Sahbi Keraani

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ON THE BLOWUP THEORY FOR THE CRITICAL NONLINEAR SCHRÖDINGER EQUATIONS

SAHBI KERAANI

1. INTRODUCTION

In this talk we prove a refined version of compactness lemma adapted to the blowup analysis and we use it to give direct and simple proofs to some classical results of blowup theory for critical nonlinear Schrödinger equations. It's based on a joint work with T. Hmidi.

We consider the L^2-critical nonlinear Schrödinger equation (NLS):

$$(1) \qquad \begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{d}}u = 0, & x \in \mathbb{R}^d,\ t > 0, \\ u(0,x) = u_0(x). \end{cases}$$

Here, $\Delta = \sum_{i=1}^d \partial_{x_i}^2$ is the Laplace operator on $\mathbb{R}^d$ and $u_0 : \mathbb{R}^d \longrightarrow \mathbb{C}$. It is well known from the result by Ginibre and Velo [5] (see [3] for a review) that Cauchy problem (1) is locally well-posed in H^1: there exists $T \in (0, +\infty]$ and a solution $u \in \mathcal{C}([0,T), H^1)$, with the following blowup alternative: either $T = \infty$ (the solution is global) or $T < +\infty$ (the solution blows up in finite time) and

$$\lim_{t \uparrow T} \|\nabla u(t,\cdot)\|_{L^2} = +\infty.$$

The unique solution has the following quantities conserved as t varies

$$\begin{aligned} \mathcal{N}(t) &= \int_{\mathbb{R}^d} |u(t,x)|^2 dx, \\ E(t) &= \frac{1}{2}\int |\nabla u|^2 dx - \frac{d}{4+2d}\int |u|^{\frac{4}{d}+2} dx \end{aligned}$$

Also, if $u_0 \in \Sigma := \{f \in H^1, xf \in L^2\}$, then the solution satisfies the *Virial* identity (see [6])

$$(2) \qquad \frac{d^2}{dt}\int |x|^2 |u(t,x)|^2 dx = 16 E(0).$$

Obviously, if $E(0) < 0$ then the solution cannot exist globally and blows up in finite time. This was the starting point of the blowup theory of Schrödinger equations which has been developed in the two last decays (see [3], [14], [12] and the references therein). This theory is mainly connected to the notion of ground state: the unique positive radial solution of the elliptic problem

$$\Delta Q - Q + |Q|^{\frac{4}{d}}Q = 0.$$

In [16], M. I. Weinstein exhibited the following refined Gagliardo-Nirenberg inequality

$$(3) \qquad \|\psi\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \leq C_d \|\psi\|_{L^2}^{\frac{4}{d}} \|\nabla\psi\|_{L^2}^2 \qquad \forall \psi \in H^1,$$

with $C_d = \frac{d+2}{d}\|Q\|_{L^2}^{-\frac{4}{d}}$. Combined with the conservation of energy, the inequality above implies that $\|Q\|_{L^2}$ is the critical mass for the formation of singularities, that is if

$$\|u_0\|_{L^2} < \|Q\|_{L^2}$$

then the corresponding solution is global. Also, this bound is optimal since by using the conformal invariance one constructs

$$u(t,x) = (T-t)^{-d/2}e^{[(i/(T-t))+(-i|x|^2/T-t)]}Q(\frac{x}{T-t})$$

a singular solution of (1) with $\|u\|_{L^2} = \|Q\|_{L^2}$ that blows up in a finite time T. There is an abundant literature devoted to the study of the blowup mechanism (see [3] and [14] for a review). In this talk we prove a compactness lemma adapted to the analysis of the blowup phenomenon of the nonlinear Schroödinger equation and use it to give elementary proofs to some classical results in the field.

2. Compactness lemma

The main result of this section is

Theorem 1. *Let $\{v_n\}_{n=1}^\infty$ be a bounded family of $H^1(\mathbb{R}^d)$, such that*

$$(4) \qquad \limsup_{n\to\infty} \|\nabla v_n\|_{L^2} \leq M \qquad and \qquad \limsup_{n\to\infty} \|v_n\|_{L^{\frac{4}{d}+2}} \geq m.$$

Then, there exists $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^d$ such that, up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup V \qquad weakly,$$

with $\|V\|_{L^2} \geq (\frac{d}{d+2})^{d/4}\frac{m^{\frac{d}{2}+1}}{M^{d/2}}\|Q\|_{L^2}$.

Remark 2. *The lower-bound on the L^2 norm of V is optimal. In fact, if we take $v_n = Q$ then we get equality.*

Proof. In the sequel we put $2^* = \infty$ if $d = 1,2$, and $2^* = \frac{2d}{d-2}$ if $d \geq 3$. Theorem 1 is a consequence of a profile decomposition of the bounded sequences in H^1 following the work by P. Gérard [4] (see also [1] and [7]). More precisely, we have the following[1]

Proposition 3. *Let $\mathbf{v} = \{v_n\}_{n=1}^\infty$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Then, there exist a subsequence of $\{v_n\}_{n=1}^\infty$ (still denoted $\{v_n\}_{n=1}^\infty$), a family $\{\mathbf{x}^j\}_{j=1}^\infty$ of sequences in $\mathbb{R}^d$ and a sequence $\{V^j\}_{j=1}^\infty$ of H^1 functions, such that*

    i) *for every $k \neq j$,* $|x_n^k - x_n^j| \xrightarrow[n\to\infty]{} +\infty$;

    ii) *for every $\ell \geq 1$ and every $x \in \mathbb{R}^d$, we have*

$$v_n(x) = \sum_{j=1}^\ell V^j(x - x_n^j) + v_n^\ell(x),$$

        with

$$(5) \qquad \limsup_{n\to\infty} \|v_n^\ell\|_{L^p(\mathbb{R}^d)} \xrightarrow[\ell\to\infty]{} 0$$

        for every $p \in]2, 2^*[$.

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[1]The proof of this proposition is similar to the proof of Proposition 2.6 in [7].

Moreover, we have, as $n \to +\infty$,

$$\text{(6)} \qquad \|v_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|V^j\|_{L^2}^2 + \|v_n^{\ell}\|_{L^2}^2 + o(1),$$

and

$$\text{(7)} \qquad \|\nabla v_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\nabla V^j\|_{L^2}^2 + \|\nabla v_n^{\ell}\|_{L^2}^2 + o(1).$$

Let us finish the proof of Theorem 1. By extracting a subsequence we may replace $\limsup$ in the assumptions (4) by $\lim$. According to Proposition 3, the sequence $\{v_n\}_{n=1}^{\infty}$ can be written, up to a subsequence, as

$$v_n(x) = \sum_{j=1}^{\ell} V^j(x - x_n^j) + v_n^{\ell}(x)$$

such that (5) and (6) hold. This implies, in particular,

$$m^{\frac{4}{d}+2} \leq \limsup_{n\to\infty} \|v_n\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = \limsup_{n\to\infty} \|\sum_{j=1}^{\infty} V^j(\cdot - x_n^j)\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$$

The elementary inequality

$$\|\sum_{j=1}^{l} a_j|^{4/d+2} - \sum_{j=1}^{l} |a_j|^{4/d+2}| \leq C \sum_{j\neq k} |a_j||a_k|^{4/d+1}.$$

and the pairwise orthogonality of the family $\{\mathbf{x}^j\}_{j=1}^{\infty}$ leads the mixed terms in the sum above to vanish and we get

$$m^{\frac{4}{d}+2} \leq \sum_{j=1}^{\infty} \|V^j\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$$

On the one hand, in view of Gagliardo-Nirenberg inequality (3), we have

$$\sum_{j=1}^{\infty} \|V^j\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \leq C_d \sup\{\|V^j\|_{L^2}^{4/d}, j \geq 1\} \sum_{j=1}^{\infty} \|\nabla V^j\|_{L^2}^2.$$

On the other hand, from (5), we get

$$\sum_{j=1}^{\infty} \|\nabla V^j\|_{L^2}^2 \leq \limsup_{n\to\infty} \|\nabla v_n\|_{L^2}^2 \leq M^2.$$

Therefore,

$$\sup_{j\geq 1} \|V^j\|_{L^2}^{4/d} \geq \frac{m^{\frac{4}{d}+2}}{(M^2 C_d)^{d/4}}.$$

Since the series $\sum \|V^j\|_{L^2}^2$ converges then the supremum above is attained. In particular, there exists j_0, such that

$$\|V^{j_0}\|_{L^2} \geq \frac{m^{\frac{d}{2}+1}}{(C_d M^2)^{d/4}} = (\frac{d}{d+2})^{d/4} \frac{m^{\frac{d}{2}+1}}{M^{d/2}} \|Q\|_{L^2}.$$

On the other hand, a change of variables gives

$$v_n(x + x_n^{j_0}) = V^{j_0}(x) + \sum_{\substack{1 \le j \le \ell \\ j \ne j_0}} V^j(x + x_n^{j_0} - x_n^j) + \tilde{v}_n^\ell(x),$$

where $v_n^\ell(x) = \tilde{v}_n^\ell(x + x_n^{j_0})$. The pairwise orthogonality of the family $\{\mathbf{x}^j\}_{j=1}^\infty$ implies

$$V^j(\cdot + x_n^{j_0} - x_n^j) \rightharpoonup 0 \quad \text{weakly,}$$

for every $j \ne j_0$. Hence, we get

$$v_n(\cdot + x_n^{j_0}) \rightharpoonup V^{j_0} + \tilde{v}^\ell,$$

where $\tilde{v}^\ell$ denote the weak limit of $\{\tilde{v}_n^\ell\}_{n=1}^\infty$. However, we have

$$\|\tilde{v}^\ell\|_{L^{\frac{4}{d}+2}} \le \limsup_{n\to\infty} \|\tilde{v}_n^\ell\|_{L^{\frac{4}{d}+2}} = \limsup_{n\to\infty} \|v_n^\ell\|_{L^{\frac{4}{d}+2}} \xrightarrow[l\to\infty]{} 0.$$

Thereby, by uniqueness of weak limit, we get

$$\tilde{v}^\ell = 0$$

for every $\ell \ge j_0$. So that

$$v_n(\cdot + x_n^{j_0}) \rightharpoonup V^{j_0}.$$

The sequence $\{x_n^{j_0}\}_{n=1}^\infty$ and the function V^{j_0} fulfill the conditions of Theorem 1.

$\square$

3. BLOWUP THEORY REVISITED

3.1. **Concentration.** For $d \ge 2$ and spherically symmetric blowup solutions, it has been shown that there is a minimal amount of concentration of the L^2 norm at the origin (see [13],[15], [18] and [3]) . Below we give a direct proof for the general case.

Theorem 4. *Let u be a solution of (1) which blows up at finite time $T > 0$, and $\lambda(t) > 0$ any function, such that $\|\nabla u(t)\|_{L^2}\lambda(t) \longrightarrow +\infty$ as $t \uparrow T$. Then, there exists $x(t) \in \mathbb{R}^d$, such that*

$$\liminf_{t\uparrow T} \int_{|x-x(t)|\le\lambda(t)} |u(t,x)|^2 dx \ge \int Q^2.$$

Remark 5. *A well known scaling argument yields the following lower bound on the blowup rate*

$$\|\nabla u(t_n,\cdot)\|_{L^2} \ge \frac{C}{\sqrt{T-t}}.$$

Thus, any function $\lambda(t) > 0$, such that $\frac{\sqrt{T-t}}{\lambda(t)} \longrightarrow 0$ as $t \uparrow T$, fulfills the conditions of this theorem.

Proof. In the sequel we will use the following notations:

$$\rho(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u(t,\cdot)\|_{L^2}} \qquad \text{and} \qquad v(t,x) = \rho^{d/2}u(t,\rho(t)x).$$

Let $\{t_n\}_{n=1}^\infty$ be an arbitrary sequence such that $t_n \uparrow T$. We set $\rho_n = \rho(t_n)$ and $v_n = v(t_n,\cdot)$. Since u conserves its mass, the sequence $\{v_n\}_{n=1}^\infty$ satisfies

$$\|v_n\|_{L^2} = \|u_0\|_{L^2} \qquad \text{and} \qquad \|\nabla v_n\|_{L^2} = \|\nabla Q\|_{L^2}.$$

Furthermore, by conservation of the energy and blowup criteria, it ensues that

$$E(v_n) = \rho_n^2 E(0) \longrightarrow 0, \qquad \text{as} \quad n \to \infty,$$

which yields, in particular,

$$\|v_n\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \longrightarrow \frac{d+2}{d}\|\nabla Q\|_{L^2}^2, \qquad \text{as} \quad n \to \infty.$$

The family $\{v_n\}_{n=1}^\infty$ satisfies the conditions of Theorem 1 above with

$$m^{\frac{4}{d}+2} = \frac{d+2}{d}\|\nabla Q\|_{L^2}^2 \qquad \text{and} \qquad M^2 = \|\nabla Q\|_{L^2}^2.$$

Thus, *there exists $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^d$ such that, up to a subsequence,*

$$\rho_n^{d/2} u(t_n, \rho_n \cdot + x_n) \rightharpoonup V \in H^1 \qquad \qquad weakly$$

with $\|V\|_{L^2} \geq \|Q\|_{L^2}$.[2] From this, it follows that

$$\liminf_{n \to +\infty} \int_{|x| \leq A} \rho_n^d |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x| \leq A} |V|^2 dx,$$

for every $A > 0$. Thus,

$$\liminf_{n \to +\infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq A\rho_n} |u(t_n, x)|^2 dx \geq \int_{|x| \leq A} |V|^2 dx.$$

Since $\frac{\rho_n}{\lambda(t_n)} \longrightarrow 0$, it ensues that

$$\liminf_{n \to +\infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq \lambda(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x| \leq A} |V|^2 dx$$

for every A, which means that

$$\liminf_{n \to +\infty} \sup_{y \in \mathbb{R}^d} \int_{\{|x-y| \leq \lambda(t_n)\}} |u(t_n, x)|^2 dx \geq \int_{\mathbb{R}^d} |V|^2 dx \geq \int Q^2.$$

Since the sequence $\{t_n\}$ is arbitrary we get finally

$$\liminf_{t \to T} \sup_{y \in \mathbb{R}^d} \int_{\{|x-y| \leq \lambda(t)\}} |u(t, x)|^2 dx \geq \int Q^2.$$

Since, for every t, the function $y \longmapsto \int_{\{|x-y| \leq \lambda(t)\}} |u(t, x)|^2 dx$ is continuous and goes to 0 at infinity, then there exists a family $x(t)$ such that

$$\sup_{y \in \mathbb{R}^d} \int_{\{|x-y| \leq \lambda(t)\}} |u(t, x)|^2 dx = \int_{\{|x-x(t)| \leq \lambda(t)\}} |u(t, x)|^2 dx,$$

which concludes the proof of Theorem 4. $\square$

3.2. **Universality of the profile with critical mass.**

If, in the context of the proof of Theorem 4 we assume also that $\|u_0\|_{L^2} = \|Q\|_{L^2}$, we get

$$\|V\|_{L^2} = \|Q\|_{L^2}.$$

Thus, $\|v_n\|_{L^2} = \|V\|_{L^2}$, which means

$$v_n(\cdot + x_n) \to V \qquad \text{strongly in} \qquad L^2.$$

Also, since it's bounded in H^1, we have

$$v_n(\cdot + x_n) \to V \qquad \text{strongly in} \qquad L^{\frac{4}{d}+2}.$$

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[2]Note that this asymptotic is proved by Weinstein [17] via Concentration-Compactness Lemma by Lions [9].

In view of Gagliardo-Nirenberg inequality, this leads to
$$\|\nabla V\|_{L^2} \geq \|\nabla Q\|_{L^2}.$$
Since $\|\nabla V\|_{L^2} \leq \limsup \|\nabla v_n\|_{L^2} = \|\nabla Q\|_{L^2}$, then
$$\|\nabla v_n\|_{L^2} \longrightarrow \|\nabla V\|_{L^2}.$$
This means that the strong convergence holds in H^1 and $E(V) = 0$. Let us summarize the properties of the profile V:
$$V \in H^1, \quad \|V\|_{L^2} = \|Q\|_{L^2}, \quad \|\nabla V\|_{L^2} = \|\nabla Q\|_{L^2} \quad \text{and} \quad E(V) = 0.$$
The variational characterization of the ground state implies that
$$V(x) = e^{i\theta} Q(x + x_0),$$
for some $\theta \in [0, 2\pi[$ and $x_0 \in \mathbb{R}^d$. This result, which is due to Weinstein [17] and Kwong [8], can be rewritten as follows: *if u is a singular solution with critical mass then there exist $x(t)$ and $\theta(t)$, such that*

(8) $$(\rho(t))^{d/2} e^{i\theta(t)} u(t, \rho(t)x + x(t)) \longrightarrow Q, \qquad as \quad t \to T,$$

strongly in H^1.

3.3. **Determination of the singular solutions with minimal mass.** In the sequel we need the following notation:
$$\mathcal{A} = \{\rho^{d/2} e^{i\theta} Q(\rho x + y), \quad y \in \mathbb{R}^d, \rho \in \mathbb{R}_*^+, \theta \in [0, 2\pi[\}.$$
The characterization of the singular solutions with minimal mass is due to F. Merle [10]. Below we give a direct and short proof to this fundamental result.

Theorem 6. *Let u be a blowing up solution of (1) at finite time $T > 0$ such that $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Then there exists $x_0 \in \mathbb{R}^d$ such that $e^{i\frac{|x-x_0|^2}{4T}} u_0 \in \mathcal{A}$.*

Proof. Let $t_n \to T$ be an arbitrary sequence. It is clear that (8) implies
$$|u(t_n, x)|^2 dx - \|Q\|_{L^2}^2 \delta_{x=x_n} \rightharpoonup 0.$$
Up to extract a subsequence and translation, one assumes $x_n \to x_0 \in \{0, \infty\}$. Let ϕ be a nonnegative radial $C_0^\infty(\mathbb{R}^d)$ function, such that
$$\phi(x) = |x|^2, \quad \text{if} \quad |x| < 1 \quad \text{and} \quad |\nabla\phi(x)|^2 \leq C\phi(x).$$
For every $p \in \mathbb{N}^*$ one defines
$$\phi_p(x) = p^2 \phi(\frac{x}{p}) \quad \text{and} \quad g_p(t) = \int \phi_p(x) |u(t, x)|^2 dx.$$
Using the Cauchy-Schwartz estimates by V. Banica [2][3], we get
$$|\dot{g}_p(t)| = |2\Im \int \bar{u}(x) \nabla u(x) \nabla \phi_p(x) dx| \leq \left(8E(u_0) \int |u|^2 |\nabla\phi_p|^2 dx\right)^{1/2}$$
$$\leq C_0\sqrt{g_p(t)},$$

[3]The argument of Banica is as follows: since u has critical mass and ϕ_p is a real valued function then $E(e^{is\phi_p}u) \geq 0$, for every $s \in \mathbb{R}$. The Cauchy-Schwartz estimates is the non positivity of the discriminant of the polynomial $s \mapsto E(e^{is\phi_p}u)$.

for every $t \in [0, T[$. In the last line we have used the inequality $|\nabla \phi_p|^2 \leq C\phi_p$. By integration we obtain, for every $t \in [0, T[$,

$$|\sqrt{g_p(t)} - \sqrt{g_p(t_n)}| \leq C_0 |t_n - t|.$$

We let n go to infinity; and we get (since $\phi_p(x_0) = 0$ for both finite and infinite case)

$$g_p(t) \leq C_0 (T - t)^2.$$

We fix $t \in [0; T[$ and let p go to infinity[4] to obtain

$$(9) \qquad 8t^2 E(e^{i\frac{|x|^2}{4t}} u_0) = \int |x|^2 |u(t, x)|^2 dx \leq C(u_0)(T - t)^2.$$

The first identity is just another way of writting the Virial identity (2). The uniform bound (9) implies that $\lim x_n \neq \infty$ and then equal, up to a translation, to 0. Now, we let t go to T and get

$$E(e^{i\frac{|x|^2}{4T}} u_0) = 0.$$

Since $\|e^{i\frac{|x|^2}{4T}} u_0\|_{L^2} = \|u_0\|_{L^2} = \|Q\|_{L^2}$, then the variational characterization of Q implies that $e^{i\frac{|x|^2}{4T}} u_0 \in \mathcal{A}$. This ends the proof of this theorem. $\qquad \square$

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[4]We can take first the limit at $t = 0$ to get $u_0 \in \Sigma$. This implies that $\int |x|^2 |u(t, x)|^2 dx$ is well defined for every $t \in [0, T[$. We take then the limit in all $t \in [0, T[$.

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