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Igor Rodnianski

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U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

DECAY OF A LINEAR SCALAR FIELD ON SCHWARZSCHILD SPACE-TIME

IGOR RODNIANSKI

1. Result

In this paper we describe a result obtained in a joint work with M. Dafermos (University of Cambridge), see [6]. The problem concerns the long term behavior of solutions of the linear wave equation

$$\Box_g \phi = 0$$

in the domain of outer communication of the Schwarzschild space-time $(\mathcal{M}, g)$. In this region ($r \geq 2M$) the metric g can be written in the form

$$g = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\sigma_{\mathbb{S}^2}$$

relative to the coordinates (t, r) or, alternatively, in the form

$$g = -4(1 - \mu)dudv + r^2(u, v)d\sigma_{\mathbb{S}^2}$$

relative to the null coordinates (u, v), which can be chosen in such a way[1] that $\mu = 2M/r$. The parameter M is the ADM mass of the Schwarzschild space-time and metric $d\sigma_{\mathbb{S}^2}$ is the standard metric on $\mathbb{S}^2$. The manifold $(\mathcal{M}, g)$ is spherically symmetric, i.e., the group $SO(3)$ acts by isometry and $4\pi r^2$ is the area of an orbit. The vectorfield $\frac{\partial}{\partial t}$, defined in the exterior region relative to the coordinates (r, t), is time-like Killing. The wave equation for a scalar field takes the following explicit form relative to the (r, t) coordinates:

$$\Box_g \phi = -(1 - \mu)^{-1}\partial_t^2 \phi + \frac{1}{r^2}\partial_r \left(r^2(1 - \mu)\partial_r \phi\right) + \Delta\!\!\!/ \phi = 0.$$

Our main result is the following

Theorem 1.1 ([6]). *Let ϕ be a sufficiently regular solution of the wave equation*

$$\Box_g \phi = 0 \tag{1}$$

on the (maximally extended) Schwarzschild spacetime $(\mathcal{M}, g)$, decaying suitably at spatial infinity on an arbitrary complete asymptotically flat Cauchy surface Σ. For any achronal hypersurface $\mathcal{S}$ in the closure of this region, let $F(\mathcal{S})$ denote the flux of the energy through $\mathcal{S}$, where

[1] These are the Eddington-Finkelstein coordinates $v = t + r^*$, $u = t - r^*$ with $r^* = r + 2M \log(r - 2M)$.

energy is here measured with respect to the timelike Killing vector field. Let $v_+ = \max\{v, 1\}$, $u_+ = \max\{u, 1\}$, *and* $v_+(\mathcal{S}) = \max\{\inf_{\mathcal{S}} v, 1\}$, $u_+(\mathcal{S}) = \max\{\inf_{\mathcal{S}} u, 1\}$. *We have*

$$F(\mathcal{S}) \leq C((v_+(\mathcal{S}))^{-2} + (u_+(\mathcal{S}))^{-2}). \tag{2}$$

(We also allow $\mathcal{S}$ *to be a subset of null infinity, interpreted in the obvious limiting sense.) In addition, we have the pointwise decay rates to the future*[2] *of the Cauchy hypersurface* Σ

$$\begin{aligned} |\phi| &\leq C v_+^{-1}, \quad \text{in} \quad \{r \geq 2M\}, \\ |r\phi| &\leq C_{\hat{R}}(1 + |u|)^{-\frac{1}{2}} \quad \text{in} \quad \{r \geq \hat{R} > 2M\}. \end{aligned} \tag{3}$$

Remark 1.1. *Various constants* C *appearing in the statement of the theorem above depend on the initial data for the scalar field* ϕ *prescribed on* Σ. *In a particular case where* $\Sigma = \{t = 1\}$ *these constants depend on the weighted Sobolev norms of the initial data with weights determining the rate of decay of the initial data at space-like infinity* $r = \infty$. *An example of such norm is given by the expression*

$$\int_{-\infty}^{\infty} \int_{\mathbb{S}^2} r^2 \left(u^2 (\partial_u (r^4 \nabla\!\!\!/^4 \phi))^2 + v^2 (\partial_v (r^4 \nabla\!\!\!/^4 \phi))^2 \right) dr^* \, d\sigma_{\mathbb{S}^2}.$$

Here $\nabla\!\!\!/$ *denotes covariant differentiation along the 2-dimensional sphere of radius* r. *The norms impose no assumptions on vanishing of the initial data for* ϕ *on the bifurcate sphere* $r = 2M$. *In particular, the theorem holds for any sufficiently smooth initial data vanishing for all sufficiently large* r.

Note also the decay rate (3) *holds uniformly in all of the domain of outer communication.*

Independently of us, a version of this problem is being studied in [1].

Acknowledgments: The author would like to thank Mihalis Dafermos for valuable discussions concerning this paper.

2. Previous work

Heuristics: The heuristic arguments for the decay of linear fields on Schwarzschild background were first put forward in the work of R. Price, [15], where a linearized perturbation theory on a fixed (Schwarzschild) background was used to study the problem of a non-spherical gravitation collapse. It was argued there that linear fields should develop "power tails", i.e. decay polynomially at time-like infinity (as opposed to the linear fields on Minkowski space, where the fields vanish in the neighborhood of time-like infinity). Gundlach-Price-Pullin, [8], extended the work of Price by predicting power law decay (for each fixed spherical harmonic) along null infinity and the event horizon.

[2] Similar decay rates hold to the past of Σ.

Boundedness: Uniform boundedness

$$|\phi| \le C$$

of solutions of the wave equation $\Box_g\phi = 0$ in the domain of outer communication of the Schwarzschild space-time was established by Kay-Wald, [9]. For the wave equation on Minkowski space the boundedness statement can be shown using conservation of energy and Sobolev inequalities. The existence of a time-like Killing vectorfield $\frac{\partial}{\partial t}$ in the domain of outer communication of the Schwarzschild space-time leads to a conserved quantity

$$E_\phi = \int_{\Sigma_t} \left((\partial_t\phi)^2 + (\partial_{r^*}\phi)^2 + (1-\mu)|\nabla\!\!\!/\phi|^2\right),$$

where Σ_t is the hypersurface $\{t = const\}$. Commuting the equation with Killing vectorfields: $\frac{\partial}{\partial t}$ and angular momentum $\Omega = r\nabla\!\!\!/$ allows one to generate "higher" energy conserved quantities, which would lead to the boundedness of the scalar field ϕ if it was not for a *loss of control* of the $\nabla\!\!\!/\phi$ part of the energy near $r = 2M$ arising due to the presence of the $(1-\mu)$ factor. This problem was overcome in [9] by applying the inverse of the Laplace-Beltrami operator Δ on Σ_t and using the discrete isometries of the maximally extended Schwarzschild space-time $\mathcal{M}$.

Spectral aspects of the problem: Sá Barreto-Zworski in [17] studied the spectral properties of the corresponding Hamiltonian

$$H = \frac{1-\mu}{r^2}\partial_r\left(r^2(1-\mu)\partial_r\right) + (1-\mu)\Delta\!\!\!/$$

defined as a self-adjoint operator with respect to the measure $(1-\mu)^{-1}r^2 dr d\sigma_{\mathbb{S}^2}$. They showed that the poles (quasi-normal modes) of the meromorphic continuation of the resolvent $(H-\lambda)^{-1}$ outside of a conic neighborhood of the origin and for all sufficiently $|\lambda|$ are in one-to-one corresponding with the lattice points

$$(3^{\frac{3}{2}}M)^{-1}\left(\pm\ell \pm \frac{1}{2} - \frac{1}{2}i(k+\frac{1}{2})\right), \qquad \ell = 0, 1, ..., \; k = 0, 1...$$

The structure of the poles is connected with the photosphere $r = 3M$ containing unstable closed geodesics of the metric $(1-\mu)^{-1}dr^2 + r^2 d\sigma_{\mathbb{S}^2}$ on the manifold $\{r > 2M\} \times \mathbb{S}^2$.

The connection between the behavior of a resolvent and solutions of a wave equation can be in some cases established via the Laplace transform. However even in these cases the problem of decay can not be easily translated into a resolvent question and is very sensitive to its behavior at the bottom of the spectrum $\lambda = 0$.

Scattering problem: the problem of existence and asymptotic completeness of the wave operators for a linear scalar field on Schwarzschild space-time was considered by Dimock, [7].

Spherical symmetry: The problem of decay of a linear scalar field on Schwarzschild space-time can be considered in the context of spherical symmetry, where the equation can be cast in the form

$$\partial_t^2\psi - \partial_{r^*}^2\psi + \frac{2M}{r^3}(1-\frac{2M}{r})\psi = 0, \qquad \psi = r\phi$$

relative to the Regge-Wheeler coordinates (t, r^*) with $r^* = r + 2M\log(r-2M)$. However, in these coordinates the event horizon $r = 2M$ corresponds to $t = \infty, r^* = -\infty$ and its geometry, which is crucial for the problem of decay, is much less apparent. From this point of view the use of the null Eddington-Finkelstein coordinates (u, v) is more suitable.

The decay rates stated in Theorem 1.1 for a spherically symmetric scalar field follow from a very special case of the results of Dafermos-Rodnianski in [5], where the decay question was investigated for a nonlinear problem of a *self-gravitating* spherically symmetric scalar field[3]. There, it was shown that the decay of a scalar field, both linear and nonlinear, along the event horizon parametrized by v obeys the *Price law* (see heuristics above)

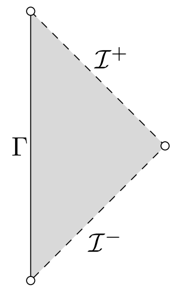
$$|\phi| \le C_\epsilon v^{-3+\epsilon}, \quad \forall\epsilon > 0.$$

For the linear problem in the spherically symmetric context some results on the decay of a scalar field were also obtained by Machedon-Stalker, [12].
Decay without a rate for solutions of the linear wave equation on Schwarzschild background was obtained by Twainy, [18].

We now quickly review the problem of decay of a linear scalar field on Minkowski space.

3. Minkowski space-time $(\mathbb{R}^{3+1}, m)$

The conformal picture of the quotient manifold $\mathbb{R}^{3+1}/SO(3)$ is given by the following diagram:



The metric m on Minkowski space-time can be written in the form

$$m = -dt^2 + \sum_{i=1}^{3}(dx^i)^2,$$

[3] The problem studied in [5] was that of coupled Einstein-Maxwell-scalar field equations in the context of spherically symmetric gravitational collapse.

or, alternatively, relatively to the null coordinates $2u = t - r$ and $2v = t + r$ in the form

$$m = -4dudv + r^2 d\sigma_{\mathbb{S}^2}.$$

The wave equation for a field ϕ reads

$$\partial_t^2 \phi - \Delta\phi = r^{-1}\left(\partial_u \partial_v (r\phi) - \not\Delta(r\phi)\right) = 0. \tag{4}$$

The long term behavior of ϕ is effectively captured by the statement that to the future of the initial hypersurface Σ_0, on which the initial data is assumed to have compact support, the function $r\phi$ has compact support[4] with respect to the null coordinate u. In some way this can be viewed as a consequence of the *strong Huygens' principle* and can be obtained using the exact form of the *fundamental solution* for (4): $\delta(t^2 - r^2)$.

While the above picture is undoubtedly accurate it is highly unstable as even small metric perturbations destroy the compact support property, which also does not survive applications to nonlinear stability problems. The answer to this problem is provided by deriving the rates of decay stable under perturbations or better yet by finding robust[5] methods for establishing such decay rates

3.1. **Robust methods for proving decay on Minkowski space.** The right approach to the problem of decay on Minkowski space is provided by the *vectorfield method* based on isometries of Minkowski space-time and their infinitesimal generators

$$\mathcal{Z} = \{\partial_\alpha, \Omega_{\alpha\beta}, S, K\},$$

where $\Omega_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha$ combines rotations and Lorentzian boosts, $S = x^\alpha \partial_\alpha$ is the scaling and $K = (t^2 + r^2)\partial_t + 2tr\partial_r$ is the Morawetz vectorfield. To be precise there are two versions of the vectorfield method: one that uses vectorfield K as a multiplier in the energy identity and goes back to the work of Morawetz on the obstacle problem, [13]. In the second approach, developed[6] by Klainerman [11], vectorfields $\Gamma = \{\partial_\alpha, \Omega_{\alpha\beta}, S\}$ are commuted through the wave equation and the decay is deduced from the energy estimates for $\Gamma\phi$ via so called *global Sobolev inequalities*.

The success of the vectorfield method is illustrated by its applications to nonlinear problems examples of which include the small data global existence result for quasilinear wave equations satisfying the null condition [11] and the proof of stability of Minkowski space [3], [10].

In the vectorfield approach the optimal decay rates for a linear scalar field on Minkowski space-time can be derived with just a Morawetz vectorfield K used as a multiplier with the energy-momentum tensor:

$$T_{\alpha\beta} = \partial_\alpha \phi \, \partial_\beta \phi - \frac{1}{2} m_{\alpha\beta} \, \partial^\mu \phi \, \partial_\mu \phi.$$

[4]In fact, the function $r\phi$ also has a limit at future null infinity as $v \to \infty$ along the outgoing null rays $u =$const.

[5]in particular independent of the exact form of the fundamental solution.

[6]as well as its combination with the multiplier method of Morawetz

One has the identity

$$\partial^\alpha \left(T_{\alpha\beta}K^\beta\right) = 2t\,\Box(\phi^2),$$

which can be integrated over the region $[0,t] \times \mathbb{R}^3$ to derived boundedness of the *conformal energy*:

$$E_K[\phi](t) = \int_{\Sigma_t} \left(v^2(\partial_v\phi)^2 + u^2(\partial_u\phi)^2 + (v^2+u^2)|\nabla\!\!\!/\phi|^2 + (1+\frac{t^2}{r^2})\phi^2\right),$$

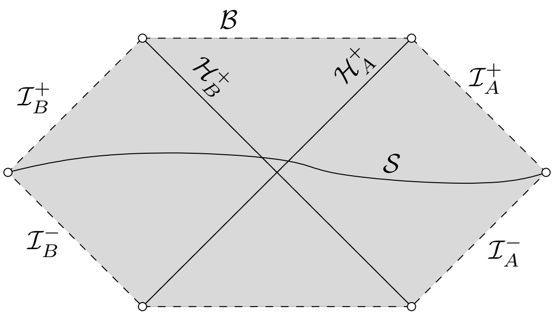
which when complemented by the boundedness of the conformal energies for the derivatives of ϕ leads to the

Decay Rate on Minkowski space-time: $|r\phi| \le Cu^{-\frac{1}{2}}$.

Note that the rate of decay stated in Theorem 1.1 in the region $r \ge R > 2M$ is precisely the same as above.

4. Schwarzschild space-time $(\mathcal{M}, g)$

We now review basic geometric properties of the Schwarzschild space-time. The conformal picture of the Lorentzian quotient $\mathcal{M}/SO(3)$ of a maximally extended Schwarzschild space-time $\mathcal{M}$ is given by the following Penrose diagram:



The two exterior regions (domains of outer communication) are formed by intersecting causal pasts of the future null infinities $\mathcal{I}_A^+$ ($\mathcal{I}_B^+$) with causal futures of the past null infinities $\mathcal{I}_A^-$ ($\mathcal{I}_B^-$) and are characterized by the values of the parameter $r > 2M$. The *black hole* region is bounded by the components of the future event horizon $\mathcal{H}_A^+$ and $\mathcal{H}_B^+$, on which $r = 2M$.

On each of the two exteriors the metric g may be written:

$$g = -(1-\mu)\,dt^2 + (1-\mu)^{-1}\,dr^2 + r^2 d\sigma_{\mathbb{S}^2} = -4(1-\mu)dudv + r^2 d\sigma_{\mathbb{S}^2},$$
$$\mu = \frac{2M}{r}, \quad v = t + r^*, \; u = t - r^*, \quad r^* = r + 2M\log(r - 2M).$$

The apparent singularity of g at $r = 2M$ is nothing else but a singularity of a particular coordinate system, i.e. (r, t). The vector field $\frac{\partial}{\partial t}$ is time-like Killing but becomes null on the event horizon: $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -(1-\mu)$.

We now restrict the attention to the right domain of outer communication A on the diagram. In the exterior region the wave equation for a linear scalar field can be written in the form

$$\Box_g \phi = -\frac{1}{1-\mu}\partial_t^2\phi + \frac{1}{r^2}\partial_r\left((1-\mu)r^2\partial_r\phi\right) + \not{\Delta}\phi = 0$$

The initial data for ϕ is prescribed on a 3-dimensional manifold $S \cap A$. It completely determines the evolution of ϕ in A provided that $S \cap A$ contains the *bifurcate sphere* $\mathcal{H}_A^+ \cap \mathcal{H}_B^+$. The point here is that unlike in the boundary value problems the event horizon $r = 2M$ is a null characteristic surface – the solution of the wave equation there is completely determined from the initial data. The initial data is assumed to be sufficiently smooth on S and vanish at a sufficiently fast rate at the asymptotically flat end. No vanishing assumption is imposed on the bifurcate sphere.

From the point of view of decay of a solution to the wave equation the two important distinct regions of the domain of outer communication A (to the future of S) are the future null infinity and the future event horizon $\mathcal{H}_A^+$. Relative to the Eddington-Finkelstein coordinates (u, v) they are characterized by (u, ∞) and (∞, v) respectively [7]. In particular, the decay of ϕ on $\mathcal{H}_A^+$ should be measured with respect to the variable v. For the same reason, unlike in Minkowski space-time where the decay of a solution of the wave equation is sometime phrased in terms of the variable t, a solution of the wave equation on Schwarzschild space-time can not decay uniformly in t: the event horizon $\mathcal{H}_A^+$ corresponds to $t = \infty$ and the decay in t would erroneously imply that ϕ has to vanish on $\mathcal{H}_A^+$.

We finally recall that $(\mathcal{M}, g)$ is a one-parameter family of spherically symmetric solutions (unique by Birkhoff's theorem) of the vacuum Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0.$$

It is also a sub-family of the 2-parameter Kerr family of solutions.

5. Motivation

While the problem the long term behavior of linear fields on Minkowski space-time has a very good quantitative description achieved by robust methods the analogous problem on space-times "far away" from Minkowski is much less understood. Among such Lorentzian manifolds a particular important class is formed by space-times arising in General Relativity, as one can expect (and indeed this is often the case) that the long term behavior of linear fields is connected with physical phenomena and that such

[7] Note the difference however: future null infinity is not part of the space-time while the event horizon is.

space-times possess “natural" geometric properties. Black hole space-times (such as Schwarzschild and Kerr families) are particularly interesting from the either point of view. Their global geometry is drastically different from Minkowski space-time while the long term behavior of a linear scalar field is (heuristically) believed to be relevant for a variety of phenomena.

Among them is a properly formulated problem of nonlinear stability of the Kerr family in a neighborhood of the Schwarzschild solution. Linearization and traditional simplifications lead to the linear problem: $\Box_g\phi = 0$ on the Schwarzschild background. In the physics literature just the boundedness of solutions of this problem is often referred to as *linear stability* of Schwarzschild, [9]. Note that that the boundedness of linear fields on a Kerr space-time is not known, see however [16].

The heuristics of Price and Price-Gundlach-Pullin for the linear problem $\Box_g\phi = 0$ on Schwarzschild (and Reissner-Nordström) space-time had been developed in an attempt to shed light on the picture of non-spherical gravitational collapse. At this level a power law decay of a linear scalar field along the event horizon becomes relevant for the problem of the internal structure of black holes.

Internal structure of black holes: the decay of external fields along the event horizon determines the amount of radiation entering a black hole at late times and thus has a direct influence on the *mass inflation* scenario proposed by Poisson and Israel, [14] and the problem of stability of Cauchy horizons (strong cosmic censorship conjecture).

The above problem is much better understood in the context of the gravitational collapse in spherical symmetry which is governed by a coupled system of Einstein-scalar field-(Maxwell)[8] equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = T_{\alpha\beta}, \tag{5}$$

where $T_{\alpha\beta}$ is the energy-momentum of a scalar field (Maxwell)

$$T_{\alpha\beta} = \partial_\alpha\phi\,\partial_\beta\phi - 1/2\,g_{\alpha\beta}, \partial^\mu\phi\,\partial_\mu\phi$$

and the scalar field satisfies the equation $\Box_g\phi = 0$ on a dynamic background with a metric g determined by the equation (5). Weak cosmic censorship for this model was established in the work of Christodoulou, [2]. His analysis had left open the problem of decay of a scalar field ϕ along the event horizon. Under assumption of a certain power law decay the problem of internal structure of black holes for the Einstein-scalar field-Maxwell equations has been rigorously understood in the work of M. Dafermos, [4]. In particular, the work confirmed the mass inflation scenario and showed a certain stability of the Cauchy horizons (in contrast with the picture suggested by the strong cosmic censorship conjecture). The required rate of decay (Price law) was then established in [5].

[8]The Maxwell field in spherical symmetry is non-dynamic. Its coupling to the Einstein equations in spherical symmetry is designed to simulate angular momentum.

The decay rates stated in Theorem 1.1 are sufficiently fast so as to suggest that the picture established in [4] may remain valid in the absense of symmetry assumptions[9].

6. Outline of the proof of Theorem 1.1

Consider a solution of the wave equation

$$\Box_g \phi = -\frac{1}{1-\mu}\partial_t^2\phi + \frac{1}{r^2}\partial_r\left((1-\mu)r^2\partial_r\phi\right) + \not\Delta\phi = 0.$$

The proof of the decay rates stated in Theorem 1.1 is based on an energy-momentum approach and several geometrically constructed vectorfields used as multipliers in the energy identities.

The energy-momentum tensor for a linear scalar field ϕ takes the form

$$T_{\alpha\beta} = \partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\,\partial^\mu\phi\,\partial_\mu\phi.$$

It has the property that contracted with a vectorfield $V = V^\alpha\partial_\alpha$

$$D^\alpha\left(T_{\alpha\beta}V^\beta\right) = \frac{1}{2}T_{\alpha\beta}(D^\alpha V^\beta + D^\beta V^\alpha). \tag{6}$$

For a vectorfield $V = V^v\partial_v + V^u\partial_u$ whose components depend only on (u,v) the right hand side of the above identity takes the form

$$\begin{aligned}\frac{1}{2}T_{\alpha\beta}(D^\alpha V^\beta + D^\beta V^\alpha) = \frac{1}{4(1-\mu)}\bigg((\partial_u\phi)^2\partial_v\frac{V_v}{1-\mu} + (\partial_v\phi)^2\partial_u\frac{V_u}{1-\mu} + |\not\nabla\phi|^2(\partial_u V_v + \partial_v V_u)\bigg)\\ - \frac{1}{2r}(V_u - V_v)(|\not\nabla\phi|^2 - \partial^\mu\phi\,\partial_\mu\phi)\end{aligned}$$

The expression $(D^\alpha V^\beta + D^\beta V^\alpha)$ vanishes for a Killing vectorfield. However the Killing fields $\frac{\partial}{\partial t}$ and angular momentum Ω are not sufficient by themselves to provide any decay information, e.g. the use of the Killing field $\frac{\partial}{\partial t}$ generates the energy identity, which can only lead to boundedness of ϕ. The identity (6) is integrated either in a region bounded by two time slices $t =$const or in a *characteristic rectangle* $[u_1, u_2] \times [v_1, v_2]$.

The crucial role in the proof of Theorem 1.1 is played by the following 3 vectorfields:

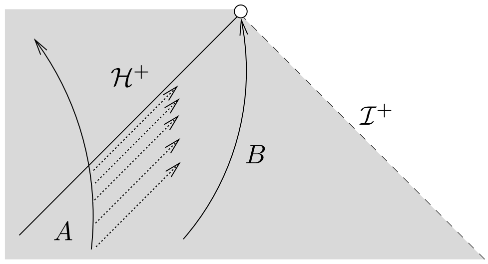
Red-shift: $Y \sim \frac{1}{1-\mu}\frac{\partial}{\partial u}$

Morawetz: $K = v^2\frac{\partial}{\partial v} + u^2\frac{\partial}{\partial u}$

Local energy: $X \sim f(r)\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)$

[9]For even the weakest results of [4], one still requires the analog of $\phi \le Cv^{-\frac{1}{2}-\epsilon}$ along the event horizon, for an $\epsilon > 0$.

The red shift vectorfield has no analog in Minkowski space-time. It captures a well-known physical phenomena taking place in a black hole space-time, where a frequency of signal sent by an observer traveling towards a black hole is shifted to the red part of the spectrum when received by another observer to the future of the first one.



The red shift vectorfield captures the geometry of the event horizon and is used in “characteristic rectangles" $[u_1, u_2] \times [v_1, v_2]$ with the side $\{u_2\} \times [v_{1,2}\,]$ on the event horizon, e.g. $u_2 = \infty$.

The associated Y flux densities have the form

$$y_u = \frac{(\partial_u \phi)^2}{1-\mu}, \qquad y_v = |\nabla\!\!\!/ \phi|^2$$

The corresponding characteristic energy identity with Y leads to the inequality (where all the integrations over $\mathbb{S}^2$ have been suppressed):

$$\int_{v_1}^{v_2} y_v(u_2) dv + \int_{u_1 \cap r < r_0}^{u_2} y_u(v_2) du + \int_{v_1}^{v_2} \int_{u_1 \cap r < r_0}^{u_2} y_u du dv \lesssim \int_{v_1}^{v_2} \int_{u_1 \cap r_0 < r < R}^{u_2} e_t + \int_{v_1 \cap r < R}^{v_2} y_v(u_1) dv + \int_{u_1}^{u_2} y_u(v_1) du$$

where

$$e_t = (\partial_v \phi)^2 + (\partial_u \phi)^2 + (1-\mu) |\nabla\!\!\!/ \phi|^2$$

is the energy density, associated with a Killing vectorfield $\frac{\partial}{\partial t}$.

The analog[10] of the Morawetz vectorfield K allows one to associate with it a *conformal energy*:

$$E^K_\phi(t) = \int_{\Sigma_t} \left(v^2 (\partial_v \phi)^2 + u^2 (\partial_u \phi)^2 + (1-\mu)(v^2 + u^2) |\nabla\!\!\!/ \phi|^2 + (1-\mu) \left(\frac{(r^*)^2}{r^2} + \frac{t^2}{r^2} \right) \phi^2 \right)$$

The energy identity with K in place of V obtained by integrating (6) between times slices $t = 0$ and t leads to the inequality

$$E^K_\phi(t) \le E^K_\phi(0) + \int_0^t \int_{[r_0, R]} \tau e_\tau, \qquad 2m < r_0 < R < \infty,$$

[10] Note that the generalization of the vectorfield K from Minkowski space-time is based on the representation of the latter relative to the null coordinates (u, v) rather than its form relative to (t, r).

On the other hand one can show that

$$E_\phi^K(t) \geq t^2 \int_{[r_1, r_2]} e_t, \qquad r_1^* > -.9t,\ r_2^* < .9t.$$

To bound space-time integrals of a local energy density we use vectorfield X. In fact vectorfields X are constructed separately for each spherical harmonic ϕ_ℓ:

$$-\frac{1}{1-\mu}\partial_t^2 \phi_\ell + \frac{1}{r^2}\partial_r\left(r^2(1-\mu)\partial_r \phi_\ell\right) - \frac{\ell(\ell+1)}{r^2}\phi_\ell = 0$$

and then put together to bound space-time integrals of a local energy density.

Vectorfields X_ℓ are found according to the formula

$$X_\ell = f_\ell(r)\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)$$

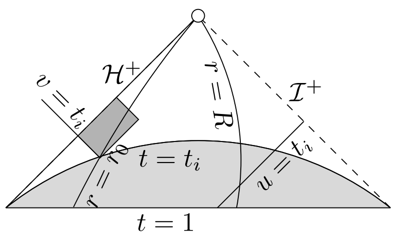
with a special choice of a function f_ℓ. With that choice the energy identity obtained by integrating (6), with X_ℓ in place of V, between times slices $t = 0$ and t leads to the inequality

$$\int_0^t \int_{[r_0, R]} \left((\partial_{r^*}\phi_\ell)^2 + (2 - 3\mu) f_\ell\ |\nabla\!\!\!/ \phi_\ell|^2 + \phi_\ell^2\right) \leq E_{\phi_\ell},$$

where E_{ϕ_ℓ} is the total energy of ϕ_ℓ. The function f_ℓ is non-negative but required to vanish at a point r_ℓ with the property that $r_\ell \to 3M$ (photosphere) as $\ell \to \infty$. To bound a space-time integral of the local energy density one needs to sum the above inequality in ℓ and the corresponding inequality for $\Omega\phi$.

$$\int_0^t \int_{[r_0, R]} e_t \leq E_\phi + E_{\phi_\omega}$$

6.1. **Putting it all together.** We now list the essential steps of the remaining parts of the argument.



(1) First round of application of K and X implies

$$E_\phi^K(t) \leq Ct, \qquad \int_{r \in [r_1, r_2]} e_t \leq Ct^{-1}$$

(2) First round of applications of Y and X: implies that fluxes associated with the red shift vectorfield Y are bounded

(3) Partitioning the interval $[0,t]$ dyadically (with the base $10/9$), using X in $[t_i, t_{i+1}]$, finite speed of propagation and the obtained bounds for the conformal energy implies that

$$\int_0^t \int_{r\in[r_1,r_2]} \tau e_\tau \le C \log t$$

(4) Second round of application of K and X implies

$$E^K_\phi(t) \le C \log t, \qquad \int_{r\in[r_1,r_2]} e_t \le C t^{-2} \log t$$

(5) Partition the event horizon dyadically and use the red-shift effect combined with the pigeonhole principle followed by

(6) Second round of application of Y and X: implies that Y fluxes decay like v^{-1}

... Third round of estimates leads

(7) to the claimed decay of energy and fluxes. Pointwise decay follows from Sobolev type estimates.

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY