}$, $\omega, a, \alpha \in \mathbb{R}$, such that

$$f(t) = ce^{2i\pi\omega t} e^{-\pi\alpha|t-a|^2} \quad a.e.$$

The following proof, using the spectral theory of the Hermite operator, seems to be du to De Bruijn [dB].

Proof. First, replacing f by $f(t) = e^{2i\pi\omega t} f(t-a)$, it is enough to assume that $\mu(f) = \mu(\widehat{f}) = 0$. Further, it is enough to prove that

$$\Delta(f)^2 + \Delta(\widehat{f})^2 \geq \frac{1}{2\pi} \|f\|_2^2$$

with equality only for $f = c\, h_0$. Indeed, it is enough to apply this to $f_\lambda(t) = \frac{1}{\sqrt{\lambda}} f(t/\lambda)$ and then minimize the left hand side over λ.

But then,

$$\begin{aligned} \Delta^2(f) + \Delta^2(\widehat{f}) &= \langle Hf, f \rangle \\ &= \sum_{k=0}^{+\infty} \frac{2k+1}{2\pi} |\langle f, h_k \rangle|^2 \\ &\geq \sum_{k=0}^{+\infty} \frac{2\times 0+1}{2\pi} |\langle f, h_k \rangle|^2 = \frac{1}{2\pi} \|f\|^2. \end{aligned}$$

Moreover, equality can only occur if $\langle f, h_k \rangle = 0$ for all $k \neq 0$, that is, for $f = ch_0$. □

Remark : The above proof actually shows slightly more. If f is orthogonal to $h_0, \ldots, h_{n-1}$, then

$$\Delta^2(f) + \Delta^2(\widehat{f}) \geq \frac{2n+1}{2\pi} \|f\|^2$$

with equality if and only if $f = ch_n$. This shows that the elements of the Hermite Basis are succesive optimizers of Heisenberg's Uncertainty Principle.

We would like to stress that one of the strength of Heisenberg's Uncertainty Principle is its optimality and the fact that we know the optimizers. There are various generalizations of this principle for which the optimal constant is not known. Let us cite the following:

Theorem 2.1 (Cowling-Price [CP]). *Let $p, q \in [1, \infty]$ and $a, b > 0$. There exists a constant K such that*

$$\left\| |x|^a f \right\|_p + \left\| |\xi|^b \widehat{f} \right\|_q \geq K \|f\|_2 \tag{2.1}$$

for all $f \in L^2(\mathbb{R})$ if and only if

$$a > \frac{1}{2} - \frac{1}{p} \quad \text{and} \quad b > \frac{1}{2} - \frac{1}{q}.$$

If this is the case, let γ be given by

$$\gamma \left(a - \frac{1}{2} + \frac{1}{p} \right) = (1 - \gamma) \left(b - \frac{1}{2} + \frac{1}{q} \right)$$

then there exists a constant K such that

$$\left\| |x|^a f \right\|_p^{\gamma} \left\| |\xi|^b \widehat{f} \right\|_q^{1-\gamma} \geq K \|f\|_2.$$

The necessary condition is easily obtained with the standard scaling argument. A proof of a particular case of this theorem will be given in Section 3.3.

2.3. **A quantitative form of Shapiro's Mean-Dispersion Theorem.**
In an unpublished note, H. Shapiro [Sh1, Sh2] asked what conditions should be put on sequences (a_k), (b_k), (c_k), (d_k) for the existence of an orthonormal basis (e_k) of $L^2(\mathbb{R})$ such that

$$\mu(e_k) = a_k, \ \mu(\widehat{e_k}) = b_k, \ \Delta(e_k) = c_k, \ \Delta(\widehat{e_k}) = d_k.$$

Further he proved, using a compactness argument, that an orthonormal sequence such that all four $\mu(e_k)$, $\mu(\widehat{e_k})$, $\Delta(e_k)$, $\Delta(\widehat{e_k})$ are bounded is necessarily finite. These results were subsequently improved by A. Powell [Po] who proved that there is no orthonormal basis for which $\mu(e_k)$, $\Delta(e_k)$, $\Delta(\widehat{e_k})$ are bounded and modified a construction of Bourgain to get an orthonormal basis for which $\mu(e_k)$, $\mu(\widehat{e_k})$, $\Delta(e_k)$ are bounded.

The question was motivated by the developement of Gabor and wavelet analysis. More precisely:

(1) A *Gabor basis* is a basis $\{g_{k,l}\}_{k,l \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ of the form $g_{k,l}(x) = e^{2i\pi kx} g(x - l)$. It follows that $\mu(g_{k,l}) \to \infty$ when $l \to \infty$ and $\mu(\widehat{g_{k,l}}) \to \infty$ when $k \to \infty$. Moreover, a theorem of Balian [Ba] and Low [Lo] asserts that if $(g_{k,l})_{k,l}$ is an orthonormal basis then $\Delta(g)\Delta(\widehat{g}) = +\infty$.

(2) A *Wavelet* is a basis $\{g_{k,l}\}_{k,l \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ of the form $g_{k,l}(x) = 2^{k/2} g(2^k x - l)$. Again $\mu(g_{k,l}) \to \infty$ when $l \to \infty$, $\Delta(g_{k,l}) \to \infty$ when $k \to -\infty$ and $\Delta(\widehat{g_{k,l}}) \to \infty$ when $k \to +\infty$.

We will now show the following quantitative form of Shapiro's result:

Theorem (Jaming-Powell [JP]**).** *Let $\{e_k\}_{k \geq 0}$ be an orthonormal sequence in $L^2(\mathbb{R})$. Then for all $n \geq 0$,*

$$\sum_{k=0}^{n} \left(\Delta^2(e_k) + \Delta^2(\widehat{e_k}) + |\mu(e_k)|^2 + |\mu(\widehat{e_k})|^2 \right) \geq \frac{(n+1)^2}{2\pi}.$$

If equality holds for all $n \leq n_0$, then for $k = 0, \ldots, n_0$, $e_k = c_k h_k$, $|c_k| = 1$.

A similar result also extends to Riesz basis that are near enough to an orthogonal one by other techniques similar to those used in Section 4.2. However, the growth of the left hand side is only in $O(n)$.

As a corollary, we immediatly get the following:

Corollary (Jaming-Powell [JP]**).** *Let $\{e_k\}_{k\geq 0}$ be an orthonormal sequence in $L^2(\mathbb{R})$ such that $\Delta(e_k)$, $\Delta(\widehat{e_k})$, $|\mu(e_k)|$, $|\mu(\widehat{e_k})|$ are all $\leq C$ then the sequence has at most $8\pi C^2$ elements.*

In the case of orthonormal sequences, the proof is a direct application of the Rayleigh-Ritz technique which we now recall:

Theorem (Rayleigh-Ritz). *Let H be a positive self-adjoint operator with domain $D(H)$. Define*

$$\lambda_k(H) = \sup_{\varphi_0,\cdots,\varphi_{k-1}} \inf_{\psi\in[\varphi_0,\cdots,\varphi_{k-1}]^\perp, ||\psi||=1, \psi\in D(H)} \langle H\psi,\psi\rangle.$$

Let V be a $n+1$ dimensional subspace $V\subset D(H)$ and let P_V be the orthogonal projection on V. Define $H_V = P_V H P_V$, and consider H_V as an operator $\widetilde{H_V}: V\to V$. Let $\mu_0\leq\mu_1\leq\cdots\leq\mu_n$ be the eigenvalues of $\widetilde{H_V}$. Then

$$\lambda_k(H)\leq\mu_k,\quad k=0,\cdots,n.$$

Taking the trace of $\widetilde{H_V}$, we then get that, for (φ_k) orthonormal and in $D(H)$,

$$\sum_{k=0}^n \lambda_k(H)\leq\sum_{k=0}^n\langle H\varphi_k,\varphi_k\rangle.$$

If one of the φ_k's is not in $D(H)$, then the right-hand side is infinite and this is trivial. To conclude the proof of the theorem, it is thus enough to take H the Hermite operator and $\varphi_k = e_k$. □

3. Qualitative uncertainty principles

3.1. **Annihilating pairs.**

Notation: We will use the following notations. All sets considered will be measurable and the measure of a set S will be denoted by $|S|$. We denote by χ_S the characteristic function of S, that is $\chi_S(x)=1$ if $x\in S$ and $\chi_S(x)=0$ otherwise.

A way of measuring the concentration of a function in the time-frequency plane is by asking that the function and its Fourier Transform have both small support. For instance, it is well known that a function and its Fourier Transform can not both have compact support. Indeed if a non zero function has compact support, then its Fourier Transform is analytic and thus can not have compact support. This leads naturally to the following property:

Definition. *Let S,Σ be two measurable subsets of $\mathbb{R}^d$. Then*

— *(S,Σ) is a (weak)* annihilating pair *if,*

$$supp\, f\subset S\quad and\quad supp\,\widehat{f}\subset\Sigma$$

implies $f=0$.

— *(S,Σ) is called a* strong annihilating pair *if there exists $C=C(S,\Sigma)$ such that*

$$\|f\|_{L^2(\mathbb{R}^d)}\leq C\big(\|f\|_{L^2(\mathbb{R}^d\setminus S)}+\|\widehat{f}\|_{L^2(\mathbb{R}^d\setminus\Sigma)}\big). \tag{3.2}$$

To prove that a pair (S, Σ) is a strong annihilating pair, one usually shows that there exists a constant $D = D(S, \Sigma)$ such that, for all functions $f \in L^2(\mathbb{R}^d)$ whose Fourier Transform is supported in Σ,

$$\|f\|_{L^2(\mathbb{R}^d \setminus \Sigma)} \geq D \|f\|_{L^2(\mathbb{R}^d)}. \tag{3.3}$$

Moreover, the best constants are related by $\frac{1}{D(S,\Sigma)} \leq C(S, \Sigma) \leq \frac{1}{D(S,\Sigma)} + 1$.

As seen as above, if S, Σ are compact, then they form an annihilating pair. More generally, if S, Σ are sets of finite measure, then they form an annihilating pair. This is a particular case of the following:

Theorem (Benedicks [Be]**).** *Assume that, for almost every $x \in (0,1)$, the set $(x + \mathbb{Z}) \cap S$ is finite and, for almost every $\xi \in (0,1)$, the set $(\xi + \mathbb{Z}) \cap \Sigma$ is finite. Then the pair (S, Σ) is a weak annihilating pair.*

Proof. The main tool is the Poisson Summation Formula, which implies that

$$e^{-2i\pi x\xi} \sum_{j \in \mathbb{Z}} f(x + j) e^{2i\pi j\xi} = \sum_{k \in \mathbb{Z}} \widehat{f}(\xi + k) e^{2i\pi kx}. \tag{3.4}$$

Assume first that there exists a set $E \subset (0,1)$ of positive measure such that, for all $x \in E$, the set $(x + \mathbb{Z}) \cap S$ is empty. Then the left hand side of (3.4) vanishes on E. Since the right hand side is a trigonometric polynomial in the x variable, for almost every ξ, it is identically 0. In particular

$$\|f\|_2^2 = \int_0^1 \sum_{k \in \mathbb{Z}} |\widehat{f}(\xi + k)|^2 = 0$$

and the conclusion follows in this case.

Let us now prove the general case. It is easy to find a set $E \subset (0,1)$ of positive measure and an integer $N > 0$ such that $(x + N\mathbb{Z}) \cap S$ is empty. We will use the previous case after a change of scale. We are thus lead to proving that, for almost all $\xi \in (0, N^{-1})$, the set $(\xi + N^{-1}\mathbb{Z}) \cap \Sigma$ is finite, which is done by writing it as a finite union of sets $(\xi_j + \mathbb{Z}) \cap \Sigma$ where $\xi_j = \xi + \frac{j}{N}$, $j = 0, \ldots, N$. □

It has been proved by Amrein-Berthier [AB] that pairs of sets of finite measure are actually strongly annihilating. As noticed in [BD], this can be deduced from Benedick's theorem.

Theorem (Amrein-Berthier [AB]**).** *Let S, Σ be sets of finite measure, then (S, Σ) is a strong annihilating pair.*

Proof. Assume there is no such constant C. Then there exists a sequence $f_n \in L^2(\mathbb{R})$ of norm 1 and with spectrum Σ such that $f_n \chi_{\mathbb{R} \setminus S}$ converges to 0. Moreover, we may assume that $\widehat{f_n}$ is weakly convergent in $L^2(\mathbb{R})$ with some limit $\widehat{f}$. As $f_n(x)$ is the scalar product of $e^{2i\pi x\xi}\chi_\Sigma$ and $\widehat{f_n}$, it follows that f_n converges to f. Finally, as $|f_n|$ is bounded by $|\Sigma|^{1/2}$, we may apply Lebesgue's Theorem, thus $f_n \chi_S$ converges to f in $L^2(\mathbb{R})$ and the limit f has norm 1. But the function f has support in S and spectrum in Σ so by Benedick's Theorem, it is 0, a contradiction. □

It is much more difficult to have an estimate of the constant in Amrein-Berthier's Theorem. Let us mention a theorem of Nazarov [Na] which states that one can take $C = c_0 e^{c_1 |S||\Sigma|}$. The same problem in higher dimension is not yet solved in a completely satisfactory way.

For particular Σ, the range of possible S may be widened. Let us mention the following theorem, which improves estimates of Logvinenko-Sereda [LS] and Paneah [Pa1, Pa2].

Definition. *A set E is γ-thick at scale $a > 1$ if, for all $x \in \mathbb{R}$,*

$$|E \cap [x-a, x+a]| \geq 2\gamma a.$$

Theorem (Kovrizhkin [Ko]**).** *There exists a constant C such that, for all functions $f \in L^2(\mathbb{R})$ with spectrum in $[-1,1]$ and for every set E which is γ-thick at scale $a > 1$, one has the inequality*

$$\|f\|_2^2 \leq \left(\frac{C}{\gamma}\right)^{Ca} \int_E |f(x)|^2 \mathrm{d}x.$$

3.2. **Prolate spheroidal wave functions.** A closely related question is that of concentration. This has been extensively studied by Landau, Pollak and Slepian [LP1, LP2, SP, S1], see also [S2]. More precisely, they studied the set of all functions $f \in L^2(\mathbb{R})$ such that, for some fixed $T, \Omega, \varepsilon > 0$,

$$\int_{|t|>T} |f(t)|^2 \mathrm{d}t \leq \varepsilon^2 \|f\|_2^2 \quad , \quad \int_{|\xi|>T} |\widehat{f}(\xi)|^2 \mathrm{d}\xi \leq \varepsilon^2 \|f\|_2^2. \tag{3.5}$$

In particular, they proved that there is an orthonormal basis $\{\psi_n\}_{n\geq 0}$ of the space

$$PW_\Omega = \{f \in L^2(\mathbb{R}) \,:\, \operatorname{supp} \widehat{f} \subset [-\Omega, \Omega]\}$$

formed of eigenfunctions of the differential operator:

$$L = (T^2 - x^2)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 - 2x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\Omega^2}{T^2}x^2$$

such that the following theorem holds:

Theorem (Landau-Pollak [LP2]**)** *Let $d = \lfloor 4T\Omega \rfloor + 1$. Then, for every $f \in L^2(\mathbb{R})$ such that* (3.5) *holds*

$$\|f - \mathbb{P}_d f\|_{L^2(\mathbb{R})}^2 \leq 49\varepsilon^2 \|f\|_2^2,$$

where $\mathbb{P}_d$ is the orthogonal projection on the span of $\psi_0, \ldots, \psi_{d-1}$.

The functions ψ_k are called *prolate spheroidal wave functions* and are linked to the wave equation in "prolate spheroidal" coordinates.

This theorem is intimately linked to the Shannon Sampling Theorem. Its heuristics is the following. Assume that $f \in PW_\Omega$ then, according to Shannon's sampling theorem, f can be reconstructed from its samples in the following way :

$$f(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{2\Omega}\right) \operatorname{sinc} \pi\left(x - \frac{k}{2\Omega}\right).$$

But now, if f is essentially 0 outside $[-T, T]$, then f is essentially reconstructed from its $4T\Omega$ samples at $\{\frac{k}{2\Omega}, -2T\Omega \leq k \leq 2T\Omega\}$. That is, the space of functions that are almost supported in $[-T, T]$ and have spectrum in $[-\Omega, \Omega]$ has dimension $d \simeq 4T\Omega$. The above theorem gives a precise formulation of this.

3.3. **Local uncertainty principles.** The following theorem is due to Faris [Fa] when $\alpha = 1$ and Price and Sitaram [Pr, PS] in the general case.

Theorem. *If $0 < \alpha < d/2$, there is a constant $K = K(\alpha, d)$ such that for all $f \in L^2(\mathbb{R}^d)$ and all measurable set $E \subset \mathbb{R}^d$,*

$$\int_E |\widehat{f}(\xi)|^2 \mathrm{d}\xi \le K|E|^{2\alpha/d} \||x|^\alpha f\|_2^2. \tag{3.6}$$

Proof. Let χ_r be the characteristic function of the ball $\{x : |x| < r\}$ and $\tilde{\chi}_r = 1 - \chi_r$. Then, for $r > 0$, write

$$\left(\int_E |\widehat{f}(\xi)|^2 \mathrm{d}\xi\right)^{1/2} = \|\widehat{f}\chi_E\|_2 \le \|\widehat{f\chi_r}\chi_E\|_2 + \|\widehat{f\tilde{\chi}_r}\chi_E\|_2 \le |E|^{1/2}\|\widehat{f\chi_r}\|_\infty + \|\widehat{f\tilde{\chi}_r}\|_2.$$

Now

$$\|\widehat{f\chi_r}\|_\infty \le \|f\chi_r\|_1 \le \| \, |x|^{-\alpha}\chi_r \, \|_2 \| \, |x|^\alpha f \, \|_2 \le C_\alpha r^{d/2-\alpha} \| \, |x|^\alpha f \, \|_2$$

and

$$\|\widehat{f\tilde{\chi}_r}\|_2 \le \| \, |x|^{-\alpha}\tilde{\chi}_r \, \|_\infty \| \, |x|^\alpha f \, \|_2 \le r^{-\alpha} \| \, |x|^\alpha f \, \|_2,$$

so

$$\left(\int_E |\widehat{f}(\xi)|^2 \mathrm{d}\xi\right)^{1/2} \le \left(C_\alpha |E|^{1/2} r^{d/2-\alpha} + r^{-\alpha}\right) \| \, |x|^\alpha f \, \|_2.$$

The desired result is obtained by minimizing the right hand side of that inequality over $r > 0$. □

An easy computation shows that this proof gives

$$K = \frac{(d+2\alpha)^2}{(2\alpha)^{4\alpha/d}} (d-2\alpha)^{2\alpha/d-2}.$$

A more interesting remark is the following. First, note that if we exchange f and $\widehat{f}$ in (3.6), then the right hand side becomes, $\left\|\Delta^{\alpha/2} f\right\|_2$, using Parseval. We thus get that

$$\sup_{E \subset \mathbb{R}^d, \ 0<|E|<\infty} \frac{1}{|E|^{\alpha/d}} \left(\int_E |f(x)|^2 \mathrm{d}x\right)^{1/2} \le K \left\|\Delta^{\alpha/2} f\right\|_2.$$

The left hand side is known to be an equivalent norm of the Lorentz-space $L^{p_\alpha,\infty}$ where $p_\alpha = \dfrac{2d}{d-2\alpha}$:

$$\left\|f\right\|_{L^{p_\alpha,\infty}} \le K \left\|\Delta^{\alpha/2} f\right\|_2 \quad \text{for all } f \in L^2(\mathbb{R}).$$

Faris also showed that local uncertainty principles are stronger than global ones of the Heisenberg type. For instance, for $0 < \alpha < \frac{d}{2}$ and $\beta > 0$,

$$\begin{aligned} \|f\|_2^2 = \|\widehat{f}\|_2^2 &= \int_{|\xi|<r} |\widehat{f}(\xi)|^2 \mathrm{d}\xi + \int_{|\xi|\ge r} |\widehat{f}(\xi)|^2 \mathrm{d}\xi \\ &\le K_\alpha r^{2\alpha} \left\||x|^\alpha f\right\|_2^2 + r^{-2\beta} \left\||\xi|^\beta \widehat{f}\right\|_2^2. \end{aligned}$$

Choosing r so as too minimize the left-hand side, we obtain

$$\left\|f\right\|_2^2 \le K_{\alpha,\beta} \left\||x|^\alpha f\right\|_2 \left\||\xi|^\beta \widehat{f}\,\right\|_2.$$

Nevertheless, the constant $K_{\alpha,\beta}$ obtained this way is likely not to be the optimal one. Indeed, it is not for $\alpha = \beta = 1$ when the equality case is known, and in other cases, the optimal constant is unknown.

Let us slightly modify the argument: for $0 < \alpha < d$, $\beta > 0$ and $f \in L^2(\mathbb{R}^d)$,

$$\begin{aligned}\left\langle (\Delta^\beta + |x|^\alpha) f, f \right\rangle &= \left\langle \Delta^\beta f, f \right\rangle + \||x|^{\alpha/2} f\|_2^2 \\ &\geq \||\xi|^{\beta/2} \widehat{f}\|_2^2 + \frac{1}{K} \|\widehat{f}\|_{L^2(|\xi|\leq 1)}^2 \geq \frac{1}{K} \|f\|_2^2\end{aligned}$$

where we have used Parseval's Indentity and Faris' Theorem. This shows that the operator $\Delta^\beta + |x|^\alpha$ has a spectral gap, the precise value of which is not known.

4. Fast decrease conditions

We will now measure concentration by asking that a function has fast decrease away from its mean. A typical very concentrated function is the gaussian $e^{-\pi a x^2}$ and the parameter a measures concentration.

4.1. **Hardy's uncertainty principle.**

Theorem (Hardy [Ha]**).** *Let $f \in L^2(\mathbb{R})$. Assume that*

(1) *for almost all $x \in \mathbb{R}$, $|f(x)| \leq C(1+|x|)^N e^{-\pi a |x|^2}$,*
(2) *for almost all $x \in \mathbb{R}$, $|\widehat{f}(\xi)| \leq C(1+|\xi|)^N e^{-\pi b |\xi|^2}$.*

Then, if $ab > 1$, $f = 0$ and if $ab = 1$ $f(x) = P(x) e^{-\pi a |x|^2}$, P polynomial of degree at most N.

Note that this theorem may be interpreted as follows: the space of functions satisfying conditions (1) and (2) of Hardy's Theorem is $N+1$-dimensional. There are various generalizations of this theorem. The following is an improvement of a result of Beurling whose proof was lost until a new prove was given by Hörmander. Hörmander's result was only in dimension 1 and with $N = 0$ so that it did not cover the equality case in Hardy's Theorem.

Theorem (Bonami-Jaming-Demange [BDJ]**).** *Let $f \in L^2(\mathbb{R}^d)$. Assume that*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)||\widehat{f}(\xi)| e^{2\pi |\langle x,\xi \rangle|} \frac{dx d\xi}{(1+|x|+|\xi|)^N}$$

then $f = P(x) e^{-\langle Ax, x\rangle}$, where A is positive definite and P is a polynomial of degree $< \frac{N-d}{2}$.

This theorem was further generalized to distributions by B. Demange. Its only disadvantage is that it does not give information on the degree of the polynomial, but in practice, this can be easily obtained.

Theorem (Demange). *Let $f \in \mathcal{S}'(\mathbb{R}^d)$. Assume that $e^{\pi|x|^2} f \in \mathcal{S}'$ and $e^{\pi|\xi|^2} \widehat{f} \in \mathcal{S}'$ then $f = P(x) e^{-\langle Ax, x\rangle}$, where A is positive definite and P is a polynomial.*

Sketch of proof for $d = 1$. The key tool here is to use the Bargmann Transform which transforms the distribution f into an entire function of order 2 via

$$F(z) = \mathcal{B}(f)(z) := e^{\frac{\pi}{2} z^2} \left\langle f(x), e^{-\pi(x-z)^2} \right\rangle.$$

Using the fact that a Schwartz distribution has finite order, we get from $e^{\pi|x|^2} f \in \mathcal{S}'$ that there exists C, N such that

$$|F(z)| \leq C(1+|z|)^N e^{\frac{\pi}{2}|\operatorname{Im} z|^2}$$

while, from $e^{\pi|\xi|^2}\widehat{f} \in \mathcal{S}'$ and the the fact that $F(-iz) = \mathcal{B}(f)(-iz) = \mathcal{B}(\widehat{f})(z)$, we get that

$$|F(z)| \leq C(1+|z|)^N e^{\frac{\pi}{2}|\operatorname{Re} z|^2}.$$

But then, Phragmèn-Lindelöf's Principle implies that $|F(z)| \leq C(1+|z|)^N$ which, with Liouville's Theorem, implies that F is a polynomial. Inverting the Bargmann Transform then shows that f is of the desired form. □

4.2. **The umbrella theorem.** We will now prove that the elements of an orthonormal basis and their Fourier Transforms can not all be bounded by fixed functions:

Theorem (Jaming-Powell). *Let φ, ψ be two* fixed *functions in $L^2(\mathbb{R})$. Then there exists $N = N(\varphi, \psi)$ such that, if $(e_k)_{k\in I} \subset L^2(\mathbb{R})$ is orthonormal and*

$$|e_k| \leq \varphi \quad , \quad |\widehat{e_k}| \leq \psi$$

then $\#I \leq N$.

This theorem generalizes an earlier result of H. Shapiro [Sh1, Sh2] that showed, using a compactness argument, that the sequence is finite, but with no quantitative estimate.

Proof. Let $M = \max(\|\varphi\|, \|\psi\|)$, and $0 < \varepsilon < \frac{1}{50M}$. Define

$$C_\varphi(\varepsilon) = \inf\left\{ T \in \mathbb{R} \,:\, \int_{|t|>T} |\varphi(t)|^2 \leq \varepsilon^2 \|\varphi\|_2^2 \right\}.$$

Define $C_\psi(\varepsilon)$ accordingly. For $T > \max\big(C_\varphi(\varepsilon), C_\psi(\varepsilon)\big)$, for all $n \in I$,

$$\int_{|t|>T} |e_n(t)|^2 \mathrm{d}t \leq \varepsilon^2 \|\varphi\|_2^2 \quad , \quad \int_{|\xi|>T} |\widehat{e_n}(\xi)|^2 \mathrm{d}\xi \leq \varepsilon^2 \|\psi\|_2^2.$$

Let $d = \lfloor 4T^2 \rfloor + 1$, $\eta = 7M\varepsilon$ and let ψ_n be the associated prolate spheroïdals,

$$\text{for } n \in I, \quad \|e_n - \mathbb{P}_d e_n\| \leq \eta.$$

Now write $\mathbb{P}_d e_n = \sum_{k=0}^{d-1} a_{n,k+1}\psi_k$ *i.e.* $a_n := (a_{n,k}) \in \mathbb{C}^d$. Then $1 - \eta \leq \|a_{n,k}\| = \|\mathbb{P}_d e_n\| \leq 1$ and

$$\begin{aligned}
\langle a_{n,k}, a_{m,k}\rangle &= \langle \mathbb{P}_d e_n, \mathbb{P}_d e_m\rangle \\
&= \langle \mathbb{P}_d e_n - e_n + e_n, \mathbb{P}_d e_m - e_m + e_m\rangle \\
&= \langle \mathbb{P}_d e_n - e_n, \mathbb{P}_d e_m - e_m\rangle + \langle e_n, \mathbb{P}_d e_m - e_m\rangle + \langle \mathbb{P}_d e_n - e_n, e_m\rangle \\
&= \langle \mathbb{P}_d e_n - e_n, \mathbb{P}_d e_m - e_m\rangle + \langle e_n - \mathbb{P}_d e_n, \mathbb{P}_d e_m - e_m\rangle + \langle \mathbb{P}_d e_n - e_n, e_m - \mathbb{P}_d e_m\rangle \\
&= \langle \mathbb{P}_d e_n - e_n, e_m - \mathbb{P}_d e_m\rangle \leq \eta^2.
\end{aligned}$$

So $b_n = \dfrac{a_n}{\|a_n\|}$ are vectors on the unit sphere such that the angles $|\langle b_n, b_m\rangle| \leq \frac{\eta^2}{1-\eta^2}$. A set of such vectors is called a *spherical code* in $\mathbb{C}^d$ and is known to be finite. □

There are various bounds on the number of spherical codes. For instance, one may identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$ in the standard way. One then gets that if $|\langle b_n, b_m\rangle_{\mathbb{C}^d}| \leq \alpha$ then $\langle b_n, b_m\rangle_{\mathbb{R}^{2d}} \in [-\alpha, \alpha]$ i.e. (b_n) is a $[-\alpha,\alpha]$-*spherical code* in $\mathbb{R}^{2d}$. If we call $N^{2d}(\alpha)$ the maximal number of elements of a $[-\alpha,\alpha]$-spherical codes in $\mathbb{R}^{2d}$, then the following bounds can be obtained

— if $\alpha < \frac{1}{2d}$ then $N^{2d}(\alpha) = 2d$.
This is a trivial bound as the code is then linearly independent.

— $N^{2d}(\alpha) \leq \left(\frac{2-\alpha}{1-\alpha}\right)^{2d}$.
This can be obtained by a volume counting argument.

— if $\alpha < \frac{1}{\sqrt{2d}}$ then $N^{2d}(\alpha) \leq 2\frac{1-\alpha^2}{1-2\alpha^2 d}d$.
This has been obtained by Delsarte [DGS].

If φ, ψ are explicitely given, one may then optimize the choice of T and ε in the previous proof to get more explicit estimates on $\#I$.
Example :

(1) Take $\varphi = \psi = Ce^{-\pi a x^2}$ then

$$\#I \leq 2 + \frac{2}{\pi a}\max\left(\ln\frac{C}{a^{7/2}}, \ln\frac{9C^2}{a}\right).$$

In this case, only the trivial bound is needed.

(2) Take $\varphi = \psi = \frac{C}{(1+|x|)^p}$ then, using the volume counting bound, one gets

$$\#I \leq N \leq 3^{8\left(\frac{20C}{\sqrt{2p-1}}\right)^{\frac{4}{2p-1}}}.$$

Moreover,

$$\#I \leq \begin{cases} 4\left(\frac{800C^2}{2p-1}\right)^{\frac{2}{2p-3}} & \text{if } p > 3/2 \\ 32\,(20C)^{\frac{2}{2p-3}} & \text{if } 1 < p \leq 3/2 \end{cases}$$

where we used the trivial bound for the first estimate and Delsarte-Goethals-Siedel's bound for the second.

5. Reformulation of uncertainty principles as properties of solutions of some classical PDE's

This section is motivated by recent results on unique continuation in [EKV, EKPV]. There, the qualitative properties of solutions of some partial differential (in)equation where explicitely inspired by uncertainty principles.

5.1. **The Free Heat Equation.**

Let us recall that the solution of the Free Heat Equation

$$\begin{cases} -\partial_t v + \frac{1}{4\pi}\partial_x^2 v = 0 \\ v(x,0) = v_0(x) \end{cases} \tag{5.7}$$

with initial data $v_0 \in L^2(\mathbb{R}) \cup L^1(\mathbb{R})$ has solution

$$v(x,t) = \int_{\mathbb{R}} e^{-\pi\xi^2 t + 2i\pi x\xi}\widehat{v_0}(\xi)\mathrm{d}\xi = \left(e^{-\pi\xi^2 t}\widehat{v_0}\right)\check{}.$$

We may then reformulate some uncertainty principles as follows.

Hardy's Uncertainty Principle. *Let $v_0 \in L^1(\mathbb{R})$ be non zero and let v be a solution of (5.7) with initial data v_0. Then for every $t > 0$ and for every $N > 0$, $x \mapsto \dfrac{e^{\pi x^2/t}}{(1+|x|)^N} v(x,t)$ is unbounded.*

Indeed, as $e^{-\pi\xi^2 t}|\widehat{v_0}(\xi)| \leq \|v_0\|_{L^1(\mathbb{R})} e^{-\pi\xi^2 t}$, if $|v(x,t)| \leq C(1+|x|)^N e^{-\pi x^2/t}$ then, from Hardy's Theorem, $e^{-\pi\xi^2 t}|\widehat{v_0}(\xi)| = Ce^{-\pi\xi^2 t}$, a contradiction.

Note that Demange's Theorem shows that, if $v_0 \in \mathcal{S}'(\mathbb{R})$ and if $x \mapsto e^{\pi x^2/t} v(x,t) \in \mathcal{S}'(\mathbb{R})$ then v_0 is a Hermite Function, an so is then v.

Annihilating pairs. Let $v_0 \in L^2(\mathbb{R})$ and let v be a solution of (5.8) with initial data v_0. As $x \mapsto v(x,t)$ is analytic, if it is supported in some set S such that $\mathbb{R} \setminus S$ has positive measure, then $v = 0$ thus $v_0 = 0$ *i.e.* the heat equation has the weak unique continuation property.

However, a little bit more can be said in the case (S, Σ) is a strong annihilating pair with constant $D(S,\Sigma)$ in (3.3). If $\widehat{v_0}$ is supported in Σ, then, for all $t > 0$,

$$\|v(x,t)\|_{L^2(\mathbb{R}\setminus S)} \geq D\|v(x,t)\|_{L^2(\mathbb{R})} = D(S,\Sigma)\left\|e^{-\pi\xi^2 t}\widehat{v_0}(\xi)\right\|_{L^2(\Sigma)}.$$

For instance, if $\Sigma = [-a, a]$, we thus get the lower bound

$$\|v(x,t)\|_{L^2(\mathbb{R}\setminus S)} \geq D(S,\Sigma)e^{-\pi a^2 t}\|v_0\|_{L^2(\mathbb{R})}.$$

5.2. **The Free Shrödinger Equation.** Let us recall that the solution of the Free Shrödinger Equation

$$\begin{cases} i\partial_t v + \dfrac{1}{4\pi}\partial_x^2 v = 0 \\ v(x,0) = v_0(x) \end{cases} \tag{5.8}$$

with initial data $v_0 \in L^2(\mathbb{R})$ has solution

$$v(x,t) = \int_{\mathbb{R}} e^{-i\pi\xi^2 t + 2i\pi x\xi}\widehat{v_0}(\xi)\mathrm{d}\xi = \left(e^{-i\pi\xi^2 t}\widehat{v_0}\right)^{\check{}}.$$

We may then reformulate some uncertainty principles as follows.

Hardy's Uncertainty Principle. *Let $v_0 \in L^2(\mathbb{R})$ and assume that $|\widehat{v_0}(\xi)| \leq Ce^{-\pi a\xi^2}$. Let $v(x,t)$ is a solution of (5.8) with initial data v_0 and assume that, for some $t > 0$, there exists $c = c(t)$ and $b = b(t)$ such that*

$$|v(x,t)| \leq ce^{-\pi b\xi^2}.$$

If $b > 1/a$, then $v_0 = 0$. If $b = 1/a$, then $v_0(x) = Ce^{-\pi x^2/(a-it_0)}$. for some $C \in \mathbb{C}$.

The first assertion is a direct consequence of Hardy's theorem. For the second, Hardy's theorem implies that $v(x,t_0) = ce^{-\pi\xi^2/a}$, thus $\widehat{v_0}(\xi) = ce^{i\pi\xi^2 t_0}e^{-\pi a\xi^2} = ce^{-\pi(a-it_0)\xi^2}$. Inverting the Fourier transform, we get the assertion. Note that, in this case

$$|v(x,t)| = \frac{C}{\sqrt{a^2+(t-t_0)^2}}\exp\left(-\frac{\pi a}{a^2+(t-t_0)^2}x^2\right).$$

Annihilating pairs. Let (S, Σ) be an annihilating pair. Let $v_0 \in L^2(\mathbb{R})$ and assume that $\widehat{v_0}$ is supported in Σ. Let v be a solution of (5.8) with initial data v_0 and assume that, for some $t > 0$, $v(\cdot, t)$ is supported in S, then $v_0 = 0$.

Moreover, if (S, Σ) is a strong annihilating pair with constant $D(S, \Sigma)$ in (3.3), then, for all $t > 0$,

$$\|v(x,t)\|_{L^2(\mathbb{R}\setminus S)} \geq D(S,\Sigma)\|v(x,t)\|_{L^2(\mathbb{R})} = D(S,\Sigma)\|v_0\|_{L^2(\mathbb{R})}.$$

For instance, if $\Sigma = [-1, 1]$ and E is γ-thick at scale $a > 1$, then

$$\|v(x,t)\|_{L^2(E)} \geq \left(\frac{\gamma}{C}\right)^{Ca} \|v_0\|_{L^2(\mathbb{R})}.$$

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