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Sharp Domains of Determinacy and Hamilton-Jacobi Equations

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Abstract. If $L(t, x, \partial_t, \partial_x)$ is a linear hyperbolic system of partial differential operators for which local uniqueness in the Cauchy problem at spacelike hypersurfaces is known, we find nearly optimal domains of determinacy of open sets $\Omega_0 \subset \{t = 0\}$. The frozen constant coefficient operators $L(\underline{t}, \underline{x}, \partial_t, \partial_x)$ determine local convex propagation cones, $\Gamma^+(\underline{t}, \underline{x})$. Influence curves are curves whose tangent always lies in these cones. We prove that the set of points Ω which cannot be reached by influence curves beginning in the exterior of Ω_0 is a domain of determinacy in the sense that solutions of $L\, u = 0$ whose Cauchy data vanish in Ω_0 must vanish in Ω. We prove that Ω is swept out by continuous space like deformations of Ω_0 and is also the set described by maximal solutions of a natural Hamilton-Jacobi equation (HJE). The HJE provides a method for computing approximate domains and is also the bridge from the raylike description using influence curves to that depending on spacelike deformations. The deformations are obtained from level surfaces of mollified solutions of HJEs.

§0. Introduction.

The question addressed in this note is to describe as accurately as possible the property of finite speed of propagation for solutions of general hyperbolic systems. This is a problem of propagation of zeros.

The analysis addresses m^{th} order $N \times N$ system of partial differential operators with complex

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matrix valued coefficients

$$L(t,x,\partial_t,\partial_x) \;=\; \sum_{|\beta|\le m} A_\beta(t,x)\,\partial_{t,x}^\beta \;=\; \partial_t^m \;+\; \text{lower order in } t\,. \tag{0.1}$$

Hypothesis 0.1. *The coefficients satisfy α,β,*

$$\partial_{t,x}^\alpha A_\beta \in L^\infty(\mathbb{R}\times\mathbb{R}^d)\,.$$

The characteristic polynomial is

$$P(t,x,\tau,\xi) \;:=\; \det\Big(\sum_{|\beta|=m} A_\beta(t,x)\,(\tau,\xi)^\beta\Big), \qquad \deg P = mN\,. \tag{0.2}$$

The operator is **hyperbolic** *with timelike variable t in the sense that the equation*

$$P(t,x,\tau,\xi)=0$$

has only real roots τ for real $\xi\in\mathbb{R}^d$.

Problems with less regular coefficients are discussed in the detailed paper [JMR].

Define

$$\tau_{\max}(t,x,\xi) \;:=\; \max\big\{\tau \;:\; P(t,x,\tau,\xi)=0\big\}, \tag{0.3}$$

and the associated convex cone of timelike codirections (see e.g. [Gå, Co, H03])

$$\mathcal{T}(t,x) \;:=\; \big\{(\tau,\xi) \;:\; \tau>\tau_{\max}(t,x,\xi)\big\}\,. \tag{0.4}$$

$\mathcal{T}(t,x)$ a subset of the cotangent space at (t,x) and yields, by duality, the forward propagation cone in the tangent space at (t,x)

$$\Gamma^+(t,x) \;:=\; \big\{(T,X) \;:\; \forall(\tau,\xi)\in\mathcal{T}(t,x)\,,\quad (T,X).(\tau,\xi)\;\ge\;0\big\}\,. \tag{0.5}$$

The set $\Gamma^+(t,x)$ depends only on the principal symbol of L. Both $\mathcal{T}$ and Γ^+ are convex. The former is open and the latter has compact intersection with the planes $T=\text{const}>0$.

Definitions. *An embedded hypersurface $\Sigma\subset\mathbb{R}^{1+d}$ is* **space like** *when its conormal vectors belong to $\mathcal{T}(t,x)\cup-\mathcal{T}(t,x)$ for every $(t,x)\in\Sigma$. A relatively open set $\Omega\subset[0,\infty[\times\mathbb{R}^d$ is called a* **domain of determinacy** *of the relatively open subset $\Omega_0\subset\{t=0\}$ when every $H^{m-1}_{\rm loc}\big([0,\infty[\times\mathbb{R}^d\big)$ solution of $Lu=0$ whose Cauchy data vanish in Ω_0 must vanish in Ω. A closed subset $S\subset[0,\infty[\times\mathbb{R}^d$ is called* **a domain of influence** *of the closed set $S_0\subset\{t=0\}$ if every $H^{m-1}_{\rm loc}\big([0,\infty[\times\mathbb{R}^d\big)$ solution of $Lu=0$ whose Cauchy data is supported in S_0 is supported in S. An* **influence curve** *is a lipschitzian curve $x(t):[a,b]\to\mathbb{R}^d$ so that the tangent vector to $\big(s,x(s)\big)$ belongs to $\Gamma^+\big(s,x(s)\big)$ for Lebesgue almost all s.*

The definitions imply that a set Ω is a domain of determinacy of Ω_0 if and only if $S:=\big([0,\infty[\times\mathbb{R}^d\big)\setminus\Omega$ is a domain of influence of $S_0:=\mathbb{R}^d\setminus\Omega_0$. The problems of finding large domains of determinacy and small domains of influence are therefore equivalent and amount to accurately describing the speed of propagation for solutions of $Lu = 0$.

The intersection of a family of domains of influence of a fixed set S_0 is a domain of influence. Thus there is a smallest such domain called the **the exact domain of influence** and sometimes just **the domain of influence**. For example, the exact domain of influence of the origin for the operator $\Box + m^2$ is the solid cone $|x|^2 \leq c^2t^2$ when $m \neq 0$ or if $m = 0$ and $d \neq 3, 5, \ldots$. For $m = 0$ and odd $d \geq 3$, the exact domain of influence is just the boundary of the cone, $|x|^2 = c^2t^2$. The case of $d = 3, 5, \ldots$ and $m \approx 0$ shows that the exact domain of influence depends sensitively on the operator even in the constant coefficient case. On the other hand, in the constant coefficient case the convex hull of the domain of influence of the origin is always equal to Γ^+. We prove that the bound of the domain of influence given by Γ^+ extends naturally to a domain of influence in the variable coefficient case.

The union of a family of domains of determination of a fixed set is also a domain of determination. The largest domain of determination is called the **exact domain of determination**.

The most natural description of domains of influence and determinacy for hyperbolic problems use influence curves ([Co, §VI.7], [Le, §VI.4], [La1, Thm 2.2]). The natural theorem is that if (t, x) is not connected by an influence curve to the set S_0 in $\{t = 0\}$, then the values of solutions of $Lu = 0$ at (t, x) are not influenced by the Cauchy data in S_0. This geometric description of the domain of influence does not immediately suggest a method of proof.

There is a second approach to the problem, the method of spacelike deformations, which has the opposite character of leading directly to a proof.

Hypothesis 0.2 *The operator L has the property of local uniqueness in the Cauchy problem at space like hypersurfaces, that is, for every embedded space like hypersurface $\Sigma \subset]-1, \infty[\times \mathbb{R}^d$ and point $p \in \Sigma$, if $u \in H^{m-1}_{\rm loc}(]-1, \infty[\times \mathbb{R}^d)$ satisfies $Lu = 0$ on a neighborhood of p in $\mathbb{R}^{1+d}$ and the Cauchy data of u vanish on a neighborhood of p in Σ, then u vanishes on a neighborhood of p in $\mathbb{R}^{1+d}$.*

Examples. 1. Constant coefficient systems and systems with analytic coefficients using Hölmgren's Theorem. **2.** Symmetric hyperbolic systems of first order. **3.** Strictly hyperbolic systems.

The second approach to domains of determination is that domains swept out by spacelike surfaces with their feet in Ω_0 describe domains of determination. The two simple examples of $\partial_t^2 - \partial_x^2$ and $\partial_t + \partial_x$ both in dimension $d = 1$ with an initial set $\Omega_0 =]-1, 1[$ give the essential idea of the method. The sharp domain of determinacies are the triangle $\{|x| < 1 - t\}$ and the strip $\{-1 < x - t < 1\}$ respectively. These domains are swept out by space like deformations sketched in Figure 0.1

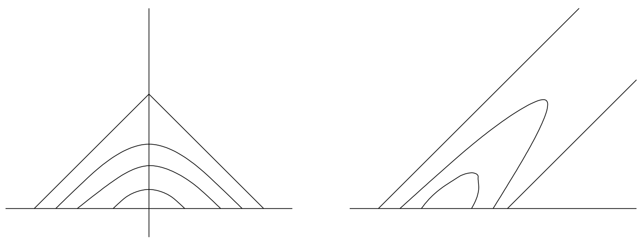

Figure 0.1

In the next result the deformation is by the level curves $\{F = c\}$ with c increasing from 0 to 1.

The straight forward proof is recalled in [JMR, Appendix].

John's Global Hölmgren Theorem [Jo]. *Suppose that L satisfies Hypothesis* 0.1-0.2, *that $\mathcal{O}$ is a relatively open subset of $\{t \ge 0\}$ and $F \in C^1(\mathcal{O})$ has the following properties.*

i. $F > 0$ *on* $\mathcal{O} \cap \{t > 0\}$.

ii. $F^{-1}([0,1]) \subset\subset \mathcal{O}$.

iii. *For all* $(t,x) \in F^{-1}\big([0,1]\big)$, $dF(t,x) \in \mathcal{T}(t,x)$.

Then, if $u \in H^{m-1}_{\rm loc}(\{t \ge 0\})$ satisfies $Lu = 0$ on $\mathcal{O}$, and, the Cauchy data of u vanish on a neighborhood of $F^{-1}([0,1]) \cap \{t = 0\}$, then $u = 0$ on $F^{-1}([0,1])$.

Neither the influence curve nor the spatial deformation approach provides a method to compute accurate approximations to the domains. The computation of first arrival times in problems of geology and elsewhere amounts to the same fundamental question and in those communities, computational strategies have been proposed based on using (maximal) solutions of Hamilton-Jacobi equations (see e.g [SF], [RMO], [FJ], and references therein). To our knowledge the relation of this Hamilton-Jacobi approach to the other two has not been investigated except in simple cases. Our main result is that all three descriptions yield the same sets. One can profit from the numerical advantages of Hamilton-Jacobi, the geometry of influence curves, or the analytic advantages of space like deformations, secure in the knowledge that they agree. We use the solution of the Hamilton-Jacobi equation to construct space like deformations. The Hamilton-Jacobi approach is the bridge between the influence curves and the space like deformations.

In the constant coefficient hyperbolic case, the convex hull of the support of the forward fundamental solution E defined by

$$L(\partial)\, E = \delta\,, \qquad \operatorname{supp} E \subset \{t \ge 0\}\,.$$

is equal to Γ^+ (see [Gå, Ho3]). The fact that it is contained in Γ^+ implies that in the constant coefficient hyperbolic case the set swept out by all influence curves starting in S_0 is a domain of influence. The set Ω defined by $(\underline{t}, \underline{x}) \in \Omega$ if and only if no influence curve $x : [0,t] \to \mathbb{R}^d$ satisfies both

$$x(0) \in S_0\,, \qquad \text{and} \qquad x(\underline{t}) = \underline{x}\,,$$

is a domain of determinacy of Ω_0 since Ω is the complement of S.

To prove a variable coefficient analogue, we need to know that $\tau_{\max}$ is lipschitzean. This is easy to prove in the symmetric hyperbolic case and is true in general.

Bronstein's Theorem [B,W]. *Hypothesis 0.1 implies that the function $\tau_{\max}(t,x,\xi)$ is uniformly lipschitzian on $\mathbb{R}^{1+d} \times \{|\xi| = 1\}$.*

The natural candidate $\Omega \subset [0,\infty[\times\mathbb{R}^d$ for a large domain of determinacy of Ω_0 is

$$\Omega \;:=\; \Big\{ (\underline{t}, \underline{x}) \;:\; \text{no influence curve with } \; x(0) \in S_0 \; \text{ can satisfy } \; x(\underline{t}) = \underline{x} \Big\}\,. \tag{0.10}$$

Theorem 0.1. *If Hypotheses* 0.1 *and* 0.2 *are satisfied, $\psi(x) \in W^{1,\infty}(\mathbb{R}^d)$ vanishes on S_0, and is strictly positive on Ω_0, then the set Ω from* (0.10) *is exactly the set $\big\{(t,x) \in [0,\infty[\times\mathbb{R}^d : \Psi(t,x) > 0\big\}$ where $\Psi \in \cap_T W^{1,\infty}([0,T] \times \mathbb{R}^d)$ is the largest uniformly lipschitzian solution of the Hamilton-Jacobi initial value problem*

$$\Psi_t + \tau_{\max}(t, x, -\nabla_x \Psi(t,x)) = 0 \quad \text{a.e. } \; (t,x)\,, \qquad \Psi(0,x) = \psi(x)\,.$$

Theorem 0.2. *If Hypotheses* 0.1 *and* 0.2 *are satisfied, then the natural* Ω *defined in* (0.10) *is a domain of determinacy of* Ω_0.

Theorem 0.3. *If Hypotheses* 0.1 *and* 0.2 *are satisfied and* $(\underline{t}, \underline{x})$ *belongs to the natural* Ω *defined in* (0.10), *then there is a deformation by spacelike hypersurfaces as in John's Theorem so that* $(\underline{t}, \underline{x}) \in F^{-1}([0, 1[)$.

Theorem 0.3 together with John's Theorem proves Theorem 0.2.

A proof of Theorem 0.2 in the strictly hyperbolic case is given in [Le]. It uses a result of Marchaud [M] asserting that if Z is a closed set in $\{t \geq 0\}$ with the property that for each point in Z there is a backward semitangent belonging to $-\Gamma^+(t, x)$ then through each point $(\underline{t}, \underline{x}) \in Z$ with $\underline{t} > 0$ there is a backward influence curve belonging to Z and reaching $t = 0$. The outline of proof in [La1] for the symmetric hyperbolic case is not quite complete. It can be completed by appealing to the above result of [M]. Appeal to the long and technical article [M] can be circumvented by proving the result cited above from scratch. Our use of Hamilton-Jacobi equations not only justifies the natural description of Theorem 0.1, but avoids recourse to [M], applies in the general case where local uniqueness in the Cauchy problem is known, and also yields Theorem 0.3.

As far as we know, Theorem 0.3 is nowhere suggested in the literature.

§1. Hamilton-Jacobi.

Let $\psi(x)$ be a uniformly lipschitzian function which is strictly positive on the nonempty open set Ω_0 and vanishes on the nonempty complement $S_0 := \mathbb{R}^d \setminus \Omega_0$. For example, $\psi(x) := \text{dist}\{x\,,\, S_0\}$. An example which tends to zero as $|x| \to \infty$ is $\psi(x) := e^{-|x|}\,\text{dist}\{x\,,\, S_0\}$.

Definitions. *Denote by* $\mathcal{X}(T, X)$ *the set of forward influence curves* $x(t) : [0, T] \to \mathbb{R}^d$ *with*

$$x(T) = X\,. \tag{1.1}$$

Define a function $\Psi(T, X)$ *in* $T \geq 0$ *by*

$$\Psi(T, X) \;=\; \inf\; \Big\{ \psi(x(0)) \;:\; x(\cdot) \in \mathcal{X}(T, X) \Big\}\,. \tag{1.2}$$

The infinum in (1.2) is an achieved minimum.

Corollary 1.1 *The set described in (0.10) is exactly the set* $\{\Psi > 0\}$. *The complementary set* $S := \big([0, \infty[\times \mathbb{R}^d\big) \setminus \Omega$ *is equal to* $\{\Psi = 0\}$.

Theorem 1.2. *If Hypotheses* 0.1 *and* 0.2 *are satisfied and* $T > 0$, *then* Ψ *is uniformly lipschitzian on* $[0, T] \times \mathbb{R}^d$ *and satisfies the Hamilton-Jacobi initial value problem*

$$\partial_t \Psi + \tau_{\max}(t, x, -\nabla_x \Psi) \;=\; 0 \;\; a.e.\,, \qquad \Psi(0, x) = \psi(x)\,. \tag{1.3}$$

It is the largest solution in the sense that if $\ell(x) \in W^{1,\infty}_{\text{loc}}\big([0, T] \times \mathbb{R}^d\big)$ *and satisfies*

$$\partial_t \ell + \tau_{\max}(t, x, -\nabla_x \ell) \;\leq\; 0 \;\; \text{a.e.}\,, \qquad \ell(0, x) \leq \psi(x)\,, \tag{1.4}$$

then,

$$\ell(t,x) \le \Psi(t,x) \quad \text{on} \quad [0,T]\times\mathbb{R}^d\,. \tag{1.5}$$

The proof of this is on one hand standard Hamilton-Jacobi Theory, and on the other hand unfamiliar to a large part of the audience of this seminar. A self contained treatment strongly influenced by [Li] can be found in the detailed version [JMR].

§2. Natural domains by spacelike deformations.

The method uses perturbations of the function Ψ. The idea is that the level sets $\{\Psi = c > 0\}$ with c decreasing from the maximum value of Ψ, almost give a smooth deformation by spacelike hypersurfaces sweeping out the natural domain of determinacy Ω from (0.10). Since the level sets of $W^{1,\infty}$ functions are ill behaved, the proof in this section uses regularization of Ψ.

In simple cases, it is clear what regions can be swept out with the constraint of remaining spacelike. In the general case we were surprised and pleased to find that the natural set Ω can be reached by such deformations.

The example of Ω_0 equal to a dumbbell shaped region in Figure 2.1 suggests some of the pitfalls. Take ψ to be equal to the distance from the boundary of Ω. Consider the case of D'Alembert's wave equation in which case $\Psi = \psi(x) - t$.

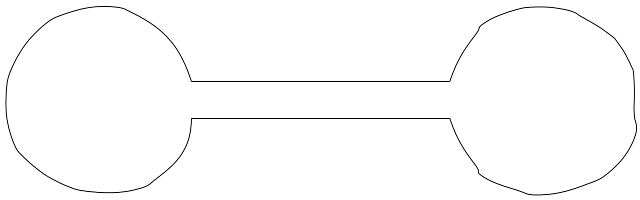

Figure 2.1

The level sets $\{\Psi = c\}$ with c small positive are dumbbell shaped for t small and then for larger t pinch off into two cusped circles. The difficulties are both the cusps and the change of topology. If one puts C^∞ wobbles in the connecting tube of the dumbbell, there can be a countable number of little bubbles pinched off in the tube. We sweep out the set $\{\Psi > \mu\}$ with $\mu > 0$ small. This strict positivity allows us just enough wiggle room to regularize the geometry.

Theorem 0.2 follows from the next general principal applied with Φ equal to the function Ψ from §1 when the initial data ψ are chosen satisfying $\lim_{|x|\to\infty}\psi(x) = 0$.

Theorem 2.1 *If L satisfies Hypotheses 0.1 and 0.2, $\Phi \in W^{1,\infty}\big([0,T]\times\mathbb{R}^d\big)$ for all $T > 0$ and satisfies*

$$\Phi_t + \tau_{\max}\big(t,x,-\nabla_x\Phi(t,x)\big) \;\le\; 0 \;\text{ a.e.}\,, \qquad \lim_{|x|\to\infty}\Phi(0,x) = 0\,, \tag{2.1}$$

and $u \in H^{m-1}_{\rm loc}\big([0,\infty[\times\mathbb{R}^d\big)$ satisfies

$$Lu \;=\; 0\,, \qquad \text{and} \qquad \Phi(0,x) \le 0 \quad \text{for} \;\; x \in \cup_{j=0}^{m-1}\,\text{supp}\,\partial_t^j u(0,.)\,. \tag{2.2}$$

then for all $|\alpha| \le m-1$,

$$\Phi(t,x) \le 0 \quad \text{on} \;\; \text{supp}\;\partial^\alpha_{t,x}u\,. \tag{2.3}$$

Theorem 2.1 in turn is proved using the next Lemma which establishes the key link between Hamilton-Jacobi equations and the method of deforming spacelike hypersurfaces.

Lemma 2.2. Spacelike deformations. *Suppose that* $\Phi \in \big(C^1 \cap W^{1,\infty}\big)\big([0,T] \times \mathbb{R}^d\big)$ *for all* $T > 0$ *satisfies*

$$\Phi_t + \tau_{\max}\big(t, x, -\nabla_x \Phi(t,x)\big) \ < \ 0\,, \qquad \limsup_{|t,x| \to \infty} \ \Phi(t,x) \le 0\,, \tag{2.4}$$

and $u \in H^{m-1}_{\rm loc}\big([0,\infty[\times\mathbb{R}^d\big)$ *satisfies*

$$Lu \ = \ 0\,, \qquad \text{and} \qquad \Phi(0,x) \le 0 \quad \text{for} \ \ x \in \cup_{j=0}^{m-1} \ \text{supp}\, \partial_t^j u(0,.)\,. \tag{2.5}$$

then for all $|\alpha| \le m-1$,

$$\Phi(t,x) \le 0 \quad \text{on} \ \ \text{supp}\ \partial^\alpha_{t,x} u\,, \tag{2.6}$$

Remark. The key additional hypotheses are the continuous differentiability of Φ and the strict Hamilton-Jacobi inequality.

Proof of Lemma 2.2. If $\Phi \le 0$ there is nothing to prove. Since Φ is nonpositive at infinity, if Φ assumes positive values it attains its maximum value. The differential inequality (2.4) shows that Φ has no critical points in $T > 0$. Thus the maximum value, $\Phi_{\max}$ is assumed only in $\{t = 0\}$.

It suffices to show that u vanishes wherever $\Phi > 0$. Therefore it suffices to show that u vanishes on $\Phi^{-1}([\epsilon, \Phi_{\max}]$ for arbitrary $\epsilon > 0$. Fix $0 < \epsilon < \Phi_{\max}$.

Let

$$\mathcal{O} \ := \ \{t \ge 0\}\,, \qquad F(t,x) \ := \ \frac{\Phi(t,x) - \Phi_{\max}}{\epsilon - \Phi_{\max}}\,. \tag{2.7}$$

Then as Φ decreases from $\Phi_{\max}$ to ϵ, F increases from 0 to 1. It suffices to show that u vanishes on $F^{-1}([0,1])$. It suffices to verify the hypotheses of John's Global Hölmgren Uniqueness Theorem.

The assertion about $\Phi_{\max}$ proves property **i.**

Property **ii.** is a consequence of the fact that $\{F \ge 0\} = \{\Phi \ge \epsilon\}$. The latter is compact thanks to the second part of (2.4).

Property **iii.** follows from the differenital inequality (2.4).

That the Cauchy data of u vanish on $\{F \ge 0\} \cap \{t = 0\} = \{\Phi \ge \epsilon\} \cap \{t = 0\}$ follows from the second part of (2.5). That $Lu = 0$ is the first part of (2.5) which completes the verification of the hypotheses of John's Theorem. ∎

Proof of Theorem 2.1. Replacing Φ by a larger function satisfying the conditions of the Theorem and with the same initial data, strengthens the conclusion (2.3). Thus it suffices to prove the Theorem for the largest such function Φ. Theorem 2.3 shows that this largest function is given by the formula

$$\Phi_{\rm upper}(t,x) \ := \ \min_{x(\cdot) \in \mathcal{X}(t,x)} \Big\{\Phi(0, x(0))\Big\}\,. \tag{2.8}$$

Then $\Phi_{\rm upper} \in W^{1,\infty}([0,\infty[\times\mathbb{R}^d)$ satisfies

$$\partial_t \Phi_{\rm upper} + \tau_{\max}\big(t, x, -\nabla_x \Phi_{\rm upper}(t,x)\big) \ = \ 0\,, \qquad \Phi_{\rm upper}(0,x) = \Phi(0,x)\,, \tag{2.9}$$

and is the largest such solution. Replace Φ by $\Phi_{\rm upper}$ and drop the subscript.

If (T,X) with $T>0$ satisfies $\Phi(T,X)>0$, we must show that u vanishes on a neighborhood of (T,X). If there are no such points there is nothing to prove. Fix (T,X) with $T>0$ and $\Phi(T,X)>0$.

With $0<\delta$ define

$$\Phi^\delta \ := \ \Phi - \delta\, t\,. \tag{2.10}$$

The Hamilton-Jacobi equation for Φ is equivalent to

$$\Phi^\delta_t + \tau_{\max}(t,x,-\nabla_x\Phi^\delta) \ = \ -\delta\,, \qquad \Phi^\delta(0,x)=\Phi(0,x)\,. \tag{2.11}$$

Fix $0<\delta$ so small that

$$\Phi^\delta(\underline{T},\underline{X})>0\,. \tag{2.12}$$

The second assertion in (2.1) together with formulas (2.8) and (2.10) imply that

$$\lim_{|t,x|\to\infty} \ \Phi^\delta(t,x) \ \le \ 0\,. \tag{2.13}$$

Regularize to construct

$$\Phi^{\epsilon,\delta} \ := \ J_\epsilon(\Phi^\delta) \ := \ \int\int \epsilon^{(-1-d)}\,\rho\Big(\frac{(t,x)-(s,y)}{\epsilon}\Big) \ \Phi^\delta(s,y)\ ds\,dy \ \in \ C^\infty([0,\infty[\times\mathbb{R}^d)\,.$$

Equations (2.12) and (2.13) imply that for ϵ small and positive

$$\lim_{|t,x|\to\infty} \ \Phi^{\epsilon,\delta}(t,x) \ \le \ 0 \quad \text{and} \quad \Phi^{\epsilon,\delta}(T,X)>0\,. \tag{2.14}$$

Carefully commute with J_ϵ (using the convexity of $\tau_{\max}$) to get one sided estimates

$$\Big\|\Phi^{\epsilon,\delta}-\Phi^\delta\Big\|_{L^\infty([0,T]\times\mathbb{R}^d)} \ < \ C\,\epsilon\,, \quad \text{and,} \quad \Phi^{\epsilon,\delta}_t + \tau_{\max}(t,x,-\nabla_x\Phi^{\epsilon,\delta}) \ \le \ -\delta + C\,\epsilon\,. \tag{2.15}$$

In addition

$$\sup\Big\{\Phi^{\epsilon,\delta}(0,x) \ : \ x\in\cup_{j\le m-1}\,\mathrm{supp}\,\partial^j u(0,x)\Big\} \ \le C\ \epsilon\,. \tag{2.16}$$

Thus for ϵ small and positive

$$\Phi^{\epsilon,\delta}_t + \tau_{\max}(t,x,-\nabla_x\Phi^{\epsilon,\delta}) < -\delta/2\,, \tag{2.17}$$

and

$$\sup\Big\{\Phi^{\epsilon,\delta}(0,x) \ : \ x\in\cup_{j\le m-1}\,\mathrm{supp}\,\partial^j u(0,x)\Big\} \ < \Phi^{\epsilon,\delta}(T,X)/2. \tag{2.18}$$

Theorem 2.1 then follows from Lemma 2.2 applied to the function $\Phi^{\epsilon,\delta}-\Phi^{\epsilon,\delta}(T,X)/2$ with ϵ small and positive. ∎

References

[B] M.D. Bronshtein, Smoothness of polynomials depending on parameters, Siberian Mat. Zh. **20**(1979)493-501 (Russian). English transl. Siberian Math. J. 20(1980)347-352.

[Co] R. Courant, *Methods of Mathematical Physics vol II*, Interscience Publ., 1962.

[FS] S. Fomel and J. Sethian, Fast phase space computation of multiple arrivals, Proc. Nat. Acad. Sci. 99(2002), 7329-7334.

[FrLa] K. O. Friedrichs and P. D. Lax, Systems of conservation laws with a convex extension, Proc. Nat. Acad. Sci. USA **68**(1971), 1686-1688.

[Gå] L. Gårding, Linear hyperbolic partial differential operators with constant coefficients, Acta Math. 85(1951), 1-62.

[Go1] S. K. Godunov , An interesting class of quasilinear systems. Dokl. Akad. Nauk SSSR 139 (1972), p. 521.

[Go2] S. K. Godunov, Symmetric form of the equations of magnetohydrodynamics. Numerical Methods for Mechanics of Continuum Medium 1 (1972), p. 26.

[Ha] A. Haar, Über eindeutigkeit und analytizität der lösungen partieller differenzialgleichungen, Atti del Congr. Intern. dei Mat., Bologna 3(1928), 5-10.

[Ho1] L. Hörmander, Uniqueness theorems and estimates for normally hyperbolic partial differential equations of the second order, Tolfte Skandinaviska Matematikerkongressen, Lunds Universitets Matematiska Institution, Lund 1954, pp 105-115.

[Ho2] L. Hörmander, *The Analysis of Linear Partial Differential Operators vol. I* (§9.6), Grundlehren der mathematischen Wissenschaften #256, Springer-Verlag 1983.

[Ho3] L. Hörmander, *The Analysis of Linear Partial Differential Operators vol. II*, Grundlehren der mathematischen Wissenschaften #257, Springer-Verlag 1983.

[JMR] J.-L. Joly, G. Métivier, and J.Rauch, Hyperbolic Domains of Determination and Hamilton-Jacobi Theory, preprint, (`www.math.lsa.umich.edu/~rauch`).

[Jo] F. John, On linear partial differential equations with analytic coefficients, Unique continuation of data, C.P.A.M. 2(1949), 209-253.

[La1] P. Lax, *Lectures on Hyperbolic Partial Differential Equations*, Stanford, 1963.

[La2] P. Lax, Shock waves amd entropy, in *Contributions to Nonlinear Functional Analysis* ed. E. Zarantonello, Academic Press NY 1971, 603-634.

[Le] J. Leray, *Hyperbolic Differential Equations*, Institute for Advanced Study, 1953.

[Li] P.-L. Lions, *Generalized Solutions of Hamilton-Jacobi Equations* Pittman Lecture Notes, 1982.

[M] A. Marchaud, Sur les champs continus de demi cônes convexes et leurs intégrales, Compositio Math. **2**(1936)89-127.

[Mo] C. B. Morrey Jr., *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, 1966.

[Ra] J. Rauch, *Partial Differential Equations*, Graduate Texts in Mathematics #128, Springer-Verlag, 1991.

[RMO] S. Ruuth, B. Merriman, and S. Osher, A fixed grid method for capturing the motion of self-intersecting wavefronts and related PDEs, J.Comp.Phys. 163(2000), 1-21.

[SFW] J. Steinfhoff, M. Fan, and L. Wang, A new Eulerian method for the computation of propagating short acoustic and electromagnetic pulses, J. Comp.Phys. 157(2000), 683-706.

[W] S. Wakabayashi, Remarks on hyperbolic polynomials, Tsukuba J. Math. **10**(1986), 17-28.