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Nonhermitian systems and pseudospectra

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Contemporary Mathematics

Nonhermitian systems and pseudospectra

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Abstract. Four applications are outlined of pseudospectra of highly nonnormal linear operators.

1. Introduction

In many applications it has been found that if a matrix or operator is "far from normal," then its eigenvalues or more generally its spectrum may be a poor guide to its behavior. Pseudospectra are sets in the complex plane that are designed to reveal such situations and give more information. The study of pseudospectra is about twenty years old, with several hundred papers now in print (see `www.comlab.ox.ac.uk/pseudospectra`), and an extensive discussion of theorems and applications can be found in my 2005 book coauthored with Mark Embree, *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators* [**28**], from which the figures and examples in this paper are taken.

What does it mean to be "far from normal"? For a problem associated with an energy represented as some kind of a sum of squares, it often makes sense to consider a linear operator in a Hilbert space; for matrices, this becomes the 2-norm. Normal matrices have a complete set of orthonormal eigenvectors; familiar examples include real symmetric, complex hermitian, and unitary matrices. In the nonnormal case, though a full set of eigenvectors may still exist, they cannot be taken to be orthogonal. In applications, the deviation from orthogonality often grows exponentially with respect to a parameter such as the matrix dimension (for nonsymmetric Toeplitz matrices), the Péclet number (for convection-diffusion problems), or the Reynolds number (for fluid flow problems). We can measure this deviation by the condition number of a matrix of normalized eigenvectors, i.e., the quantity $\kappa(V) = \|V\|\|V^{-1}\|$, where V is a matrix whose columns are linearly independent eigenvectors of unit norm. A matrix whose eigenvector matrices all have large condition number is in some sense far from normal. The same idea extends to matrices in Banach spaces (e.g. the $\|\cdot\|_1$ or $\|\cdot\|_\infty$ norms): again what matters is $\kappa(V) = \|V\|\|V^{-1}\|$, defined now with respect to the norm of the Banach space.

2. Definition of pseudospectra

Let X be a complex Banach space with norm $\|\cdot\|$, and let $\mathcal{C}(X)$ and $\mathcal{B}(X)$ denote the sets of closed and bounded linear operators in X, respectively. Let $\mathcal{D}(\boldsymbol{\mathcal{A}}) \subseteq X$ denote the domain of an operator $\boldsymbol{\mathcal{A}} \in \mathcal{C}(X)$ and $\sigma(\boldsymbol{\mathcal{A}})$ its spectrum. The following theorem asserts the equivalence of three conditions, any of which we can accordingly be taken as the definition of the ε-pseudospectrum $\sigma_\varepsilon(\boldsymbol{\mathcal{A}})$ of $\boldsymbol{\mathcal{A}}$. This statement is reproduced from [**28**, §4], but the underlying mathematics is standard material that can be found, for example, in [**17**]. For a review of the history of pseudospectra, see [**28**, §6].

THEOREM 2.1. *Let* $\boldsymbol{\mathcal{A}} \in \mathcal{C}(X)$ *and* $\varepsilon > 0$ *be arbitrary. The* ε-pseudospectrum $\sigma_\varepsilon(\boldsymbol{\mathcal{A}})$ *of* $\boldsymbol{\mathcal{A}}$ *is the set of* $z \in \mathbf{C}$ *defined equivalently by any of the conditions*

(2.1) $\|(z-\boldsymbol{\mathcal{A}})^{-1}\| > \varepsilon^{-1}$,

(2.2) $z \in \sigma(\boldsymbol{\mathcal{A}}+\boldsymbol{\mathcal{E}})$ *for some* $\boldsymbol{\mathcal{E}} \in \mathcal{B}(X)$ *with* $\|\boldsymbol{\mathcal{E}}\| < \varepsilon$,

(2.3) $z \in \sigma(\boldsymbol{\mathcal{A}})$ or $\|(z-\boldsymbol{\mathcal{A}})u\| < \varepsilon$ *for some* $u \in X$ *with* $\|u\| = 1$.

Given $\boldsymbol{\mathcal{A}}$, the pseudospectra $\sigma_\varepsilon(\boldsymbol{\mathcal{A}})$ for $\varepsilon \in (0,\infty)$ are strictly nested open sets in the complex plane whose intersection is $\sigma(\boldsymbol{\mathcal{A}})$ [**28**, Thm. 4.3]). If $\|(z-\boldsymbol{\mathcal{A}})u\| < \varepsilon$ as in (2.3), we say that z is an *ε-pseudoeigenvalue* of $\boldsymbol{\mathcal{A}}$ and u is a corresponding *ε-pseudoeigenvector* (or *pseudoeigenfunction* or *pseudomode*).

Pseudospectra can be computed efficiently (in Hilbert space) by the MATLAB package EigTool [**30**], which can handle matrices of dimensions in the hundreds quickly and in the tens of thousands still quite successfully.

We now outline four applications. Full details can be found in [**28**]. In particular, for general theorems relating pseudospectra to transient effects in discrete- and continuous-time dynamics, see [**28**, §§14–19].

3. Hydrodynamic stability

The three most successful areas of application of eigenvalues in the sciences have been quantum mechanics, vibration and acoustics problems, and stability of fluid flows. The first two involve hermitian and near-hermitian operators, respectively, but for fluid flows, at least at higher flow speeds (more precisely higher Reynolds numbers), the operators of interest are far from normal. This fact caused confusion throughout the 20th century; see [**4**], [**29**], and [**28**, §§20–23].

Consider the most famous of all problems of flow stability, first analyzed by Reynolds in 1883: the idealized flow of a Newtonian fluid through an infinite circular pipe. The flow is governed by the Navier–Stokes equations, and for any Reynolds number R, there exists a solution consisting of steady ("laminar") axial flow with a velocity profile given by a parabola. At higher Reynolds numbers, however, such flows break down to turbulence. To explain why this happens, it is natural to consider the linear operator that governs the evolution of infinitesimal perturbations about the laminar solution. Surprisingly, one finds that no matter how high R is, this operator has its spectrum in the open left half-plane. (This has not been proved, but everybody believes it.) Thus there are no eigenvalue instabilities, regardless of the Reynolds number. Nevertheless, if R is high, the laminar flow is impossible to maintain in the laboratory.

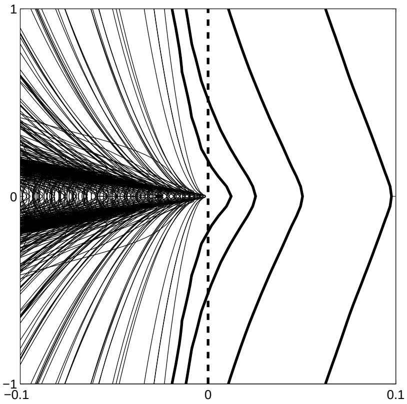

FIGURE 1. Spectrum and pseudospectra for pipe flow at Reynolds number $R = 10^4$. The spectrum consists of a collection of curves in the open left half-plane, so classically, we would expect this flow to be stable. The thicker curves in the right half-plane mark the right-hand boundaries of the ε-pseudospectra for $\varepsilon = 10^{-2}, 10^{-2.5}, 10^{-3}, 10^{-3.5}$. These curves reveal that the resolvent norm is "surprisingly large" in the right half-plane, implying that there will be transient growth effects in the associated linear dynamical system before the eventual slow exponential decay.

Figure 1 shows the spectrum and some of the pseudospectra for pipe flow with $R = 10^4$, a Reynolds number at which laminar flow would rarely be observed.

There is a vast literature on this problem, and over the years many explanations have been put forward. In the 1990s some consensus was finally reached on the essential ingredients of what is going on. The linearized operator in question is strongly nonnormal, so much so that at Reynolds numbers in the thousands, infinitesimal perturbations can be amplified by factors of hundreds before eventually, in theory, diminishing. The existence of such transient effects is implied by the pseudospectra of Figure 1; see §§14–19 [**28**]. What this means for the fluid flow is that very small perturbations may get linearly amplified enough to trigger nonlinear effects that lead to transition to turbulence.

If all fluids problems were like the circular pipe, with no unstable eigenvalues, the paradox of their instability in practice would have been so glaring that there would have been less confusion in the history of fluid mechanics. However, the most extensively analyzed problem of this kind, *plane Poiseuille flow* between two fixed infinite plates, has more complicated and distracting behavior. Here the spectrum does protrude into the right half-plane for $R > 5772$, indicating an eigenvalue instability. Surely this must have something to do with the instability observed in practice! In fact, it has little to do with it, as the caption of Figure 2 explains.

4. Wave packet pseudomodes

In 1999 E. B. Davies published a paper on the spectra and pseudospectra of a harmonic oscillator for the Schrödinger equation, but with a twist: the potential was complex [**8**]. The simplest version of Davies's example is the operator on $L^2(\mathbf{R})$

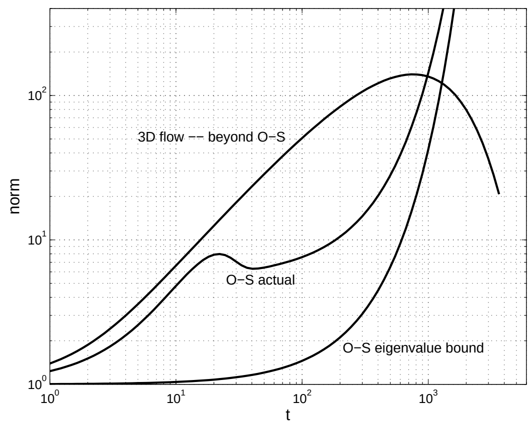

FIGURE 2. Norms of operators $\|e^{t\boldsymbol{\mathcal{A}}}\|$ on a log-log scale for plane Poiseuille flow with $R = 10^4$. This problem has an eigenvalue in the right half-plane, but its real part is only 0.0037, corresponding to exponential growth so slow as to be unobservable in practice: it would take a pipe 200 diameters long to enable amplification by even a factor of 2 (bottom curve). The nonnormal transient effects for the Orr–Sommerfeld operator, corresponding to a two-dimensional flow analysis, are an order of magnitude faster and stronger (middle curve). When three-dimensional perturbations of the laminar solution are considered, new nonnormal transient effects appear that are an order of magnitude larger still (top curve). It is mainly the latter effects, which have nothing to do with eigenmodes, that make plane Poiseuille flow unstable in practice.

defined by

$$\boldsymbol{\mathcal{A}}u = -u_{xx} + ix^2u. \tag{4.1}$$

Without the factor i, this would be the most familiar of all problems of quantum mechanics, a hermitian operator with spectrum $\{1, 3, 5, \dots\}$. With that factor, the spectrum rotates to $e^{i\pi/4}\{1, 3, 5, \dots\}$, but more importantly, the eigenvectors cease to be orthogonal and indeed they become exponentially far from orthogonal.

Davies showed that for z in the interior of the first quadrant of the complex plane, this nonnormal operator has ε-pseudoeigenvectors in the form of wave packets for values of ε that diminish rapidly as $|z| \to \infty$. In fact, ε can be taken to decrease exponentially as a function of $|z|$ along rays extending from the origin, and Davies and Kuijlaars have shown that the condition numbers of the eigenvalues $\{\lambda_n\}$ grow exponentially at a rate given by $\lim_{n\to\infty} \kappa(\lambda_n)^{1/n} = 1+\sqrt{2}$ [**10**]. The pseudospectra and an example of an associated *wave packet pseudomode* are shown in Figure 3.

Davies's example illustrates a general phenomenon of exponentially good wave packet pseudomodes for differential operators with variable coefficients, which can be analyzed by methods of WKBJ asymptotics or microlocal analysis. These effects are related to the subjects of *exponential dichotomy* [**20**] and *ghost solutions* [**12**] of ODEs and PDEs. For a one-dimensional domain as in (4.1), exponentially good wave packet pseudomodes will exist whenever a certain "twist condition" is satisfied; see [**25**] and [**28**, §§11–12], and for other examples in fluid mechanics and theoretical physics, see [**2**], [**3**], and [**7**]. But in fact the situation is much more general than this. As pointed out first by Zworski [**31**], these pseudomodes are the

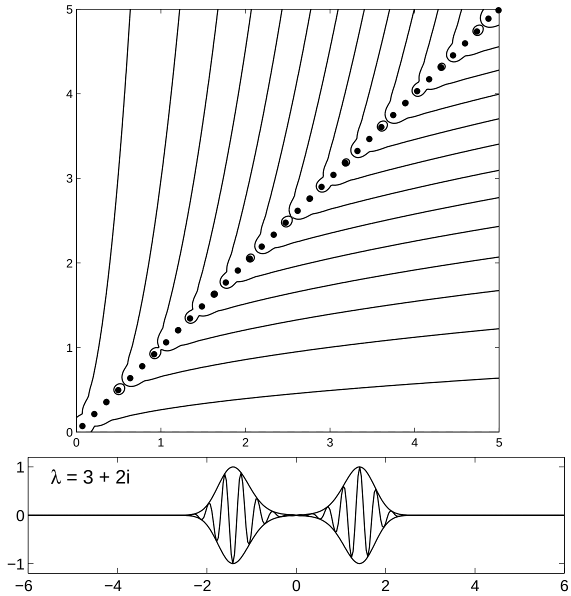

FIGURE 3. Above, eigenvalues and ε-pseudospectra of Davies's operator (4.1); the pseudospectra correspond (from bottom-left) to $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, \ldots$. The resolvent norm is exponentially large in this first quadrant of the complex plane, corresponding to pseudomodes in the form of wave packets (or pairs of wave packets). Below, an optimal pseudomode (real part and envelope) corresponding to $z = 3 + 2i$. This point lies at a distance of about 0.7 from the nearest eigenvalue, yet the corresponding value of ε is about 10^{-5}.

same solutions that arise in the theory of local solvability of linear PDE developed by Hörmander and many others over the years, including Beals, Dencker, Fefferman, Garabedian, Lerner, Nirenberg, and Treves. See Figure 4 and the explanation given in its caption. A far-reaching theorem about the existence of such pseudomodes, for both differential and pseudodifferential operators, has been published by Dencker, Sjöstrand and Zworski [**11**].

These developments for differential and pseudodifferential operators have close analogues for "Toeplitz matrices with variable coefficients" and their generalizations known as Berezin–Toeplitz operators—see [**5**], [**26**], and [**28**, §§8–9]. Again, the existence of exponentially good wave packet pseudomodes is guaranteed if a twist condition is satisfied.

5. Lasers

Laser physics involves an interplay of two eigenvalue problems, one involving the quantum mechanics of atomic states and transitions, the other involving a resonant cavity in which a monochromatic light wave accumulates. The former is hermitian and the latter, for most lasers, is close to hermitian. Some higher powered lasers, however, make use of resonant cavities associated with strongly nonhermitian operators [**23**].

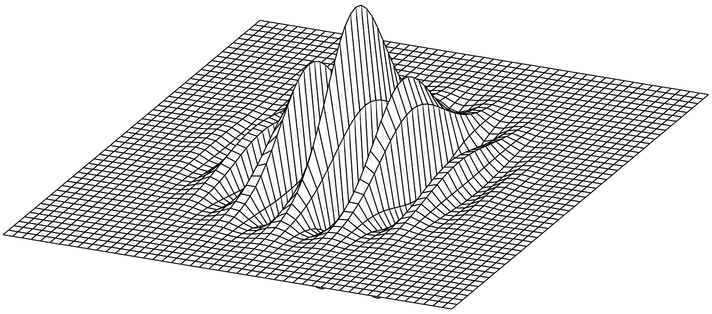

FIGURE 4. An illustration from [**28**, §13] of the kind of multidimensional wave packet pseudomode associated with Lewy–Hörmander nonexistence of solutions to certain linear PDE with variable coefficients. As pointed out by Zworski, the commutator or Ψ condition that arises in this theory is a condition for existence of exponentially good pseudosolutions to certain PDE. By an argument related to the Fredholm alternative, if one problem has solutions or approximate solutions like this (here the "Mizohata equation" $u_x + ixu_y = g$), its dual cannot be solved for all data (here $-v_x + ixv_y = f$).

The standard mathematical analysis of laser cavities, due to Fox and Li in the early 1960s [**14**], makes use of powers of a complex symmetric but nonhermitian integral operator that governs the change in shape of a packet of light as it bounces back and forth between the two mirrors at the ends of the cavity. For the simplest case of straight mirrors the operator is

$$\boldsymbol{\mathcal{A}}u(x) = \sqrt{\frac{iF}{\pi}} \int_{-1}^{1} e^{-iF(x-s)^2} u(s)\,ds \tag{5.1}$$

acting in $L^2[-1,1]$, where F is a positive parameter known as the Fresnel number. If one or both mirrors is curved outwards, on the other hand, the operator changes to

$$\boldsymbol{\mathcal{A}}u(x) = \sqrt{\frac{iFM}{\pi}} \int_{-1}^{1} e^{-iF(x/M-s)^2} u(s)\,ds, \tag{5.2}$$

where M is a parameter bigger than 1. As pointed out first by Henry Landau in the 1970s [**18**], one of the first inventors of pseudospectra, (5.1) is close to normal but (5.2) is far from normal.

It is interesting to examine stable and unstable lasers from the point of view of transient vs. asymptotic behavior. However, a physically more remarkable aspect of the nonnormality is an indirect consequence of these transients. A practical laser will never have a spectral line of zero width, i.e. a delta function: the varying phases of randomly emitted photons broaden the line to a width given by

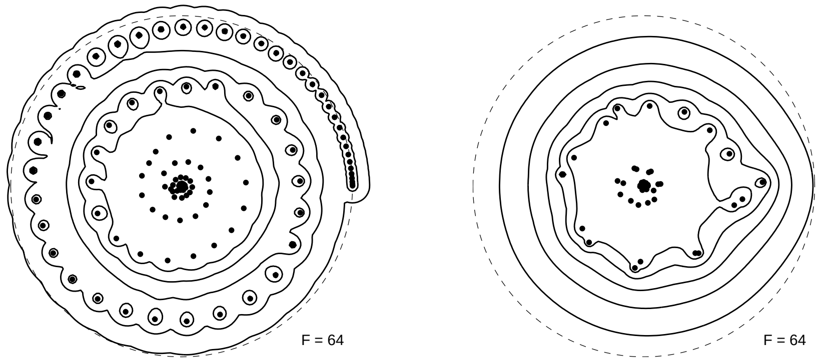

FIGURE 5. Spectra and pseudospectra for the Fox–Li laser integral operators (5.1) (left) and (5.2) (right) with $F = 64\pi$ and $M = 2$; from outside in, the contours correspond to $\varepsilon = 10^{-1}, 10^{-1.5}, 10^{-2}, \ldots$. The operator on the left is close to normal at least in the physically important region near the unit circle, as we can see from its pseudospectra that barely protrude beyond the eigenvalues. The operator on the right is far from normal, and will have transient effects much different from its asymptotic fast exponential decay.

Schawlow–Townes formula, presented in the original Nobel Prize-winning 1958 paper by Schawlow and Townes. Now for an unstable cavity governed by an operator like (5.2), the spectral lines are far wider. The reason is that for reasons of nonnormality, emitted photons with random phases can stimulate strongly enhanced output effects that distort the signal. The factor by which the line is broadened is known as the *Petermann excess noise factor* [**6, 21**], and it is equal to $\kappa(\lambda_1)^2$, the square of the condition number of the dominant eigenvalue of this operator. In the highly nonnormal case, this eigenvalue is ill-conditioned and the excess noise factor is large—for the example of the figure, about 625. See [**28**, §60].

Figure 5 shows spectra and pseudospectra for (5.1) and (5.2). This is a typical example of how pseudospectra can tell one instantly whether the eigenvalues of an operator are likely to tell the whole story.

6. Nonsymmetric random matrices of Hatano and Nelson

A famous article by Anderson in 1958 initiated the study of *localization* of eigenvectors of certain hermitian matrices with random entries [**1**]. Anderson found that if $N \times N$ tridiagonal matrices are formed with the number 1 on the sub- and super-diagonals and independent random entries on the main diagonal, then as $N \to \infty$, all of the eigenvectors will be localized in the sense that they decay exponentially away from a central point (different for each eigenvector). (Rigorous theorems concerning this effect are a delicate matter.) This phenomenon has great significance for the physical properties of disordered materials.

Four decades later, Hatano and Nelson introduced a family of analogous nonsymmetric random matrices as part of a growing interest among physicists in "nonhermitian quantum mechanics" [**15, 16**]. To give a very particular version of their matrices, consider an $N \times N$ matrix with $1/2$ on the subdiagonal and in the upper-right corner, 2 on the superdiagonal and in the lower-left corner, and independent

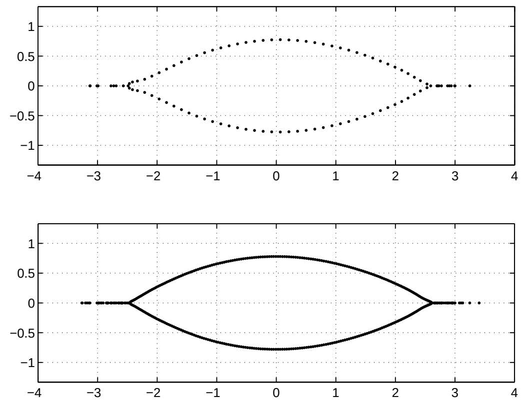

FIGURE 6. Nonsymmetric Hatano–Nelson matrices of dimensions $N = 100$ and 400, showing the mix of real eigenvalues (with localized eigenvectors) and complex ones (with global eigenvectors).

random numbers from the uniform distribution in $[-2, 2]$ on the main diagonal. Hatano and Nelson found that the eigenvalues of such a matrix delineate an elegant pattern in the complex plane, shown in Figure 6. Some are real, associated with localized eigenvectors like those of the Anderson model, while others are complex and associated with global eigenvectors.

The Hatano–Nelson matrices are just mildly nonnormal. However, this is a consequence of their upper-right and lower-left corner entries, introduced in order to make the structure periodic. If one replaces these numbers by zero, one obtains matrices that are exponentially far from normal. Now the eigenvalues are all real, as is obvious since the matrices are symmetrizable, i.e., reducible by a diagonal similarity transformation to symmetric Anderson matrices. Yet the condition number of that transformation will be of order 4^N, and it distorts the significance of eigenvalues beyond recognition. Figure 7 shows spectra and pseudospectra for these modified Hatano–Nelson matrices with zero corner entries. Where Figure 6 had a bubble of eigenvalues, we now have pseudospectra in the very same shape. All this is made precise in [**27**], where implications for localization and delocalization are discussed.

The study of pseudospectra of random matrices such as these Hatano–Nelson matrices with zero corner entries casts light on an interesting phenomenon in spectral theory. One might imagine that for a scientific problem modeled by matrices of dimensions $N \to \infty$, the essential pseudospectral effects for finite N should carry over to effects of the spectrum itself for a corresponding operator with $N = \infty$ [**9**]. Yet in these problems we see a crucial distinction for large finite N: in some parts of the complex plane the resolvent norm is exponentially large, perhaps "infinite in practice" from a physical point of view, while in others it is only algebraically large, "finite in practice" (Figure 8).

7. Prospects for pseudospectra

Pseudospectra have by now been computed for a wide range of mathematical and physical problems; some that we have not mentioned include shuffling of decks

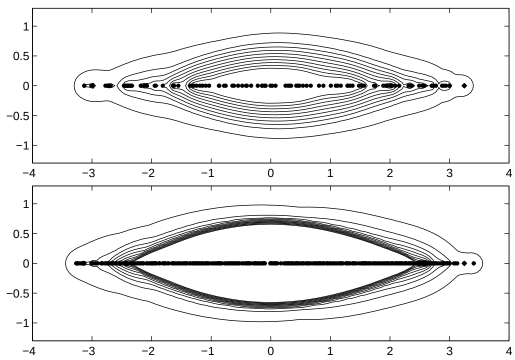

FIGURE 7. Spectra and pseudospectra for the same matrices, except with the upper-right and lower-left entries replaced by zero. Now we have strong nonnormality, with the pseudospectra revealing the same "bubble" that was shown by the eigenvalues in Figure 6.

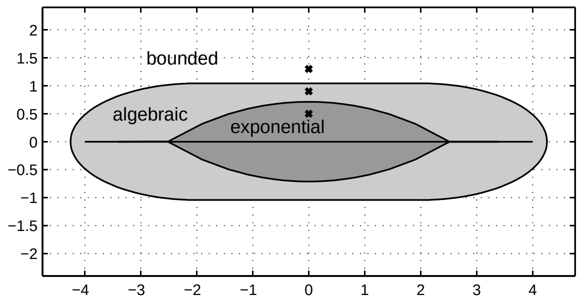

FIGURE 8. Schematic view of growth of the resolvent norm as $N \to \infty$ for Hatano–Nelson matrices with zero corner entries as in Figure 7. Both the shaded regions become spectrum in the limit $N = \infty$, but for finite N, the dark grey region will typically "behave like spectrum" in practice while the light grey region will not.

of cards, stability of food webs, and convergence of Krylov subspace iterations in numerical linear algebra [**28**]. Pseudospectral technology is well established and highly successful as a tool for diagnosing situations where eigenvalues may be misleading: a plot from EigTool may instantly reveal such a situation. When the eigenvalues fail, however, do pseudospectra provide the missing information? Are they a quantitatively helpful too, or merely a warning device? Many results pertaining to these question are collected in [**28**, §§14–19], but a general answer is not yet clear. Indeed, many gaps in our understanding remain. Here for example is a seemingly elementary problem whose solution is not known. Suppose A and B are two matrices with simple eigenvalues whose ε-pseudospectra are identical for all ε. Must $\|p(A)\| = \|p(B)\|$ for all polynomials p? Without the hypothesis of simple eigenvalues, the answer is no; see [**28**, §47].

Acknowledgments

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